Buckingham Π - Theorem

$$a = f(\underbrace{a_1, \dots, a_k}_{\text{parameters with independent dimensions}}, \underbrace{b_1, \dots, b_m}_{\text{parameters that can be expressed as products of powers of } a_1, \dots, a_k})$$

a: dimensional quantity we're interested in

i.e. $b_1 = a_1^{p_1} \dots a^{r_1}$

- sometimes $m=0$
  - all the governing parameters have independent dimensions
- sometimes $k=0$
  - all quantities/governing parameters are dimensionless

in general: $k>0$ & $m>0$

Buckingham Π–Theorem: The physical relationship between a dimensional quantity and several dimensional parameters can be rewritten as a relationship between a dimensionless parameter and several dimensionless products of the governing parameters. The number of dimensionless products is equal to the number of governing parameters minus the number of independent dimensions.

Construct dimensionless parameters:

$$\Pi = \frac{a}{a_1 \cdots a_k} \qquad \Pi_1 = \frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}}$$

$$\Pi_2 = \frac{b_2}{a_1^{p_2} \cdots a_k^{r_2}}$$

$$\Pi_m = \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}$$

$$\Pi = \Phi(\Pi_1, \Pi_2, \dots, \Pi_m)$$

Recall: drag on a sphere

$$\underset{a}{D_f} = \underset{a_1}{\rho}\,\underset{a_2}{R}^2\,\underset{a_3}{v}^2\, F\Big(\overbrace{\frac{\mu}{\rho R v}}^{b_1}\Big)$$

a_1' a_2' a_3'

$$\underbrace{\frac{D_f}{\rho R^2 v^2}}_{\Pi} = F\Big(\underbrace{\frac{\mu}{\rho R v}}_{\Pi_1}\Big) \to \Phi(\Pi_1)$$

Essential Dimensionless Products

e.g. $\frac{1}{10} < \underbrace{\Pi_m}_{\text{essential}} < 10$

- if Π_m is small or large, assume that it can be neglected

e.g. Brittle Fracture

$$D = f(P, K, \nu, d, \Delta)$$

$$D \sim \underbrace{\left(\frac{P}{K}\right)^{2/3}}_{\text{Length}} \Phi\Big(\underset{\Pi_1}{\nu}, \underset{\Pi_2}{\frac{d}{(P/K)^{2/3}}}, \underset{\Pi_3}{\frac{\Delta}{(P/K)^{2/3}}}\Big)$$

neglect

$\Pi_2 \ll 1$ & $\Pi_3 \gg 1$

more formally:

- Π_m that is small or large can be neglected if $\Phi(\dots)$ has a finite, non-zero limit as $\Pi_m \to 0$ or $\Pi_m \to \infty$

Complete Similarity or Similarity of the First Kind

Incomplete Similarity or Similarity of the Second Kind

- if $\Pi_m \to 0$ or $\Pi_m \to \infty$ the function $\Phi(\dots)$ does not have a finite and non-zero limit
  - b_m can remain essential no matter how small or large its corresponding dimensionless parameter becomes
- if $\Phi(\dots)$ has a power-like asymptote representation then you can rewrite it as

$$\Phi = \Pi_m^{\alpha}\, \Phi_1(\Pi_1, \dots, \Pi_{m-1}) + \underbrace{O(\Pi_m^{\alpha})}_{\text{small compared with first term}}$$

$$\Pi^* = \Phi_1(\Pi_1, \dots, \Pi_{m-1})$$

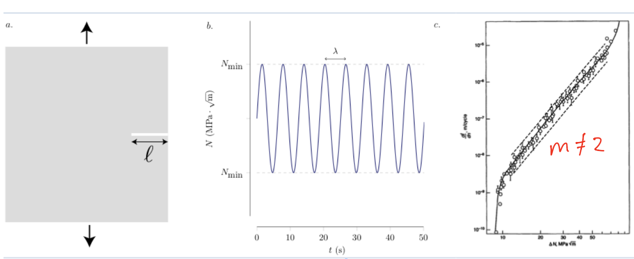
$$\frac{\Pi}{\Pi_m^{\alpha}} = \frac{a}{a_1^{p-\alpha p_m} \cdots a_k^{r-\alpha r_m}\, b_m}$$

still appears

α & Π^* cannot be determined by dimensional analysis alone

may depend on the dimensionless products in the problem

Fatigue : Incomplete Similarity



$$\frac{d\ell}{dn} = A(\Delta N)^m$$

material properties? geometry or specimen size?

$$\frac{d\ell}{dn} = f\Big(\Delta N, \underbrace{\frac{N_{max}}{N_{min}}}_{R}, \underset{\text{frequency}}{f}, \underset{\text{time}}{t}, \underset{\text{thickness}}{h}, \underset{\text{yield stress}}{\sigma_Y}, \underset{\text{stress intensity factor}}{K_{Ic}}\Big)$$

$$\left[\frac{d\ell}{dn}\right] = \left[\Delta N^a R^b f^c t^d h^e \sigma_Y^f K_{Ic}^g\right]$$

$$\frac{d\ell}{dn} = \left(\frac{\Delta N}{\sigma_Y}\right)^2 \Phi\Big(\underset{\text{small}}{\frac{\Delta N}{K_{Ic}}}, \boxed{R}, \underbrace{\frac{\sigma_Y \sqrt{h}}{K_{Ic}}}_{\boxed{Z}}, \underset{\text{neglect}}{ft}\Big)$$

$$\frac{d\ell}{dn} = \left(\frac{\Delta N}{\sigma_Y}\right)^2 \Phi_1(R, Z) = \underbrace{\frac{\Phi(R,Z)}{\sigma_Y^2}}_{A} (\Delta N)^{2 \to m} \cancel{=} A(\Delta N)^m$$

$m \neq 2$

Incomplete Similarity

$$\Phi = \left(\frac{\Delta N}{K_{Ic}}\right)^{\alpha} \Phi_1(R, Z) \longrightarrow \frac{d\ell}{dn} = \frac{(\Delta N)^{2+\alpha}}{\sigma_Y^2 K_{Ic}^{\alpha}} \Phi_1(R, Z) = \underbrace{\boxed{\frac{\Phi_1(R,Z)}{\sigma_Y^2 K_{Ic}^{\alpha}}}}_{A} (\Delta N)^{\overset{m}{2+\alpha}}$$

$\alpha = f(R, Z)$