Elastic Instabilities for Form and Function

Buckling, Wrinkling, Folding, and Snapping

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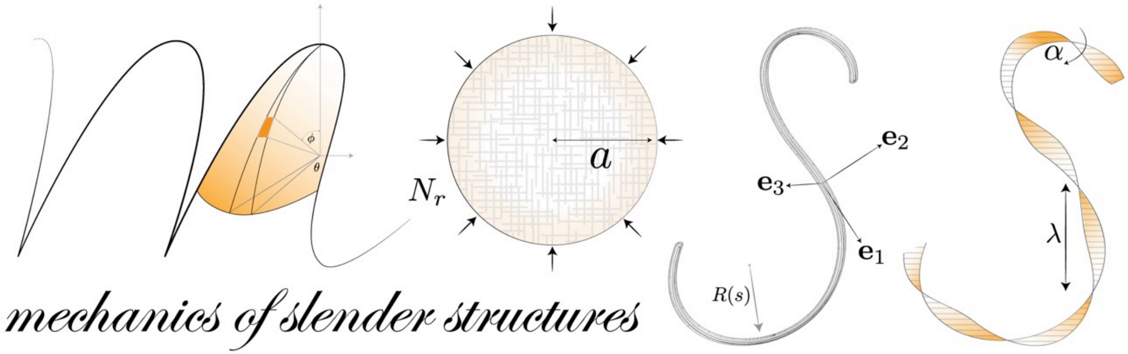
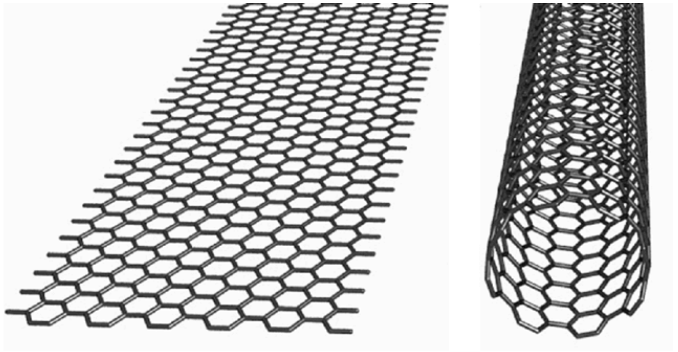
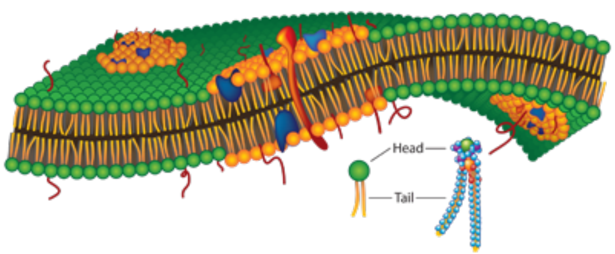
Sapienza, Università di Roma – Short Course (2015)





(a) (b)

Source: Kreupl *et al.* (2004)



http://www.lanl.gov/science/1663/august2011/images/Cell-Wave-Final.png

http://isabelleteo.deviantart.com/art/Just-hair-292904304

http://www.contactlensescomparison.com/wp-content/themes/smallbiz/images/lens.jpg

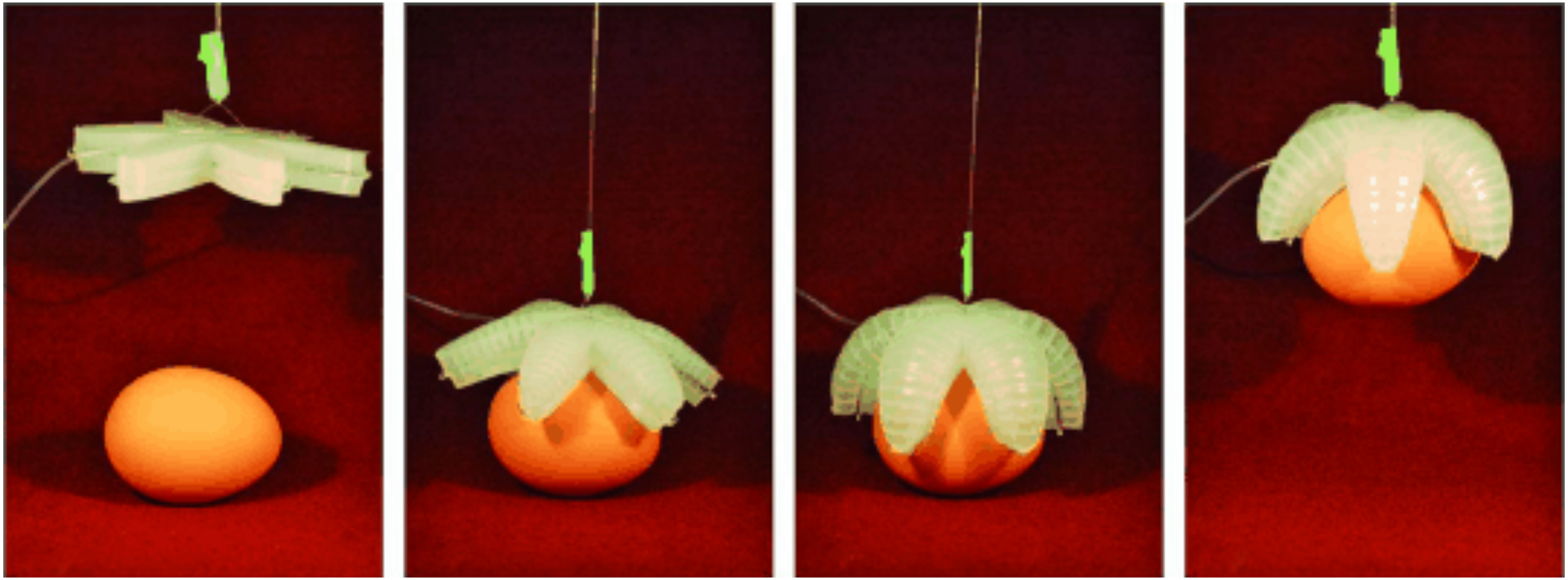
Thin Structures

Leonardo da Vinci, “Drapery study for a Seated Figure”, c1470

“British Lettuce”, http://www.britishleafysalads.co.uk/

Advanced Materials

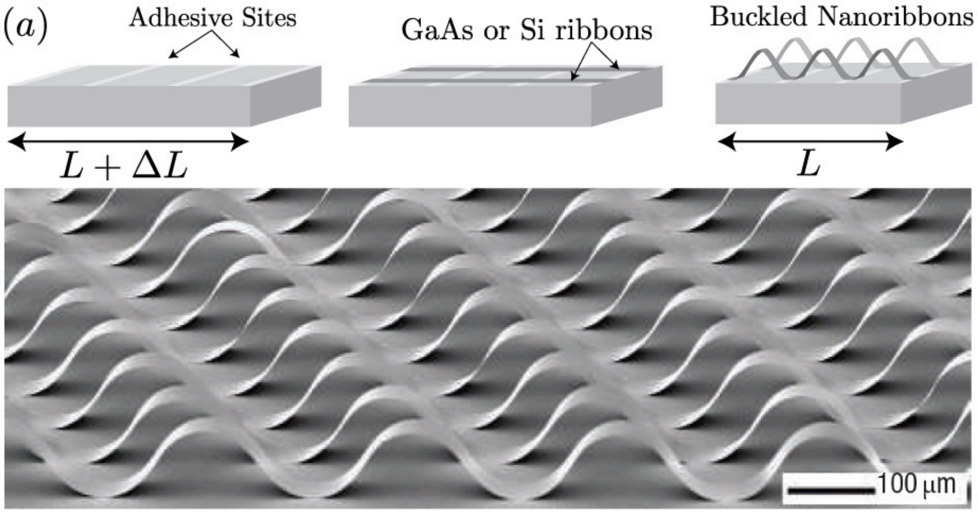
Soft Robotics



F. Ilievski, G.M. Whitesides, et al. "Soft Robotics for Chemists", *Angewandte Chemie*, **50**, 8, 1890-1895 (2011)

Advanced Materials

Flexible Electronics



mc10 Inc., Cambridge, MA

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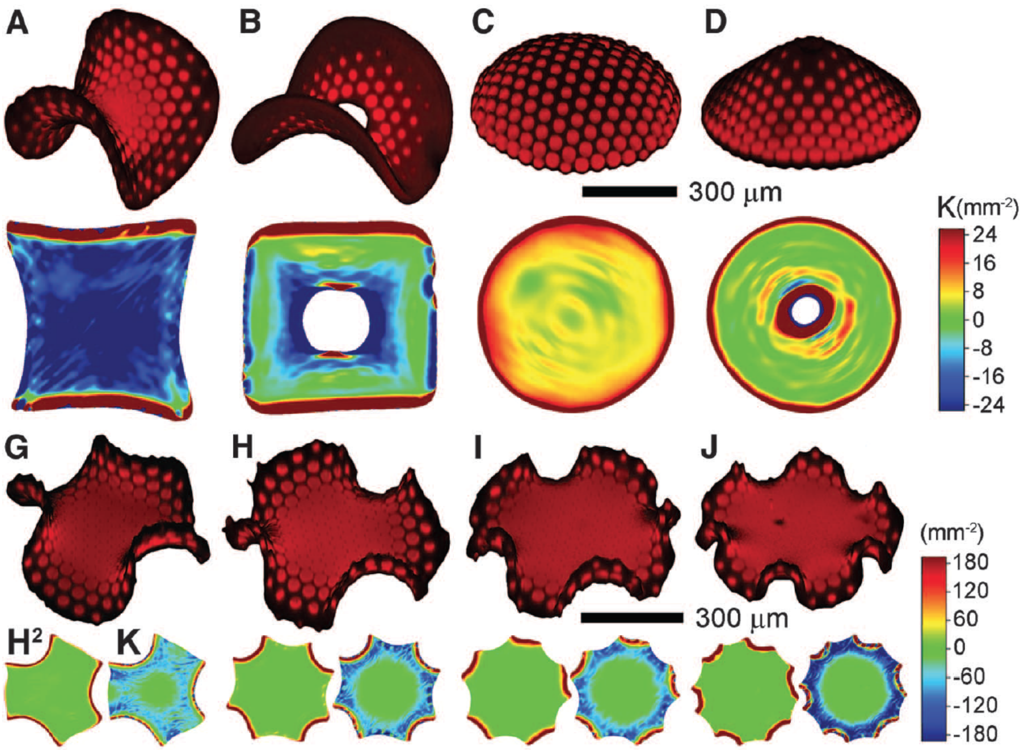
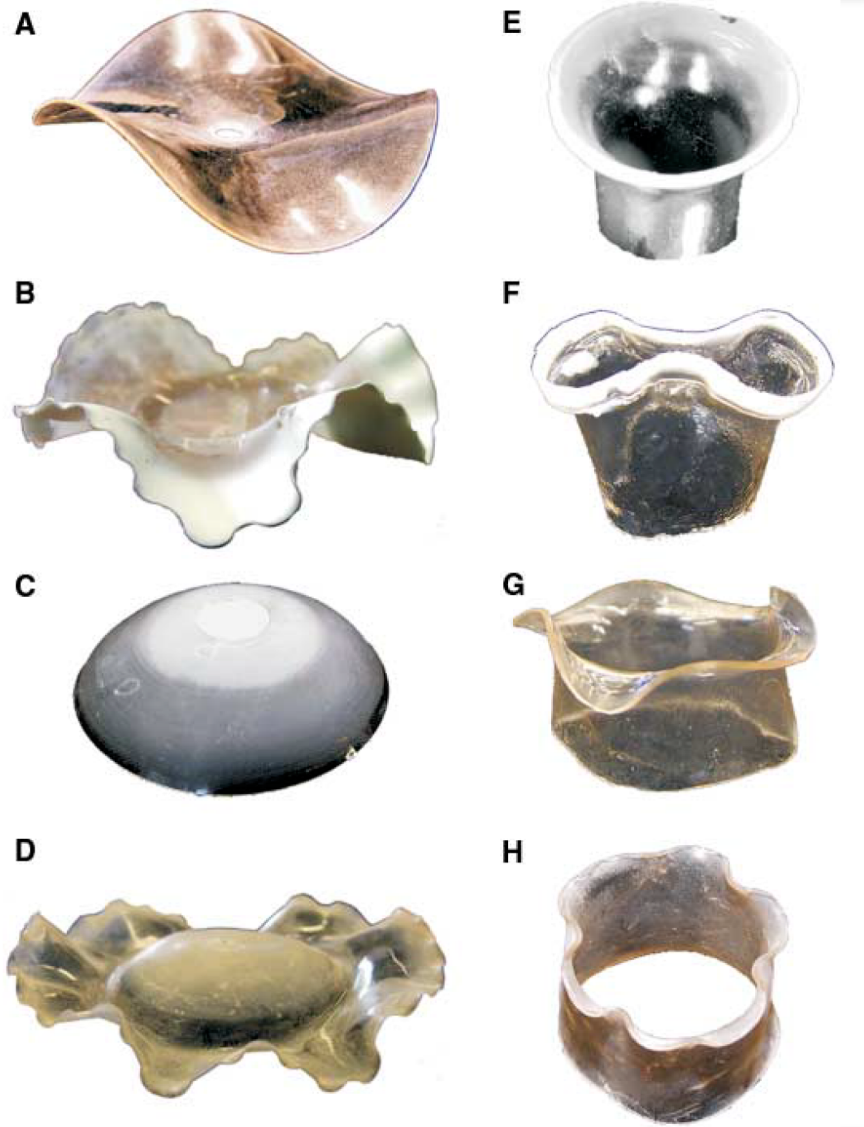
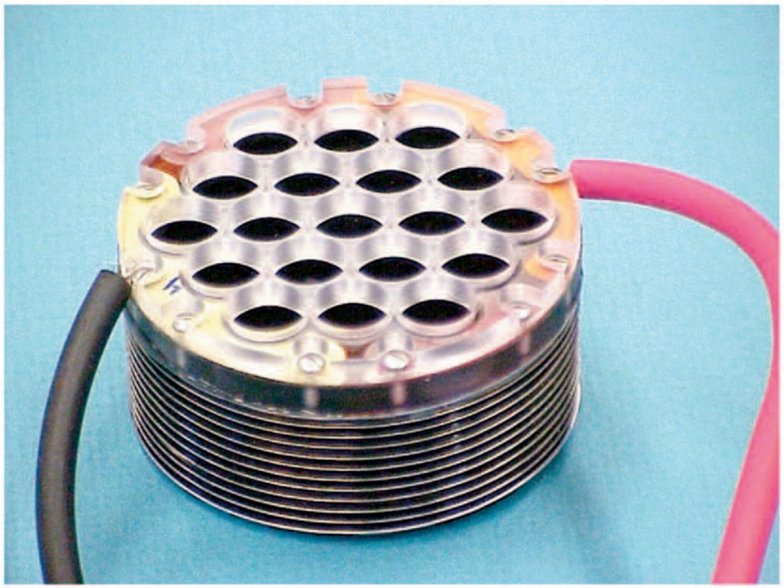
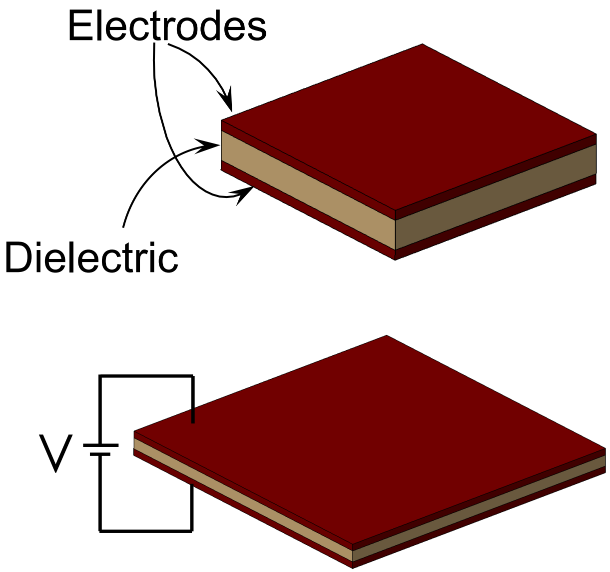
Morphing Structures





Energy Harvesting

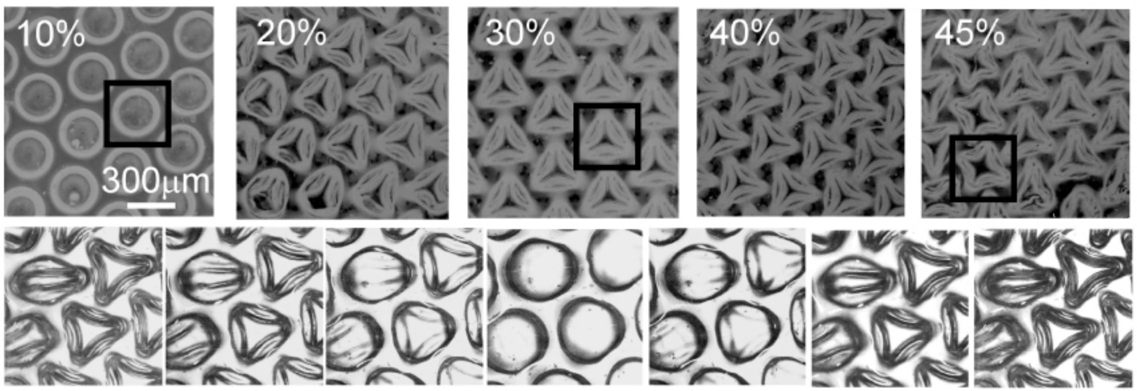


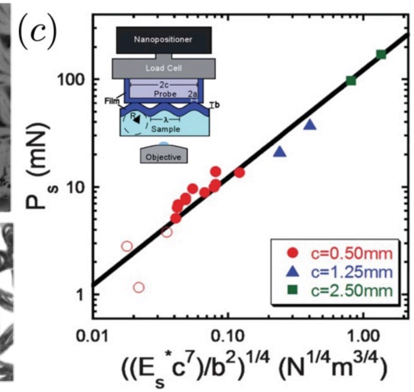


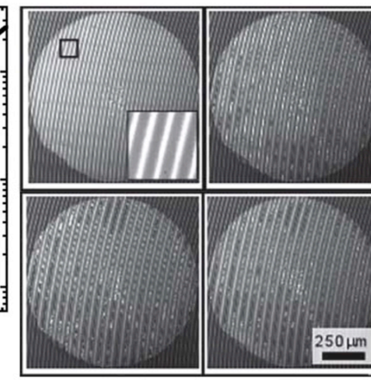
J.A. Paradiso and T. Starner, "Energy scavenging for mobile and wireless electronics", *Pervasive Computing, IEEE,* 4(1), 18, (2005).

Advanced Materials

Adaptive Surface Patterning







D.P. Holmes, and A.J Crosby, "Crumpled Surface Structures", *Soft Matter*, 4, 82, (2008).

Overview

Elastic Instabilities for Form and Function
Buckling, Wrinkling, Folding, and Snapping

Geometry and Mechanics:

- Fundamental equations, geometric rigidity, morphing.

Buckling & Wrinkling:

- Stability, wavelength, flexible electronics, mechanical metamaterials, adhesion.

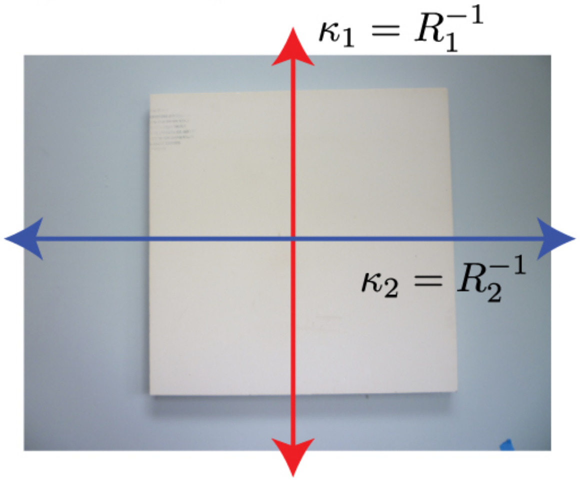
Stress Focusing – Folding & Creasing:

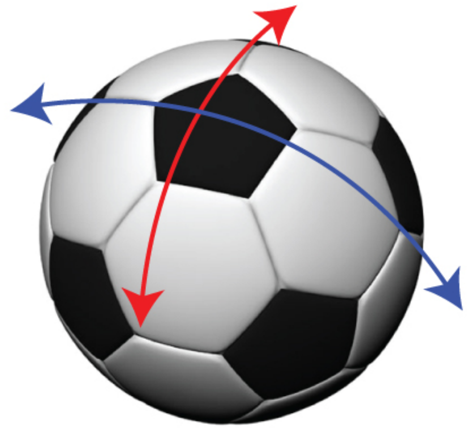
- Wrinkle-to-fold, origami.

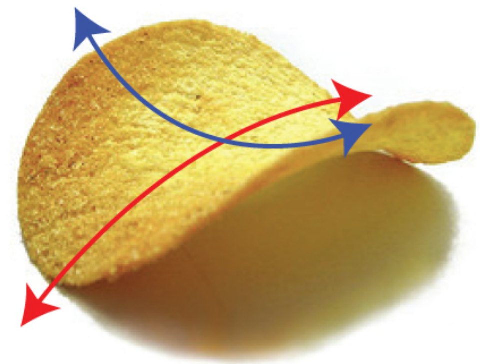
Snapping:

- Snapping surfaces.

Gauss Curvature

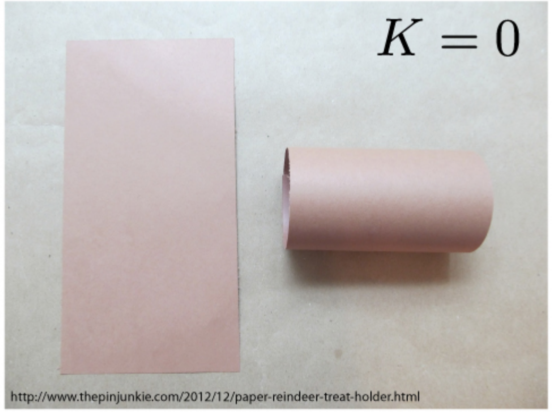


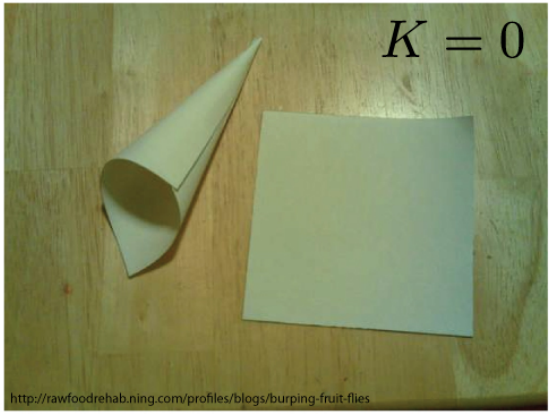




$K = \kappa_1\kappa_2 = \Diamond^4[w, w,] = 0$ $\quad$ $K = \kappa_1\kappa_2 = \Diamond^4[w, w,] > 0$ $\quad$ $K = \kappa_1\kappa_2 = \Diamond^4[w, w,] < 0$

Developable

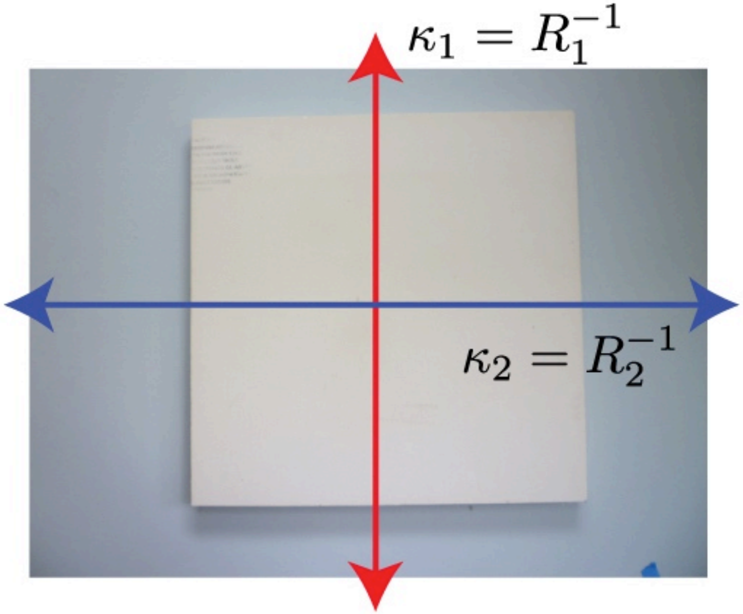




Mechanics of Plates

Elastic Energy (variational form)

$$\delta\mathcal{U}_e = \iiint_V \sigma_{ij}\delta\varepsilon_{ij}\ \mathrm{d}V$$



Elastic Energy (linear, isotropic material)

$$\mathcal{U}_e = \iiint_V \sigma_{ij}(\varepsilon_{kl})\ \varepsilon_{ij}\ \mathrm{d}V$$

Constitutive relationship:

$$\varepsilon_{ik} = \frac{1+\nu}{E}\sigma_{ik} - \frac{\nu}{E}\sigma_{jj}\delta_{ik}$$

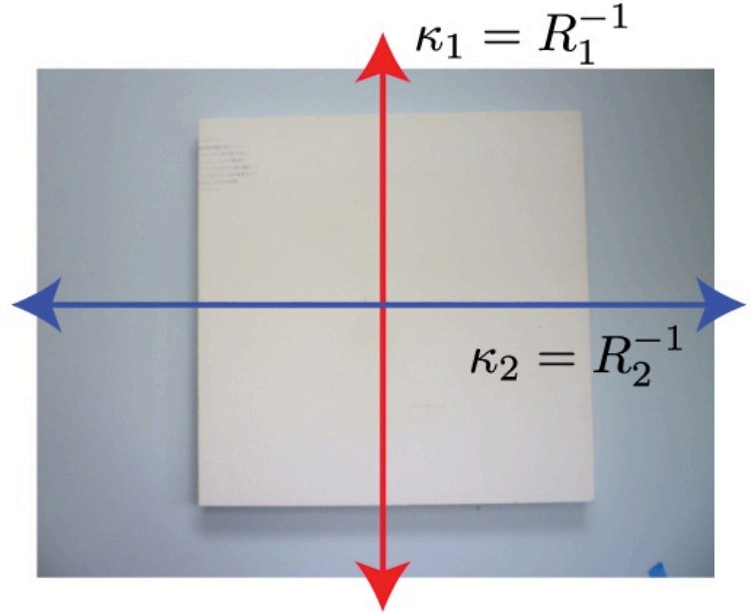
Mechanics of Plates

mechanics of slender structures

Elastic Energy (linear, isotropic material)

$$\mathcal{U}_e = \iint_A \mathrm{d}x\mathrm{d}y \int_{-h/2}^{h/2} \sigma_{ij}\varepsilon_{ij}\ \mathrm{d}z$$

Since *h* is small, it is often reasonable to **neglect** the **through thickness dependence** of the energy density.



Membrane Approximation

$$\mathcal{U}_m = \frac{h}{2} \iint_A \mathrm{d}x\mathrm{d}y \left(\sigma_{\alpha\beta}\varepsilon_{\alpha\beta}\right)_{cs}$$

Hooke's law:

$$\sigma_{\alpha\beta} \approx E\varepsilon_{\alpha\beta}$$

$$\mathcal{U}_m \sim Eh \iint_A \mathrm{d}x\mathrm{d}y\ \left(\varepsilon_{\alpha\beta}\right)^2$$

Scaling is valid to a numerical factor of order one, depending on Poisson's ratio.

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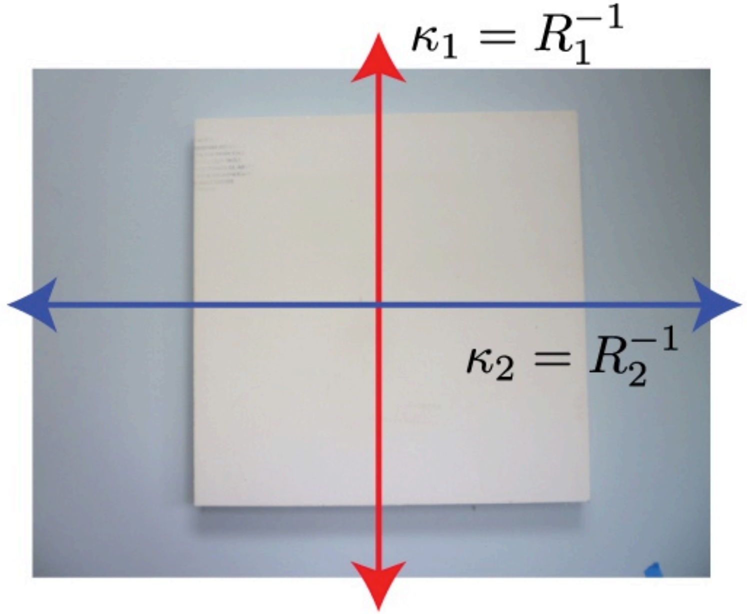
Mechanics of Plates

Membrane Approximation

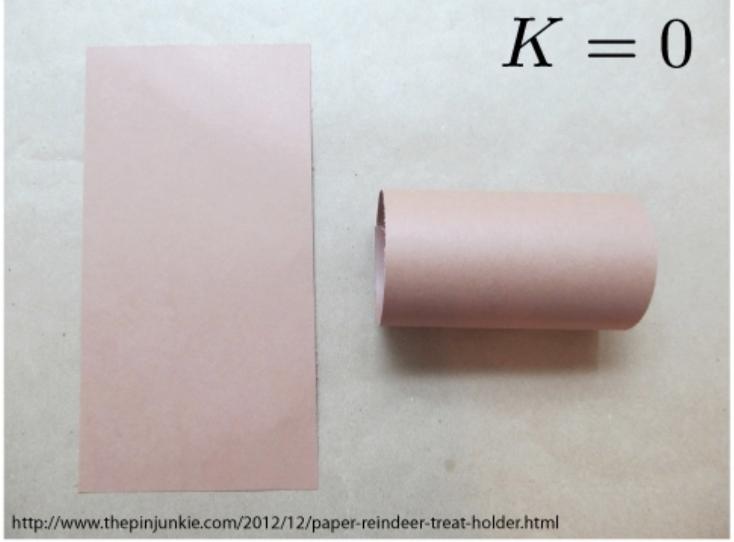
$$\mathcal{U}_m \sim Eh \iint_A \mathrm{d}x\mathrm{d}y \;\; (\varepsilon_{\alpha\beta})^2$$

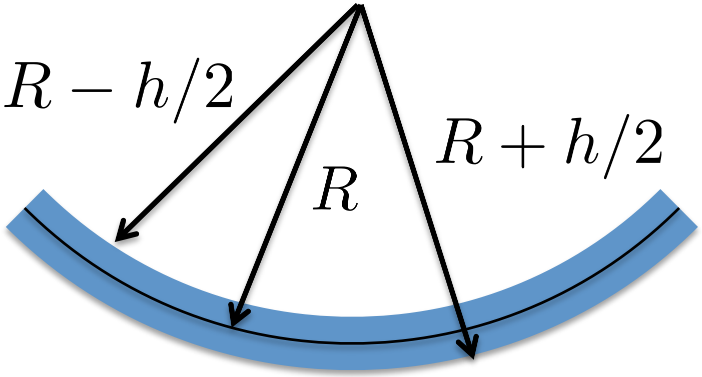


- Any reference to the transverse direction (z) has been eliminated by integration across the plate thickness.
- Membrane, or stretching energy, is linearly proportional to *E* and *h*.
- For an **isometric deformation**, the in-plane strain must vanish, therefore the membrane energy is zero.
- There are many ways to deform a plane isometrically – to break this degeneracy we need to consider the plate's **bending energy**.

Bending energy

- Curved plate is under **compression** on one side and **extension** on the other.
- This contribution is **missed** by **membrane theory** since the stress is averaged over the thickness.





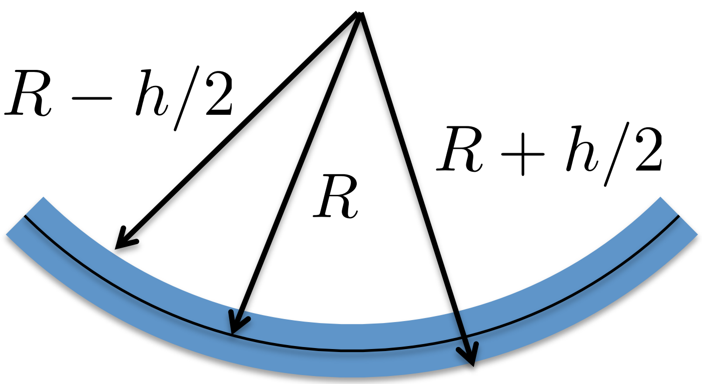
Radius of curvature of plate:

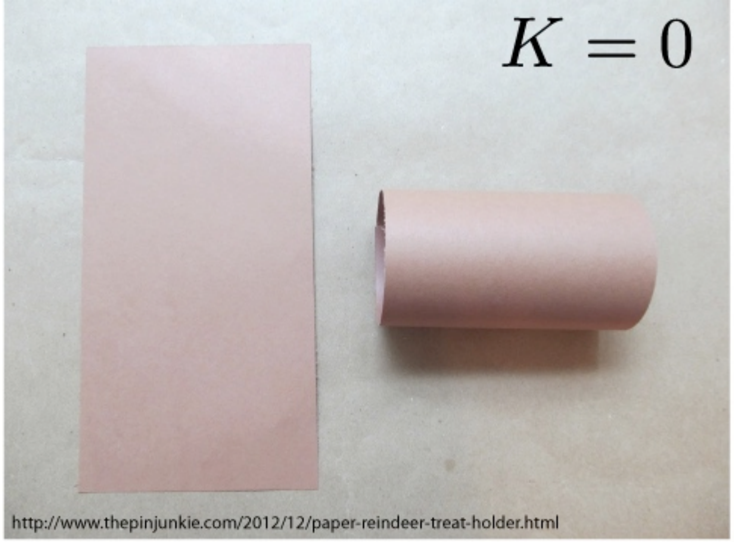
$$\frac{1}{R} \equiv \partial_x^2 w$$

Strain:

$$\varepsilon_{\alpha\beta}(x) = z\frac{\partial^2 w}{\partial x^2} = \frac{z}{R}$$

Bending energy





$$\mathcal{U}_e = \iint_A \mathrm{d}x\mathrm{d}y \int_{-h/2}^{h/2} \sigma_{ij}\varepsilon_{ij}\ \mathrm{d}z$$

Stresses & strains in bending:

$$\sigma_{ij} = Ez\partial_\alpha\partial_\beta w^2$$
$$\varepsilon_{ij} = z\partial_\alpha\partial_\beta w$$

Scaling of the bending energy:

$$\mathcal{U}_b \sim \frac{Eh^3}{24} \iint_A \mathrm{d}x\mathrm{d}y\ |\partial_\alpha\partial_\beta w|^2$$

Mechanics of Plates

Elastic Energy of a **Plate**

$$\mathcal{U} \sim \underbrace{EhA|\varepsilon_{\alpha\beta}|^2}_{\text{stretching}} + \underbrace{Eh^3A|\partial_\alpha\partial_\beta w|^2}_{\text{bending}}$$

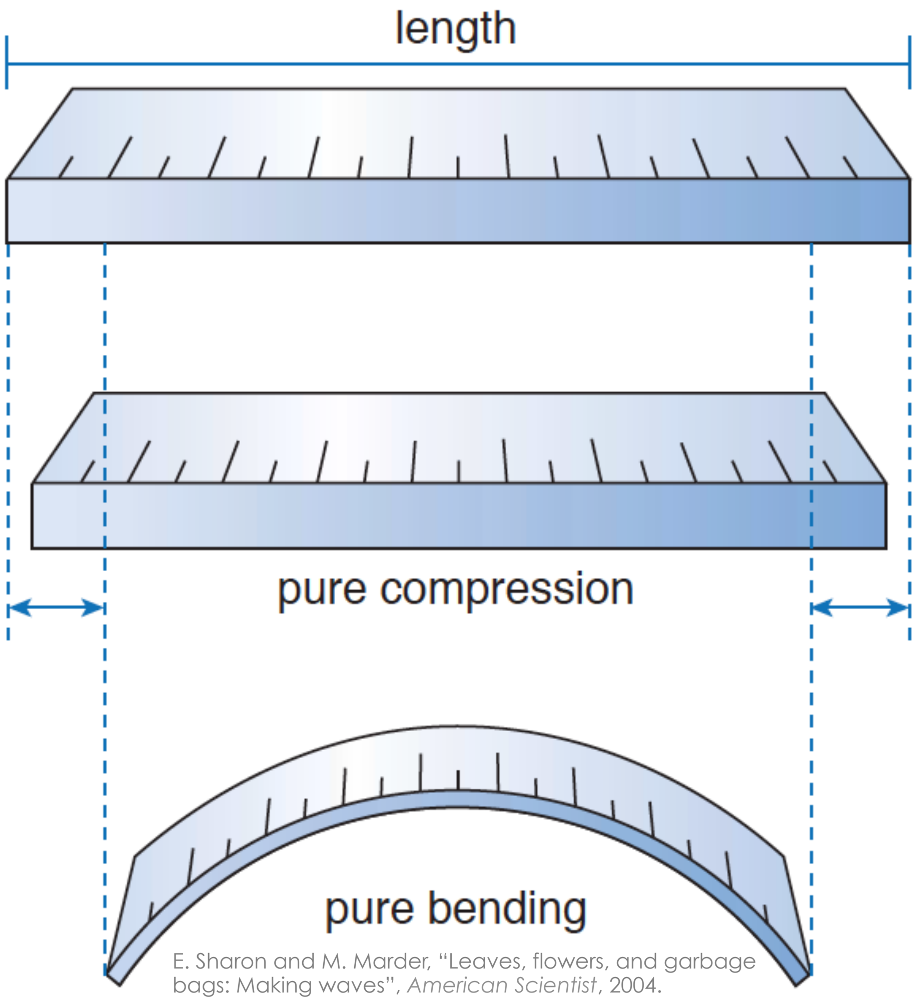
Scaling of the in-plane **strain** & out-of-plane **curvature**

$$|\varepsilon_{\alpha\beta}| \ \sim w^2/L^2 \qquad |b_{\alpha\beta}| = |\partial_\alpha\partial_\beta w| \ \sim w/L^2$$

Scaling of the elastic energy of the plate

$$\mathcal{U} \sim A\left[Eh\left(\frac{w}{L}\right)^4 + Eh^3\left(\frac{w}{L^2}\right)^2\right]$$

Mechanics of Plates



E. Sharon and M. Marder, "Leaves, flowers, and garbage bags: Making waves", *American Scientist*, 2004.

Bending vs. **Stretching**

E – Elastic Modulus
h – thickness
$\varepsilon_{\alpha\beta}$ – in-plane strain
κ – curvature

$$U_m \sim E\, h\, \varepsilon_{\alpha\beta}^2$$

Energy in Compression $\sim$ thickness

$$U_b \sim E\, h^3\, \kappa^2$$

Energy in Bending $\sim$ thickness^3

Thin structures deform by **bending** & avoid **stretching**

Mechanics of Plates

Scaling of the elastic energy of a plate

$$\mathcal{U} \sim A \left[\underbrace{Eh \left(\frac{w}{L}\right)^4}_{\text{stretching}} + \underbrace{Eh^3 \left(\frac{w}{L^2}\right)^2}_{\text{bending}} \right]$$

Ratio of stretching to bending:

$$\mathcal{U}_m / \mathcal{U}_b \sim w^2 / h^2$$

If deflections of the plate are small, i.e. $w \ll L$

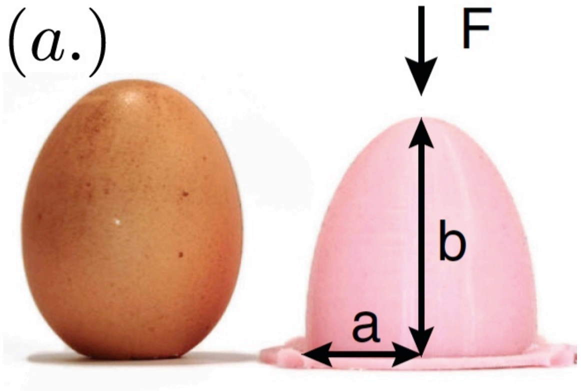
Stretching can be neglected,

Stretching can be decoupled from Bending.

Plates: Stretching and bending are decoupled.

Shells: ?

$(a.)$



Scaling of the elastic energy of a thin structure

$$\mathcal{U} \sim EhA|\varepsilon_{\alpha\beta}|^2 + Eh^3A|\partial_\alpha\partial_\beta w|^2$$

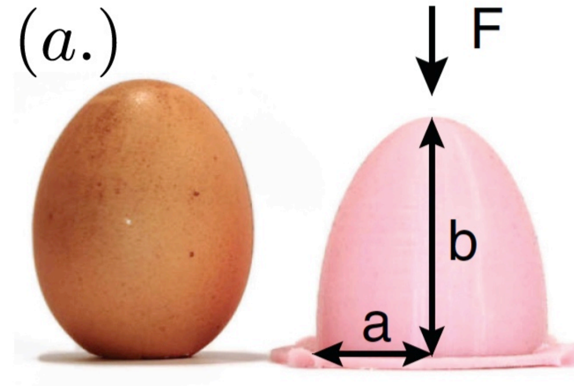
Assume deformations of an elliptical shell – material points are displaced radially by an amount w

- The relative extension, or strain, is given by:

$$|\varepsilon_{\alpha\beta}| \sim w/R$$ where *R* is the typical radius of curvature

Mechanics of Shells

Scaling of the elastic energy of a thin structure



$$\mathcal{U} \sim EhA|\varepsilon_{\alpha\beta}|^2 + Eh^3A|\partial_\alpha\partial_\beta w|^2$$

Strain: $|\varepsilon_{\alpha\beta}| \sim w/R$

Stretching energy: $$\mathcal{U}_s \sim EhA\left(\frac{w}{R}\right)^2$$

Bending energy: $$\mathcal{U}_b \sim Eh^3A\left(\frac{w}{R^2}\right)^2$$

For a uniform deformation...

Ratio of stretching to bending:

$$\mathcal{U}_s/\mathcal{U}_b \sim (R/h)^2 \gg 1$$

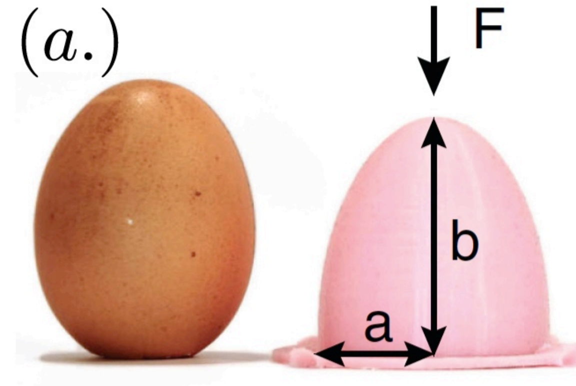
Stretching with bending for a curved shell is always a first order effect.

Mechanics of Shells

Scaling of the elastic energy of a thin structure

$$\mathcal{U} \sim EhA|\varepsilon_{\alpha\beta}|^2 + Eh^3A|\partial_\alpha\partial_\beta w|^2$$

Strain: $|\varepsilon_{\alpha\beta}| \sim w/R$



Point force applied to the shell, causing a local deformation of ℓ

Bending energy:

For a point force deformation…

$$\mathcal{U}_b \sim Eh^3A\left(\frac{w}{\ell^2}\right)^2$$

Balancing stretching and bending energies
(equivalent to minimizing total elastic energy):

Scaling law for
deformation depth:

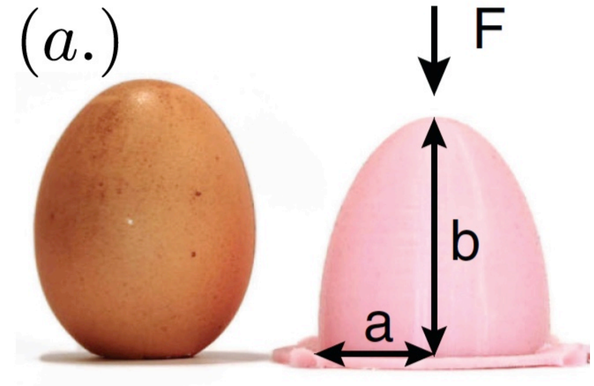
$$\ell \sim \sqrt{Rh}$$

Mechanics of Shells

Fundamental equations of **thin shell theory**

in-plane strain: $\varepsilon_{\alpha\beta} \equiv \frac{1}{2}(a_{\alpha\beta} - \overline{a}_{\alpha\beta})$

curvature strain: $\kappa_{\alpha\beta} \equiv b_{\alpha\beta} - \overline{b}_{\alpha\beta}$



Stress and moment are derived from the total elastic energy

$$\sigma^{\alpha\beta} \equiv \frac{\delta\mathcal{U}}{\delta\varepsilon_{\alpha\beta}} \equiv \frac{Eh}{1-\nu^2}\left[(1-\nu)\varepsilon^{\alpha\beta} + \nu\overline{a}^{\alpha\beta}\varepsilon^{\gamma}{}_{\gamma}\right] \qquad \sigma^{\alpha\beta} \sim \mathcal{O}(Eh\varepsilon)$$

$$\mu^{\alpha\beta} \equiv \frac{\delta\mathcal{U}}{\delta\kappa_{\alpha\beta}} \equiv \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\kappa^{\alpha\beta} + \nu\overline{a}^{\alpha\beta}\kappa^{\gamma}{}_{\gamma}\right] \qquad \mu^{\alpha\beta} \sim \mathcal{O}(Eh^3\kappa)$$

Shell equations are derived by expressions the variations in the in-plane and curvature strains.

Mechanics of Shells

Fundamental equations of **thin shell theory**

$$\underbrace{\overline{\nabla}_\alpha \sigma^{\alpha\beta}}_{\sim\mathcal{O}\left(\frac{Eh\varepsilon}{\ell}\right)} + \underbrace{2\bar{b}^\beta{}_\gamma \overline{\nabla}_\alpha \mu^{\gamma\alpha}}_{\sim\mathcal{O}\left(\frac{Eh^3\kappa}{R\ell}\right)} + \underbrace{\mu^{\gamma\alpha}\overline{\nabla}_\alpha \bar{b}^\beta{}_\gamma}_{\sim\mathcal{O}\left(\frac{Eh^3\kappa}{R\ell}\right)} + f^\beta = 0$$

Ratio of first term to next two

$$\text{ratio} \sim \mathcal{O}\left(\frac{\varepsilon}{h\kappa}\right)\,\mathcal{O}\left(\frac{R}{h}\right)$$

If $h\kappa \lesssim \varepsilon$

$$\overline{\nabla}_\alpha \sigma^{\alpha\beta} + f^\beta = 0$$

$$\overline{\nabla}_\alpha \overline{\nabla}_\beta \mu^{\alpha\beta} - \underbrace{\bar{b}_{\alpha\gamma}\bar{b}^\beta{}_\gamma \mu^{\alpha\beta}}_{\sim\mathcal{O}\left(\frac{Eh^3\kappa}{R^2}\right)} - \underbrace{\bar{b}_{\alpha\beta}\sigma^{\alpha\beta}}_{\sim\mathcal{O}\left(\frac{Eh\varepsilon}{R}\right)} - p = 0$$

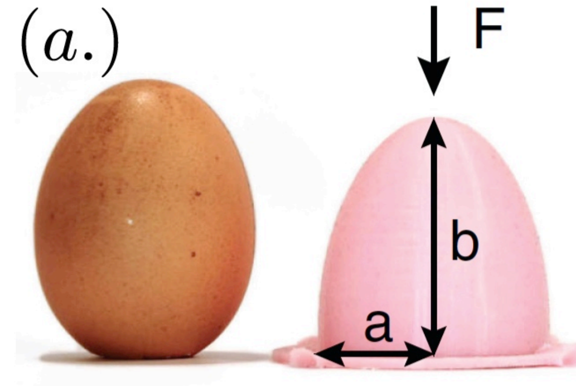
Ratio of second term to third term

$$\text{ratio} \sim \mathcal{O}\left(\frac{\varepsilon}{h\kappa}\right)\,\mathcal{O}\left(\frac{R}{h}\right)$$

If $h\kappa \lesssim \varepsilon$

$$\overline{\nabla}_\alpha \overline{\nabla}_\beta \mu^{\alpha\beta} - \bar{b}_{\alpha\beta}\sigma^{\alpha\beta} - p = 0$$

$(a.)$



$$\sigma^{\alpha\beta} \sim \mathcal{O}(Eh\varepsilon)$$

$$\mu^{\alpha\beta} \sim \mathcal{O}(Eh^3\kappa)$$

$$\ell \sim (Rh)^{1/2}$$

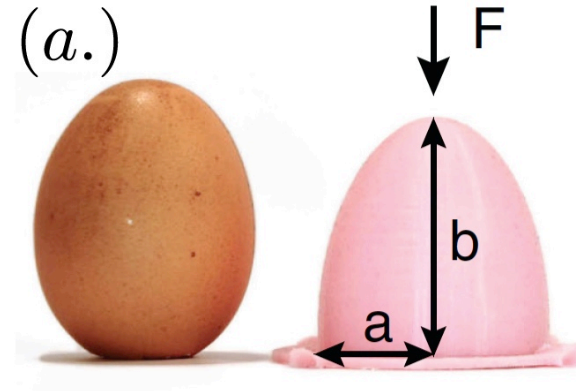
mechanics of slender structures

Mechanics of Shells

Approximate equations of **thin shell theory**

$$\overline{\nabla}_\alpha \sigma^{\alpha\beta} + f^\beta = 0$$

$$\overline{\nabla}_\alpha \overline{\nabla}_\beta \mu^{\alpha\beta} - \bar{b}_{\alpha\beta}\sigma^{\alpha\beta} - p = 0$$



$$\sigma^{\alpha\beta} \sim \mathcal{O}(Eh\varepsilon)$$

$$\mu^{\alpha\beta} \sim \mathcal{O}(Eh^3\kappa)$$

$$\ell \sim (Rh)^{1/2}$$

As long as the Gaussian curvature is much less the curvature set by the deformation length:

$$\mathcal{K} \ll 1/\ell^2$$

This problem can be solved by the introduction of particular scalar field (i.e. Airy potential), and ensuring compatibility between the strains.

$$\frac{1}{Eh}\overline{\nabla}^4\phi + \bar{b}_{\alpha\beta}\epsilon^{\alpha\gamma}\epsilon^{\beta\gamma}\overline{\nabla}_\gamma\overline{\nabla}_\lambda w = 0$$

$$B\overline{\nabla}^4 w - \bar{b}_{\alpha\beta}\epsilon^{\alpha\gamma}\epsilon^{\beta\gamma}\overline{\nabla}_\gamma\overline{\nabla}_\lambda \phi = 0$$

Donnell-Mushtaru-Vlasov (DMV) Equations

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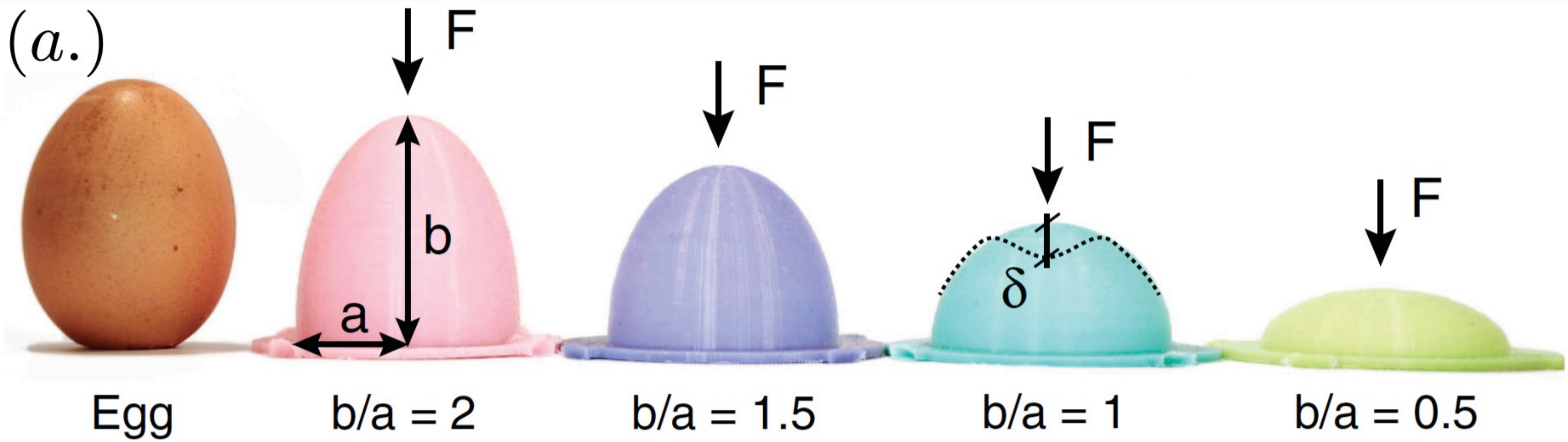
Geometric Rigidity

Geometric Rigidity



DMV Equations (point load at the apex)

$$B\nabla^4 w - \nabla_k^2 \phi - \Diamond^4[\phi, w] = p - \frac{F}{2\pi}\frac{\delta(r)}{r}$$

Introducing the operators:

$$\nabla_k^2(f) \equiv R_\beta^{-1} f_{,\alpha\alpha} + R_\alpha^{-1} f_{,\beta\beta}$$

$$\Diamond^4[f,g] \equiv f_{,\alpha\alpha} g_{,\beta\beta} - 2 f_{,\alpha\beta} g_{,\alpha\beta} + f_{,\beta\beta} g_{,\alpha\alpha}$$

Can be reduced to an ODE by assuming the pressurized shell is experiencing a uniform state of stress: $\sigma = pR/2$

$$B\nabla^4 w - \sigma \nabla_k^2 w - \frac{Eh}{R^2} w = -\frac{F}{2\pi}\frac{\delta(r)}{r}$$

Lazarus et al. 2012., Vella et al. 2012

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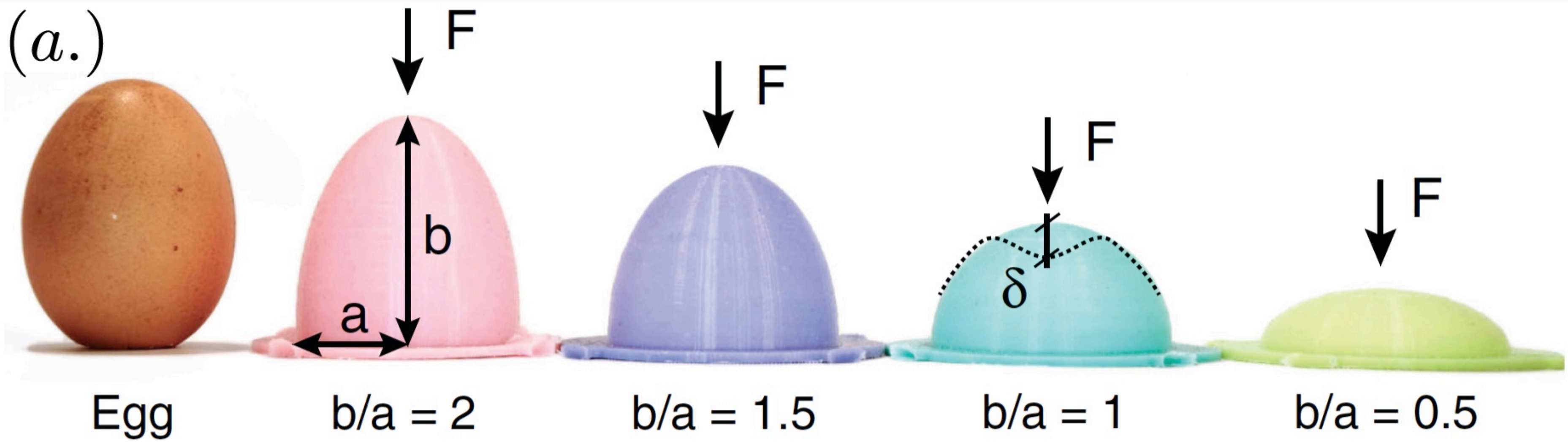
Geometric Rigidity



Integration of the ODE leads to a linear relationship between force and displacement:

$$F = k_1 w_0$$

The shell's rigidity is given by: $k_1 \sim \dfrac{4\pi B}{\ell_b^2}\dfrac{\tau}{\log 2\tau}$

Dimensionless pressure:

$\tau = \frac{1}{4}pR^2(EhB)^{-1/2}$

In the Reissner limit of $\tau \ll 1$

$$k_1 = \frac{8B}{\ell_b^2} = 8\sqrt{BEhK}$$

The shell's **rigidity** is tied to its **Gaussian curvature** – **deforming** the shell requires in-plane **stretching**, which is energeticallly costly.

Lazarus et al. 2012., Vella et al. 2012

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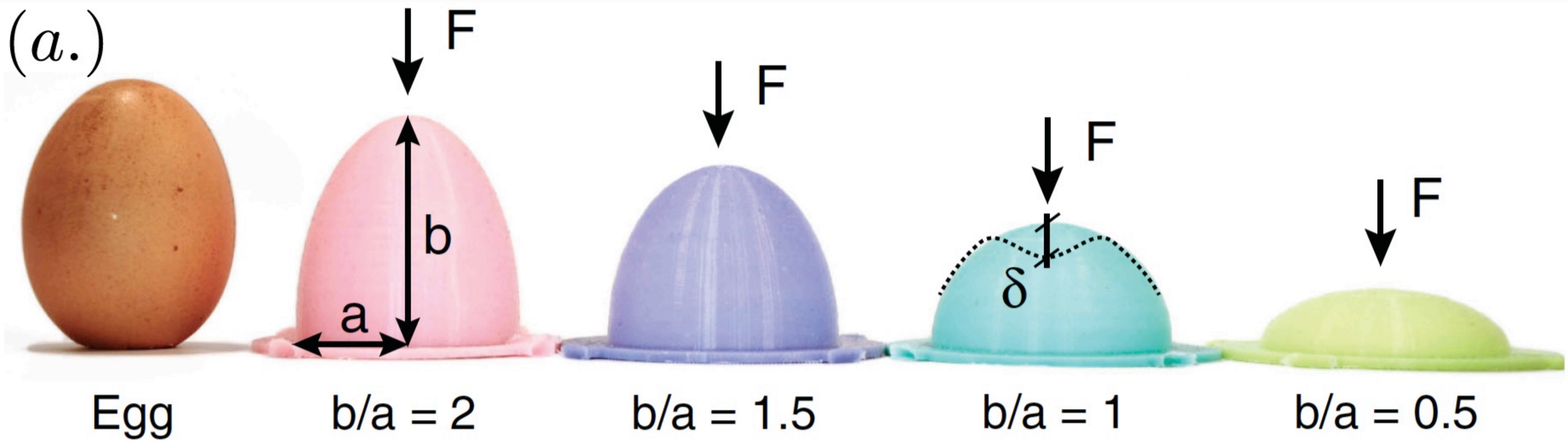
Geometric Rigidity



In the Reissner limit of $\tau \ll 1$

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The shell's **rigidity** is tied to its **Gaussian curvature** – **deforming** the shell requires in-plane **stretching**, which is energeticallly costly.

For an elliptical shell, $R = a^2/b$

$$k_1 = \frac{b}{a} k_1^s$$

Enhancement of structural rigidity by changing the shell's aspect ratio.

Lazarus et al. 2012., Vella et al. 2012

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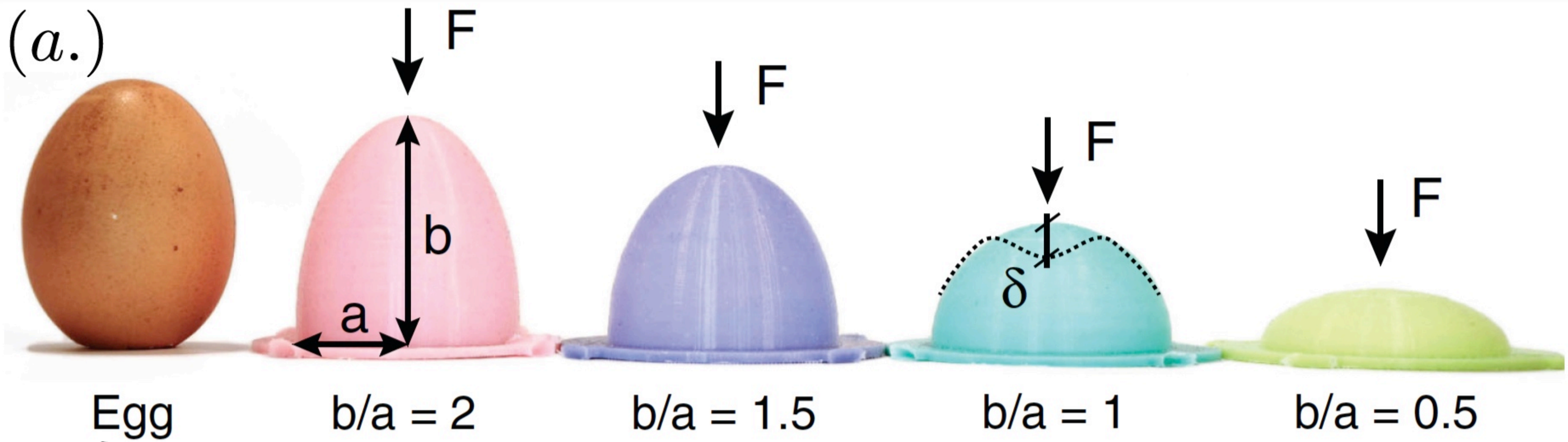
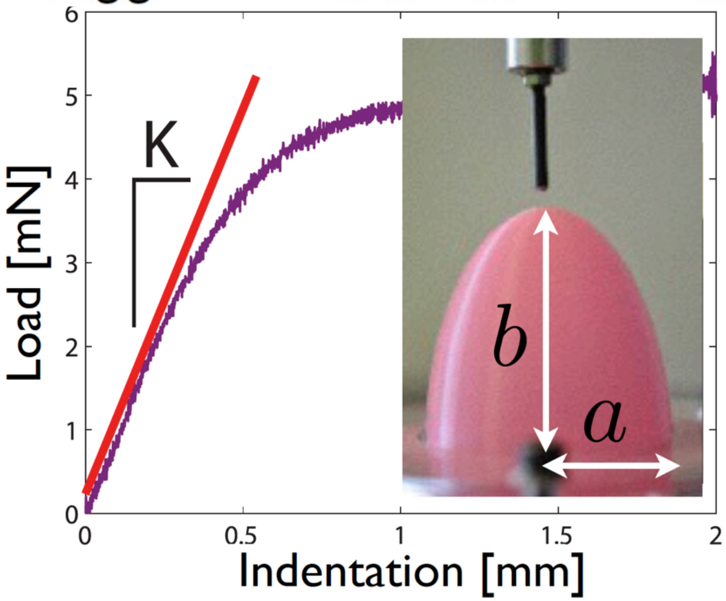
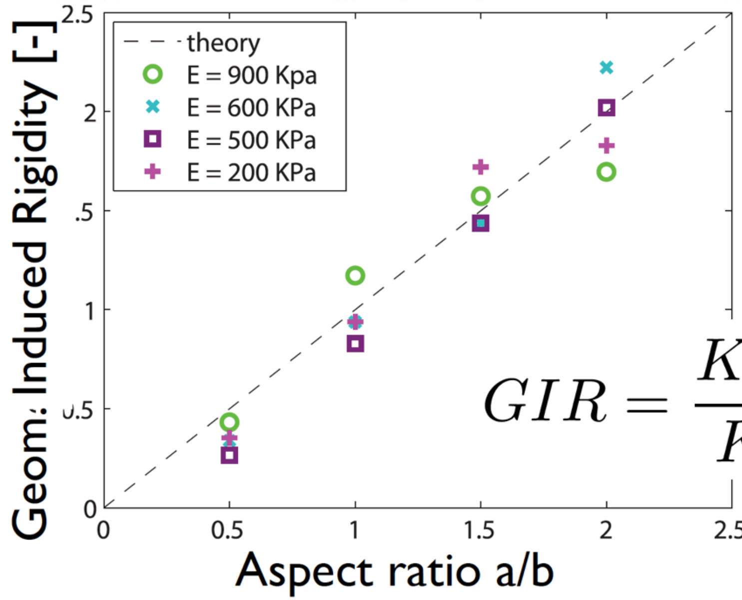
Geometric Rigidity







$$GIR = \frac{K_{ellispoid}}{K_{sphere}}$$

Lazarus et al. 2012.

Geometric Rigidity

Large deformations:
(From experiments and numerics)

A second linear relationship between force and displacement:

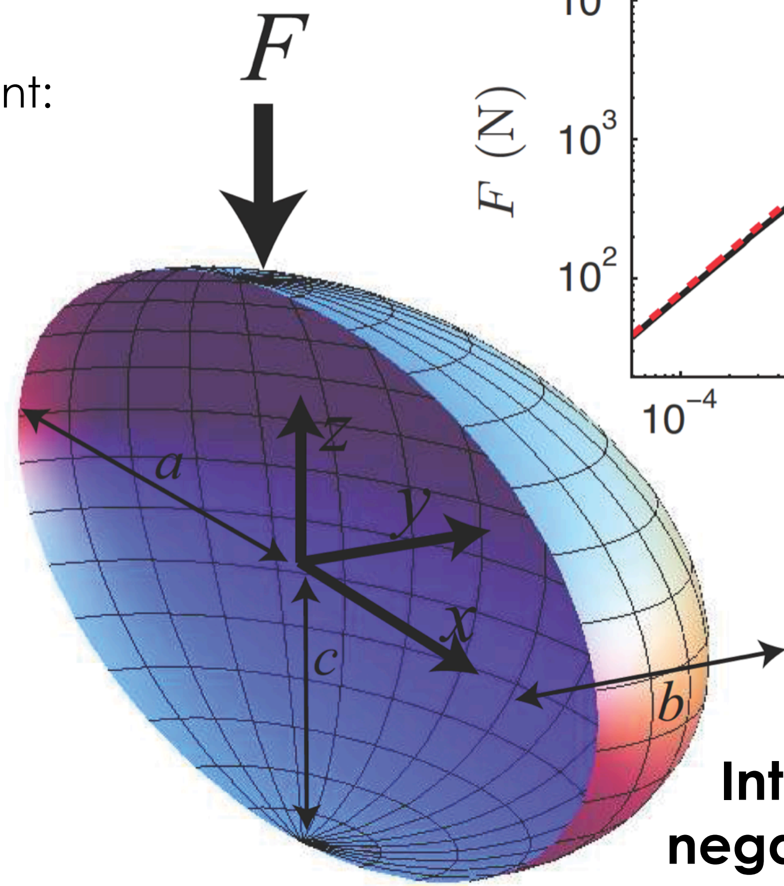
$$F = k_2 w_0$$

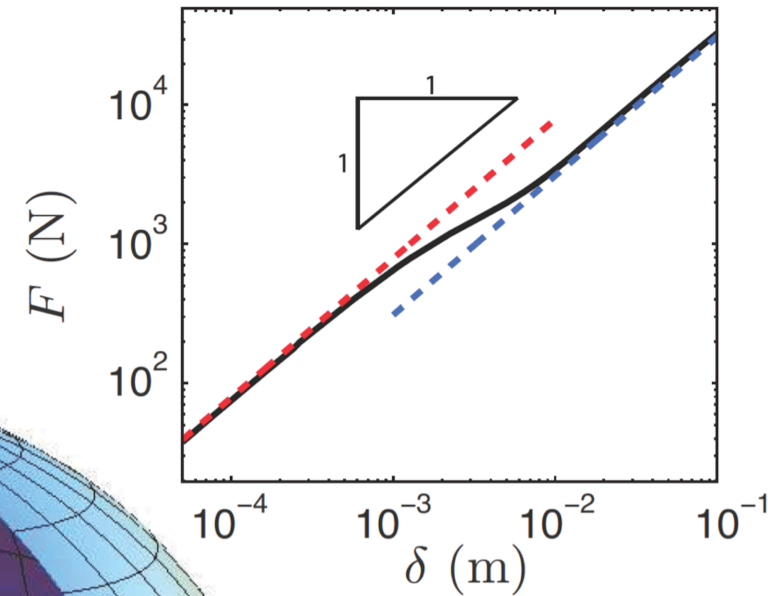
Considering the limit of large pressures, bending can be neglected:
(Balance of in-plane stretching to shell stretching from internal pressure)

$$\ell_p \sim R\left(\frac{pR}{Eh}\right)^{1/2}$$

Mean Curvature governs shell rigidity

$$k_2 = \pi p \mathcal{H}$$





Internal pressure negates the effect of geometric rigidity

Lazarus et al. 2012.

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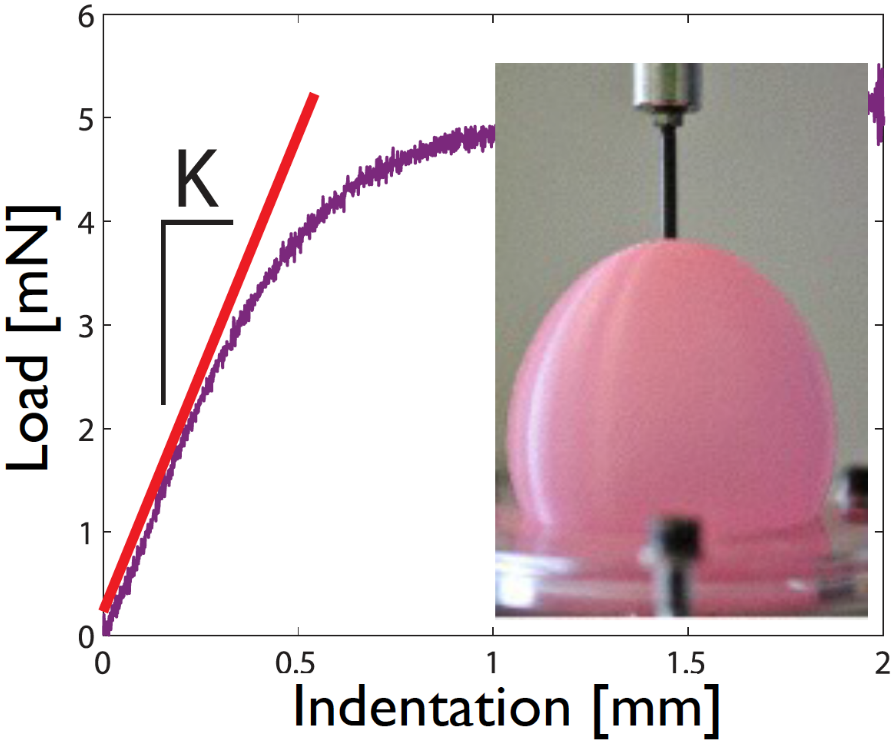
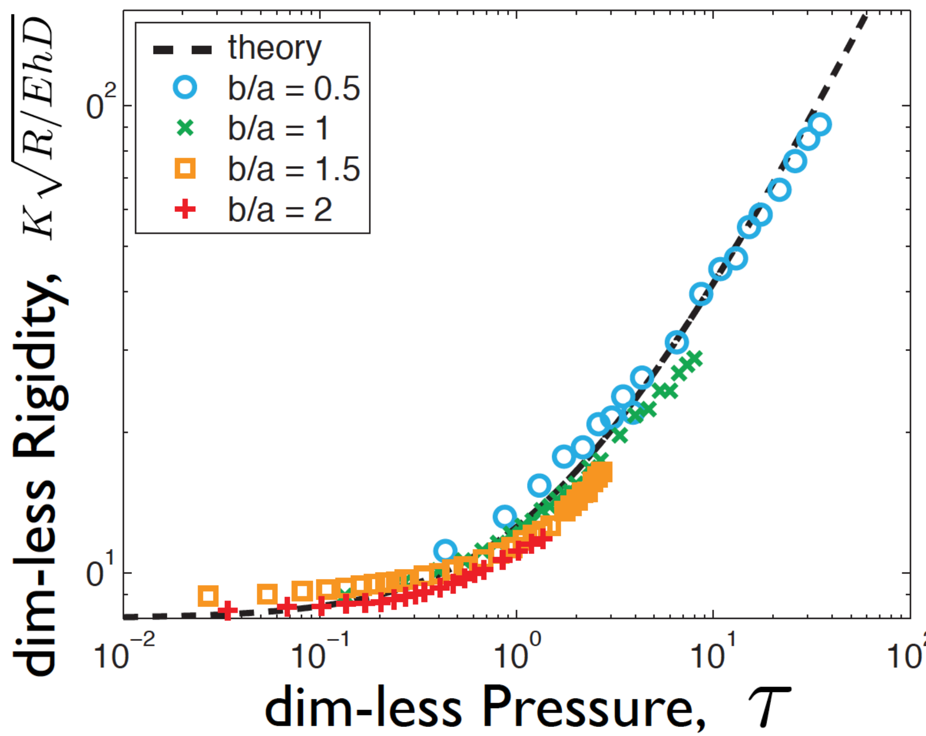
Geometric Rigidity





Lazarus et al. 2012.

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Geometric Rigidity

Saccharomyces cerevisiae

AFM experiments measured yeast cell stiffness as osmotic pressure was varied.

Deflections ~ thickness, low internal pressure.

Turgor pressure estimated of 0.1 to 0.2 MPa, consistent with measurements from other techniques.

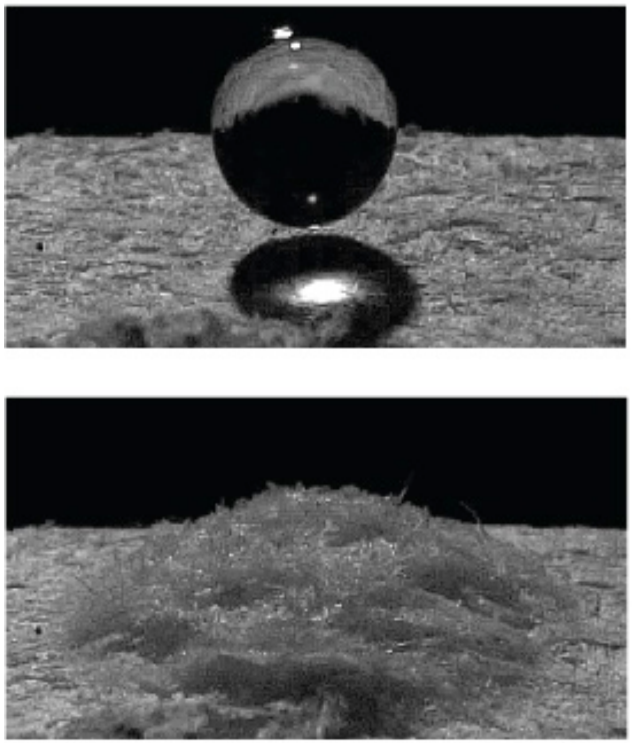
Same technique being used to measure **elastic** properties of **tomato fruit cells, plant tissures**, and **artificial biological microcapsules.**

http://what-when-how.com/wp-content/uploads/2011/09/tmp1717_thumb.jpg

Geometric Morphing

Swelling & Growth

Materials Science



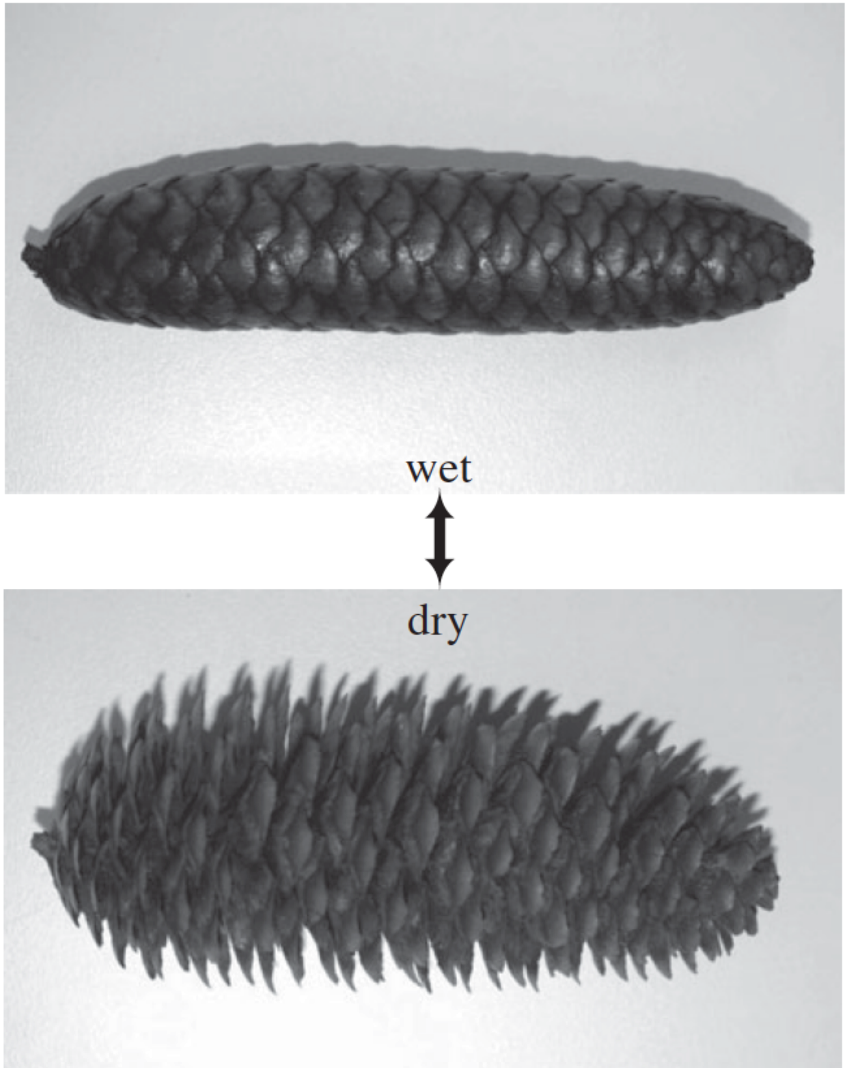
Swelling of a sponge.

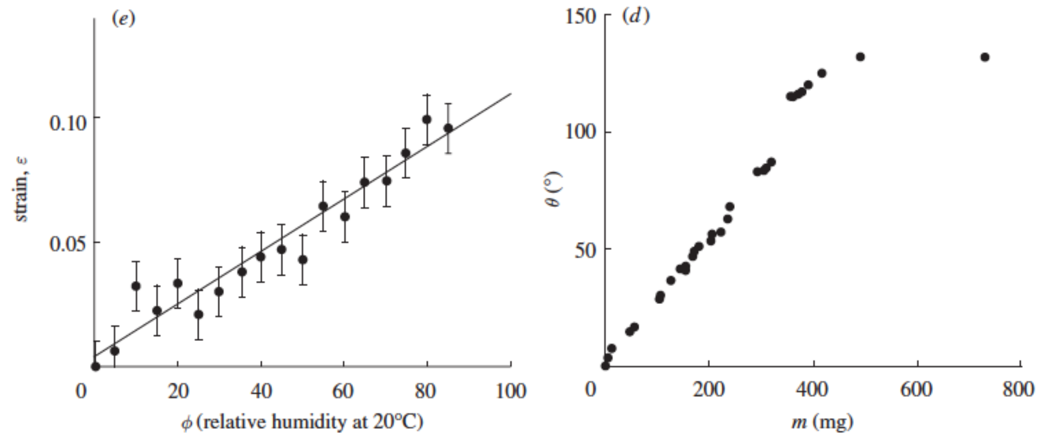
Mechanics

An almond leaf which was attacked by *Taphrina Deformans*.

E. Sharon and E. Efrati. *Soft Matter*, **6**, 5693, 2010.

Pine Cones





Tree-bound pine cones:

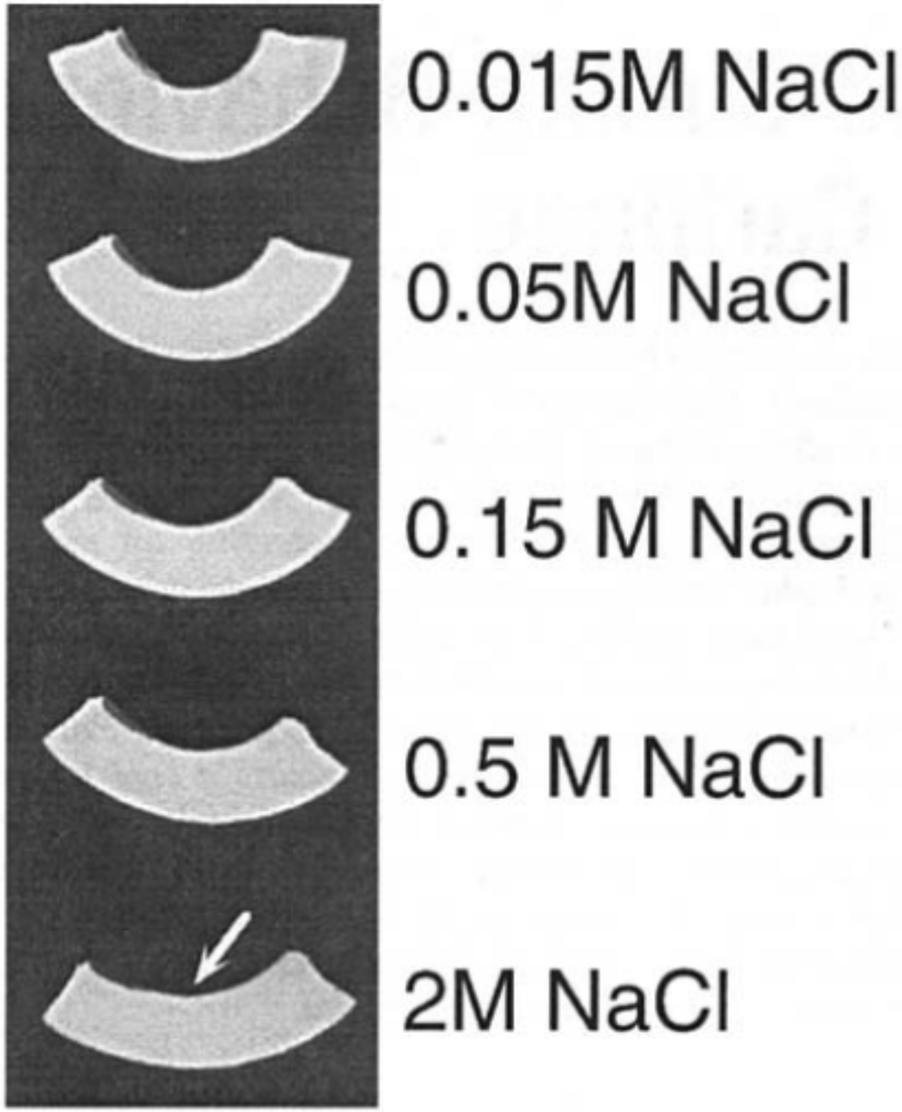
Hydrated & **closed**, protecting seeds

Fallen pine cones:

Dried out & **opened**, releasing seeds

E. Reyssat and L. Mahadevan. *Journal of the Royal Society Interface*, **6**, 951, 2009.

Articular Cartilage



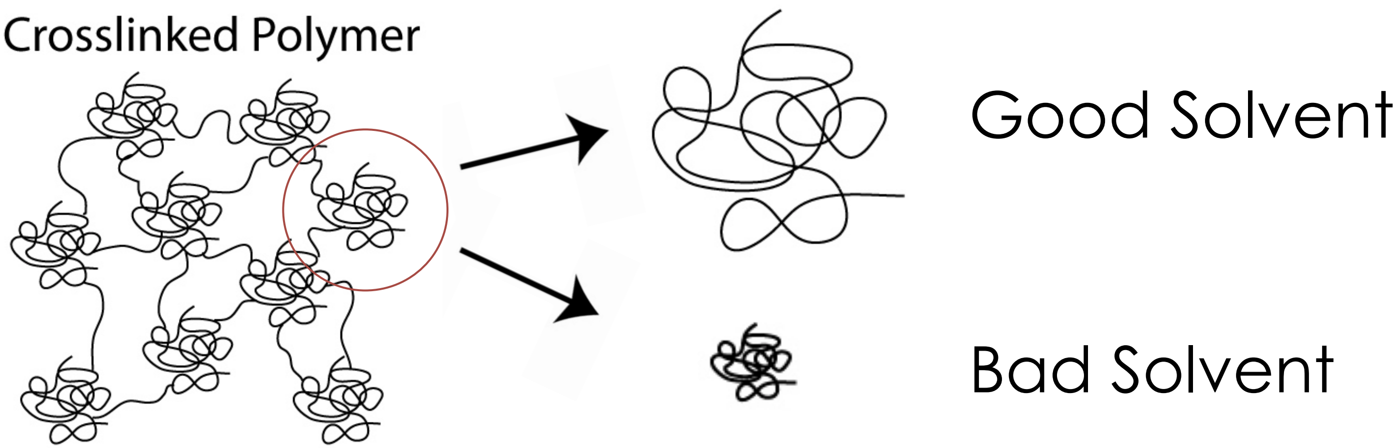
Shape change caused by ion concentration.

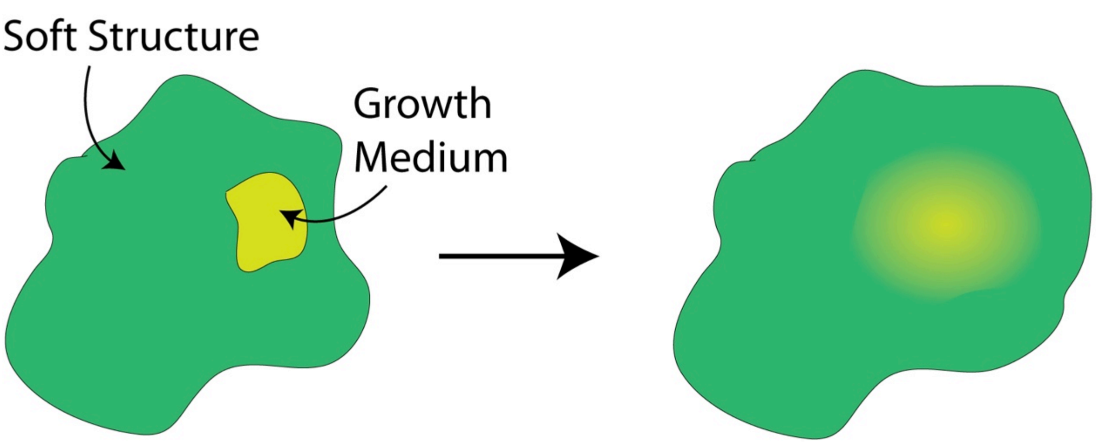
Residual strain at physiological conditions: **3-15%**

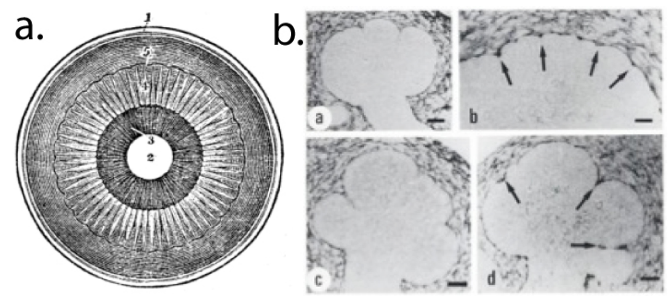
Tensile prestress in cartilage protective against frequent compresses forces.

L. A. Setton, H. Tohyama, and V. C. Mow, *Journal of Biomedical Engineering*, **120**, 355, 1998.

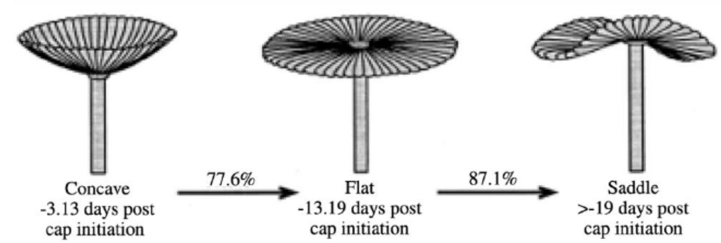
Swelling & Growth







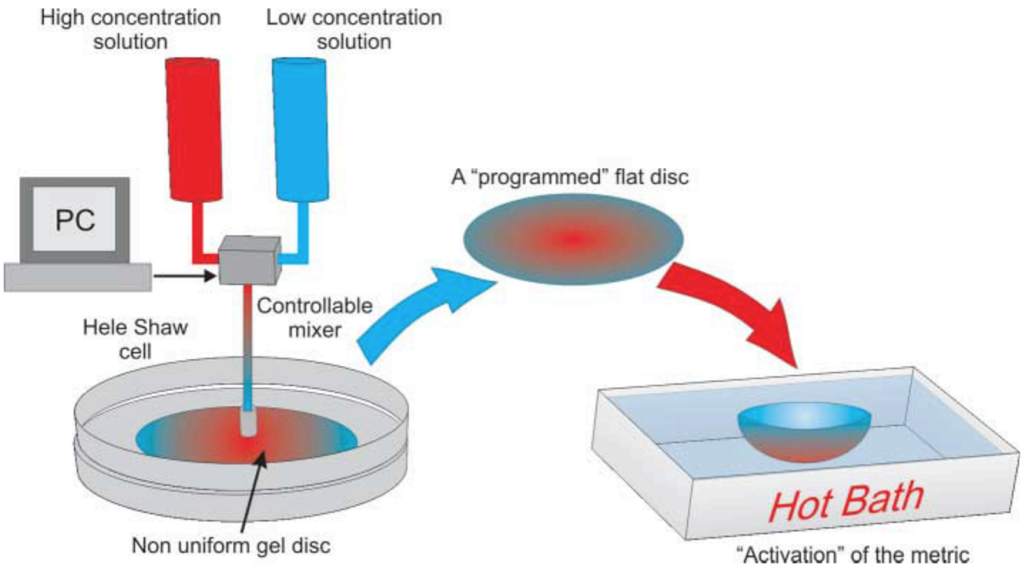
Deformation confined to surface

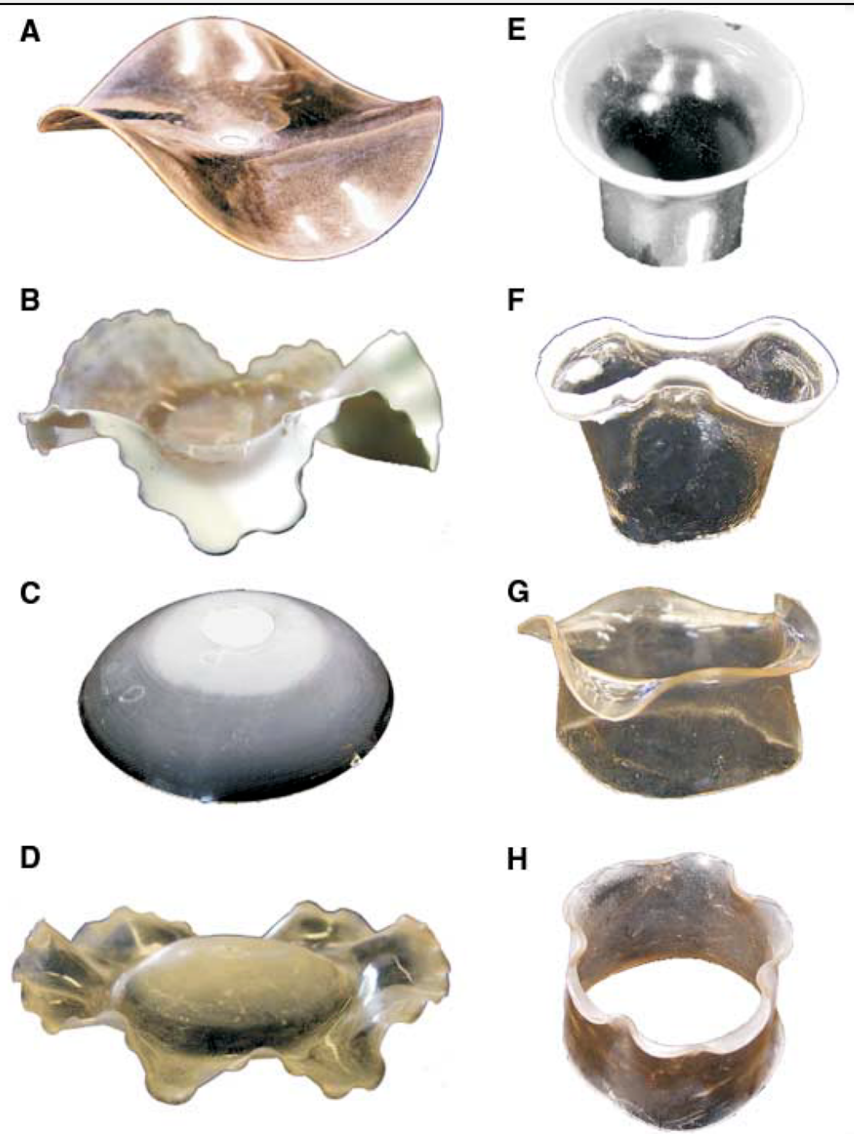


Deformation of entire structure

J. Bard. *Morphogenesis: The cellular and molecular processes of developmental anatomy*, Cambridge University Press, 1990.
J. Dervaux and M. Ben Amar. *Physical Review Letters*, **101**, 068101, 2008.

Shaping Sheets



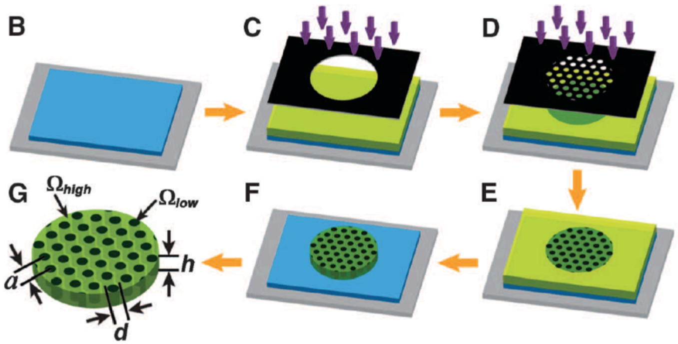


Shaping elastic sheets by prescribing non-Euclidean metrics

- Prepare gels that undergo nonuniform shrinkage.
- Buckling thin films based on chosen metrics.

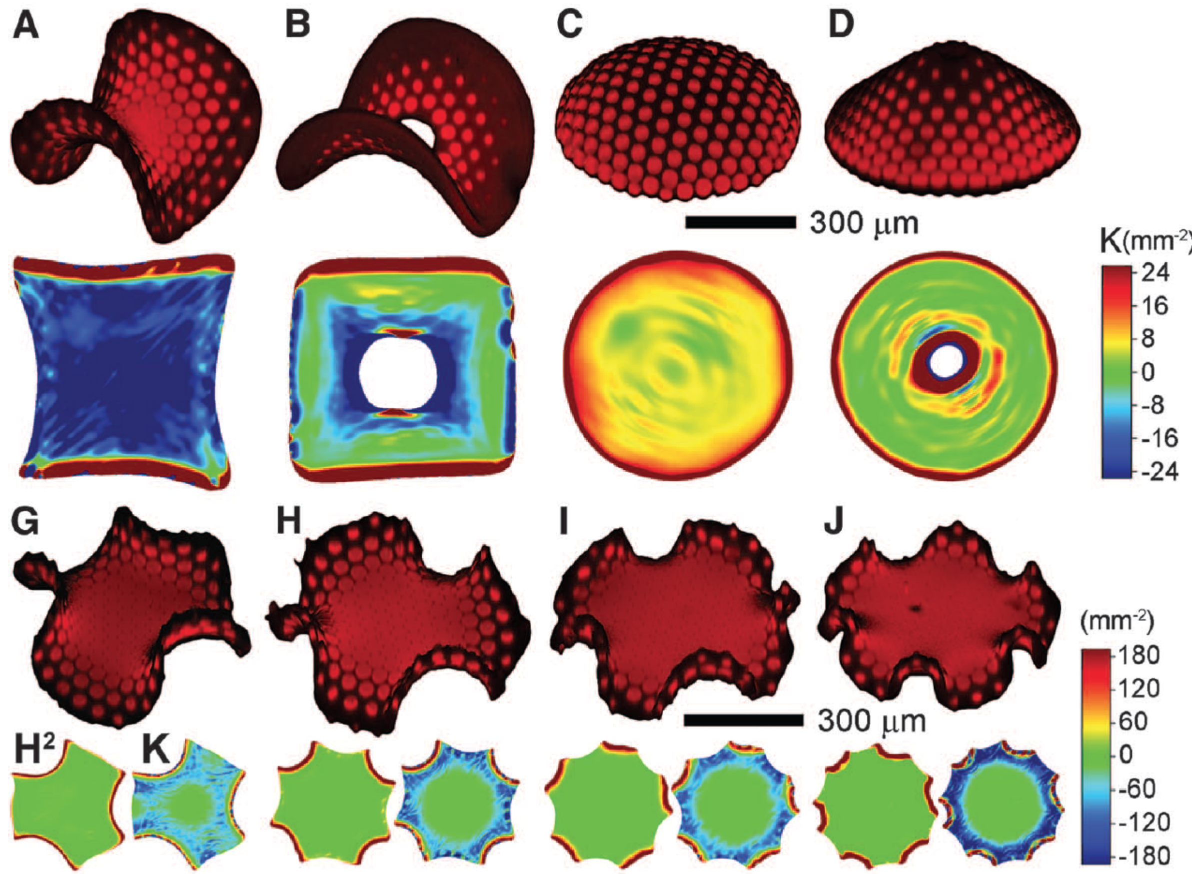
Y. Klein, E. Efrati, and E. Sharon, "Shaping of Elastic Sheets by Prescription of Non-Euclidean Metrics" *Science*, **315**, 1116, 2007.

Shaping Sheets



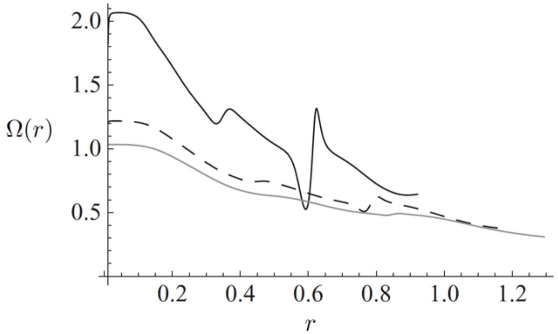
Shaping elastic sheets by halftone gel lithography

- Photopattern thin films.
- Thermal-actuated shape change.
- Swell to embedding based on prescribed metric.

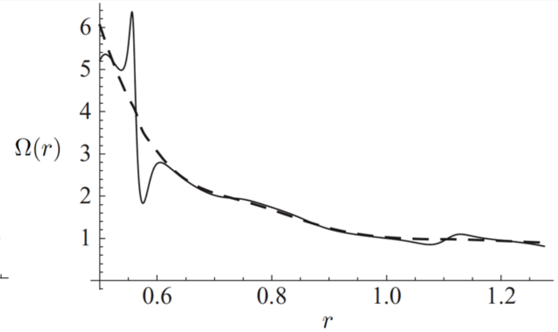


J Kim, J.A. Hanna, M. Byun, C.D. Santangelo, and R.C. Hayward, "Designing Responsive Buckled Surfaces by Halftone Gel Lithography" *Science*, **335**, 1201, 2012.

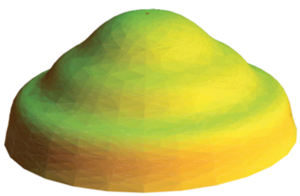
Shaping Sheets



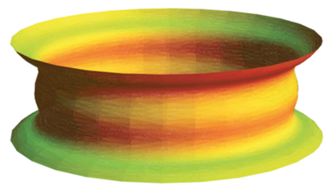
(a) Swelling factor



(a) Swelling factor



(b) Ziggurat



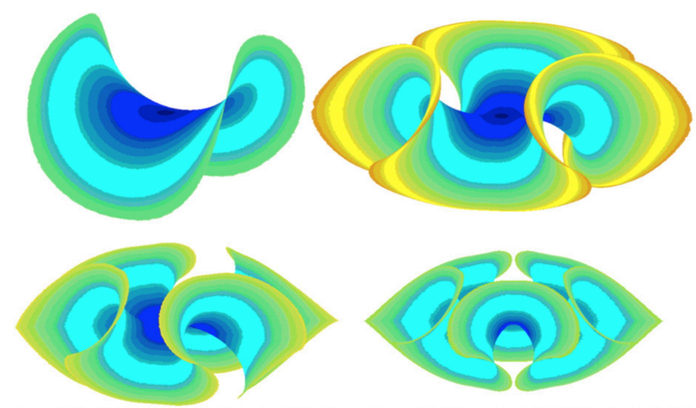
(b) Sheave

Programmed buckling by controlled lateral swelling in a thin elastic sheet

- How to prescribe a metric to produce a desire shape.
- Axisymmetric 3D structures.

Shape selection in non-Euclidean plates

- Existence of local isometries with waves that increase with radius.
- Energetically favorable to form lobes rather than saddles.



M.A. Dias, J.A. Hanna, and C.D. Santangelo "Programmed buckling by controlled lateral swelling in a thin elastic sheet" *PRE*, **84**, 036603, 2011.
J.A. Gemmer and S.C. Venkataramani, "Shape selection in non-Euclidean plates" *Physica D: Nonlinear Phenomenon*, **240**(19), 1536, 2011.

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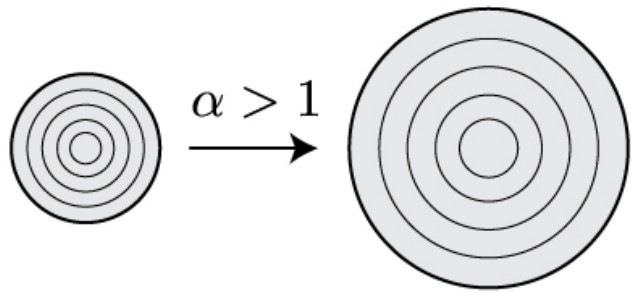
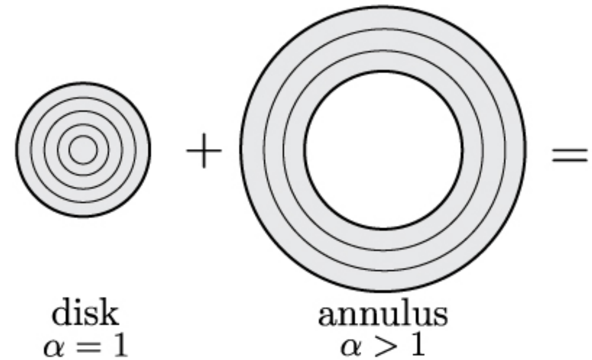
Geometric Composite

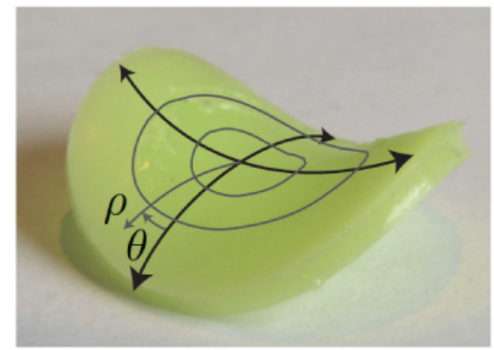
Independent Homothety

Homothetic Transformation

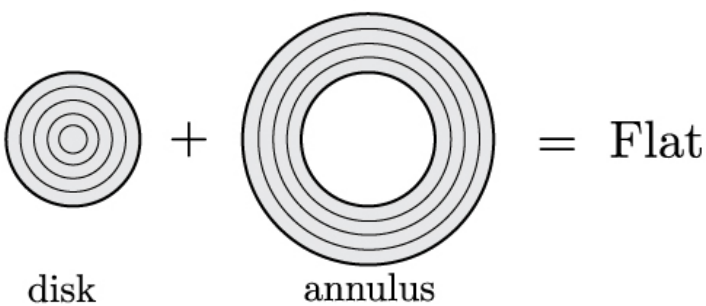


Stretched Annulus

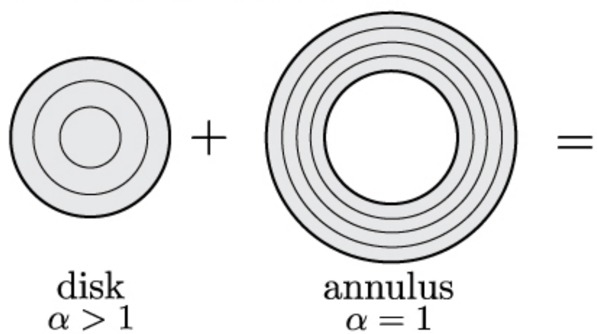


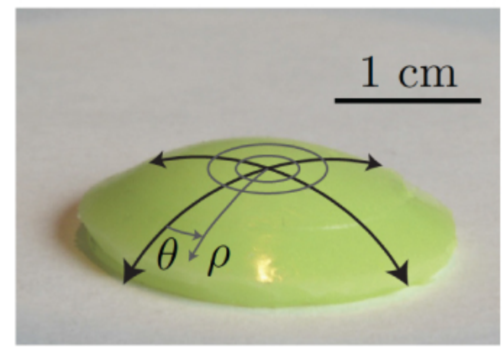


Reference Configuration



Stretched Disk

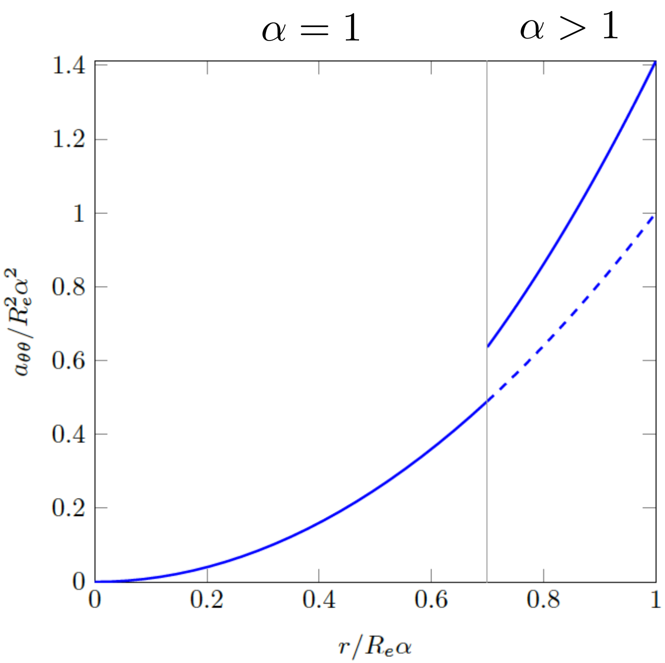


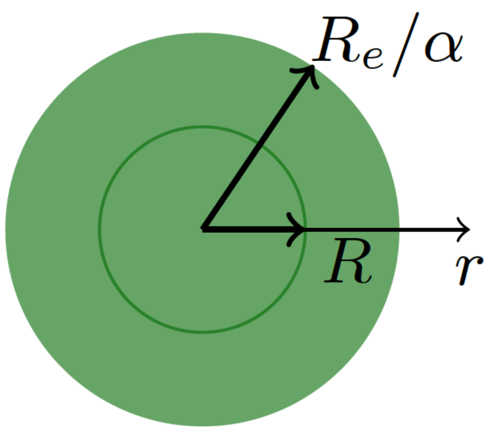


Goal: Use swelling to predictably & permanently morph plates into shells

M. Pezzulla, S.A. Shillig, P. Nardinocchi, and D.P. Holmes , "Morphing of Geometric Composites via Residual Swelling," Under Review: Soft Matter , (2015). (also: arXiv:1504.03010)

Geometric Composite





Stretching Energy of the Plate (incompressible)

$$\mathcal{U}_s \simeq h \int_A E[\mathrm{tr}^2(a - \overline{a}) + \mathrm{tr}(a - \overline{a})^2]\sqrt{|\overline{a}|}\mathrm{d}A$$

Realized metric (a) — Target metric ($\overline{a}$)

Target Metric (Polar Coordinates)

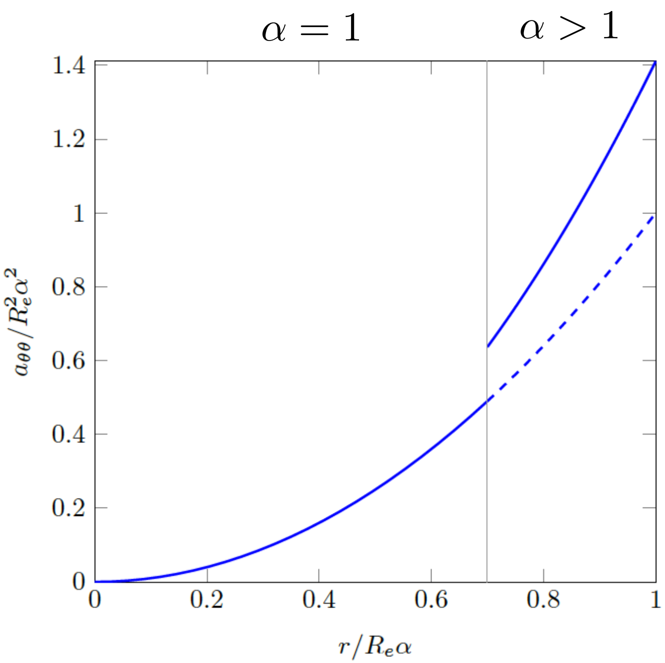
$$\overline{a} = f^2(r)\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad f^2(r) = \begin{cases} 1, & r \leq R \\ \alpha^2, & r > R \end{cases}$$

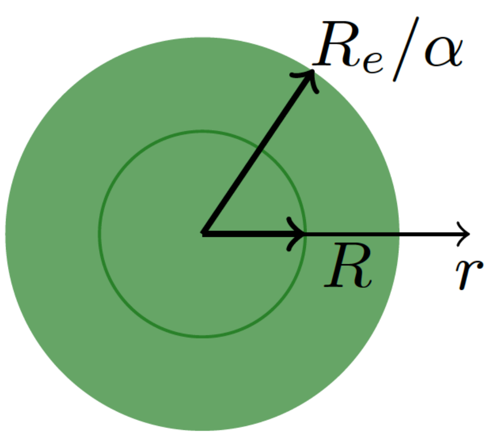
Stretching Energy (Assume all strains zero, except: $a_{\theta\theta} - \overline{a}_{\theta\theta}$)

$$\mathcal{U}_s \simeq Eh \int_0^R \frac{(a_{\theta\theta} - r^2)^2}{r^3}\,\mathrm{d}r + Eh \int_R^{R_e/\alpha} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3}\,\mathrm{d}r$$

M. Pezzulla, S.A. Shillig, P. Nardinocchi, and D.P. Holmes , "Morphing of Geometric Composites via Residual Swelling," Under Review: Soft Matter , (2015). (also: arXiv:1504.03010)

Geometric Composite





Stretching Energy (Assume all strains zero, except: $a_{\theta\theta} - \bar{a}_{\theta\theta}$)

$$\mathcal{U}_s \simeq Eh \int_0^R \frac{(a_{\theta\theta} - r^2)^2}{r^3}\,\mathrm{d}r + Eh \int_R^{R_e/\alpha} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3}\,\mathrm{d}r$$

Realized Metric: Gauss Normal Coordinates

$$\rho(r) = \int_0^r \sqrt{a_{rr}(r')}dr'$$

$\rho(r)$ measures the arc length along radial geodesics

First Fundamental Form

$$ds^2 = d\rho^2 + a_{\theta\theta}(\rho)d\theta^2$$

Gaussian curvature

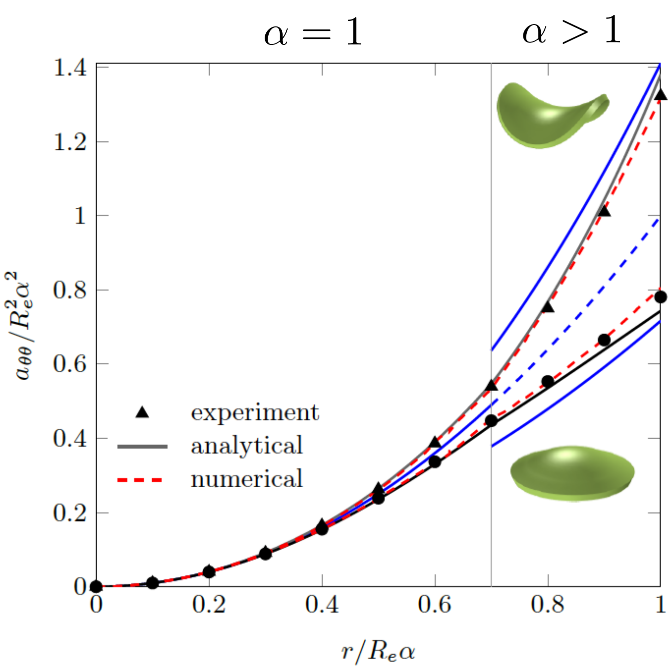
$$-\partial_{\rho\rho}\sqrt{a_{\theta\theta}}/\sqrt{a_{\theta\theta}}$$

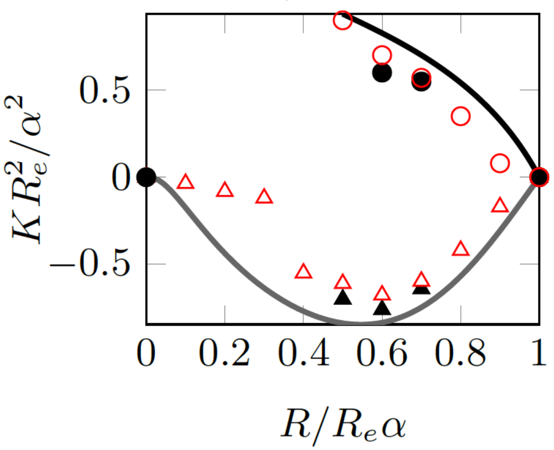
Minimize Stretching Energy (Constant *K* metric)

$$a_{\theta\theta}(\rho) = (\sin(\sqrt{K}\rho)/\sqrt{K})^2$$

M. Pezzulla, S.A. Shillig, P. Nardinocchi, and D.P. Holmes , "Morphing of Geometric Composites via Residual Swelling," Under Review: Soft Matter , (2015). (also: arXiv:1504.03010)

Geometric Composite





Minimize Stretching Energy (Constant *K* metric)

$$a_{\theta\theta}(\rho) = (\sin(\sqrt{K}\rho)/\sqrt{K})^2$$

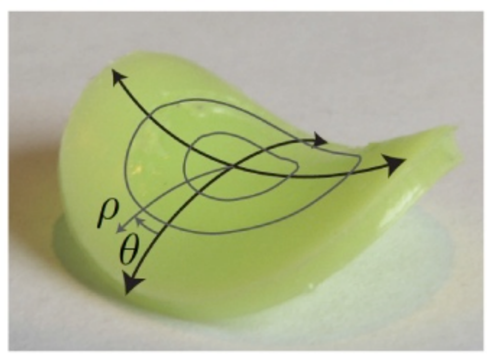
Taylor Expand $a_{\theta\theta}(\rho)$ (Assume: $|K| < \alpha^2/R_e^2$)

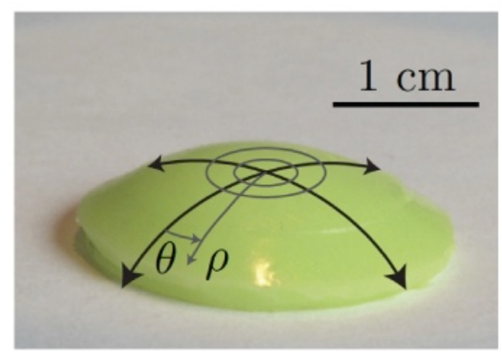
$$a_{\theta\theta}(\rho) = \rho^2 - \frac{K}{3}\rho^4 + \mathcal{O}(\rho^5)$$

Flat metric

Kind of non-Euclidean metric

Experiments: Mechanical Strain





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(also: arXiv:1504.03010)

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Residual Swelling

Polyvinylsiloxane (PVS) – Zhermack Elite Double

$$E = 0.226 \text{ MPa}$$

$$\nu = 0.5$$

$$E = 0.963 \text{ MPa}$$

$$\nu = 0.5$$

Characteristics:

- Fast curing at room temperature.
- Easily vary elastic modulus
- The elastomer contains free, uncrosslinked polymer chains.

M. Pezzulla, S.A. Shillig, P. Nardinocchi, and D.P. Holmes , "Morphing of Geometric Composites via Residual Swelling," Under Review: Soft Matter , (2015).
(also: arXiv:1504.03010)

Residual Swelling

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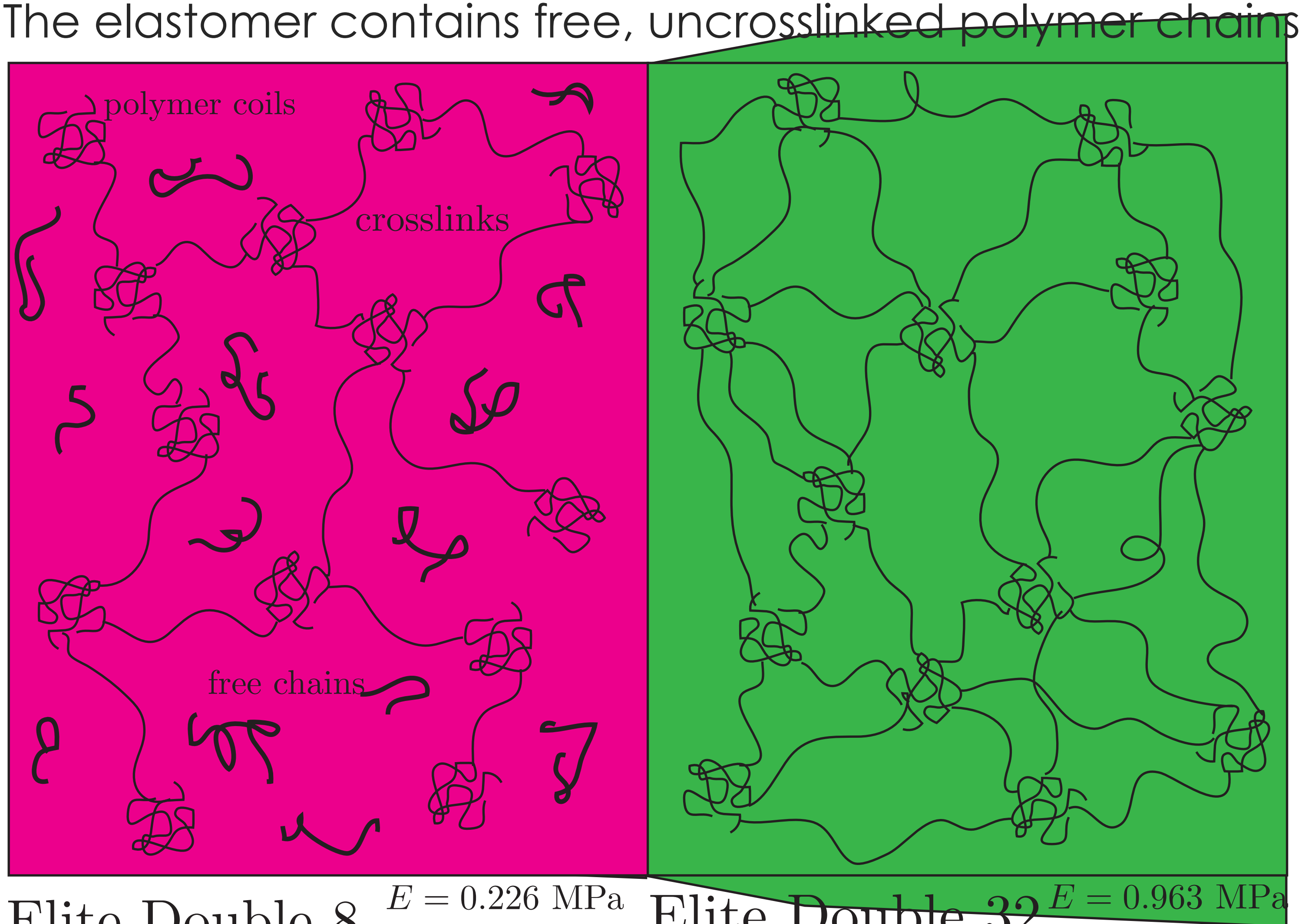

Elite Double 8 $E = 0.226$ MPa $\nu = 0.5$

Elite Double 32 $E = 0.963$ MPa $\nu = 0.5$

M. Pezzulla, S.A. Shillig, P. Nardinocchi, and D.P. Holmes , "Morphing of Geometric Composites via Residual Swelling," Under Review: Soft Matter , (2015). (also: arXiv:1504.03010)

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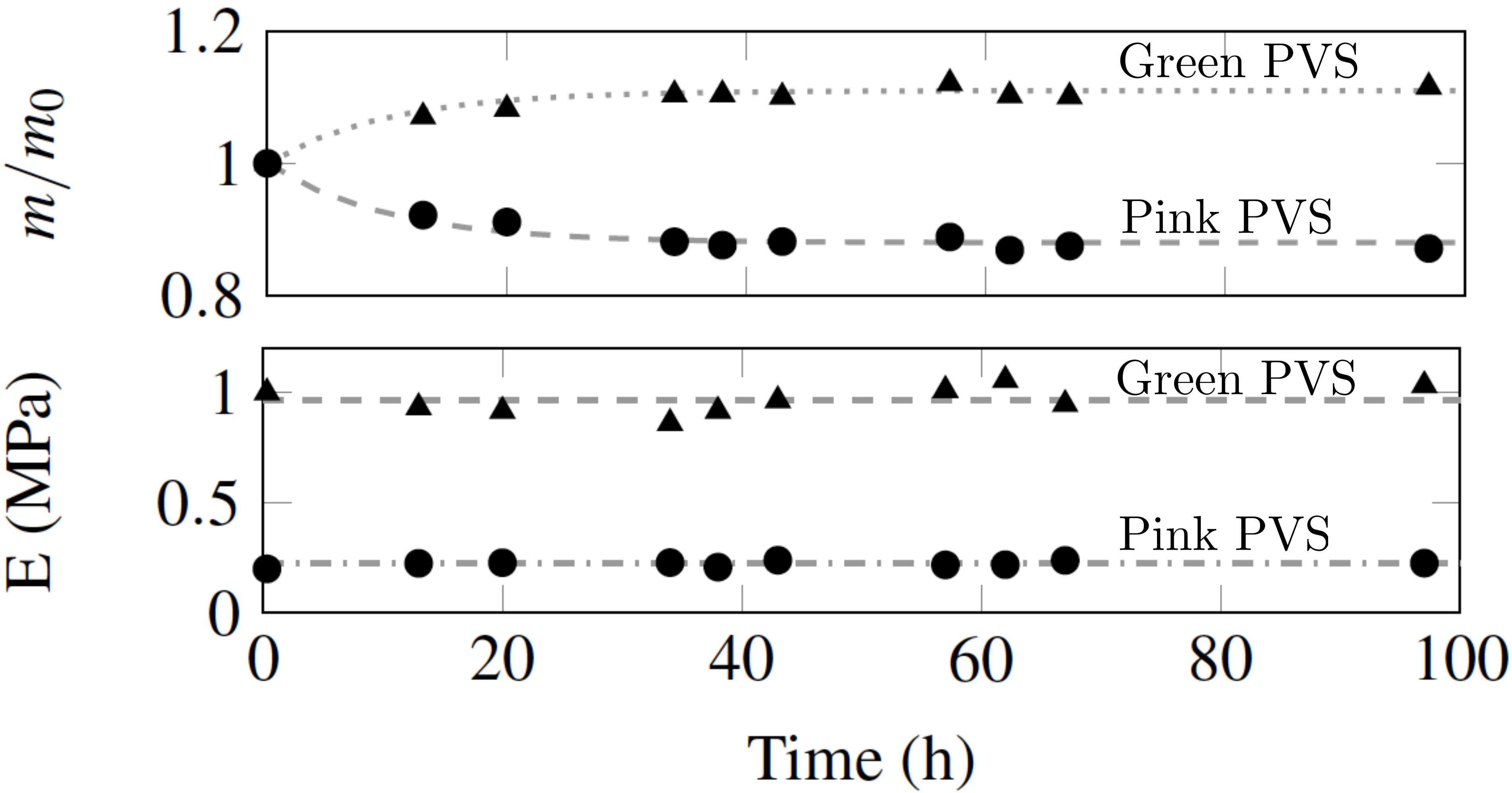
Residual Swelling



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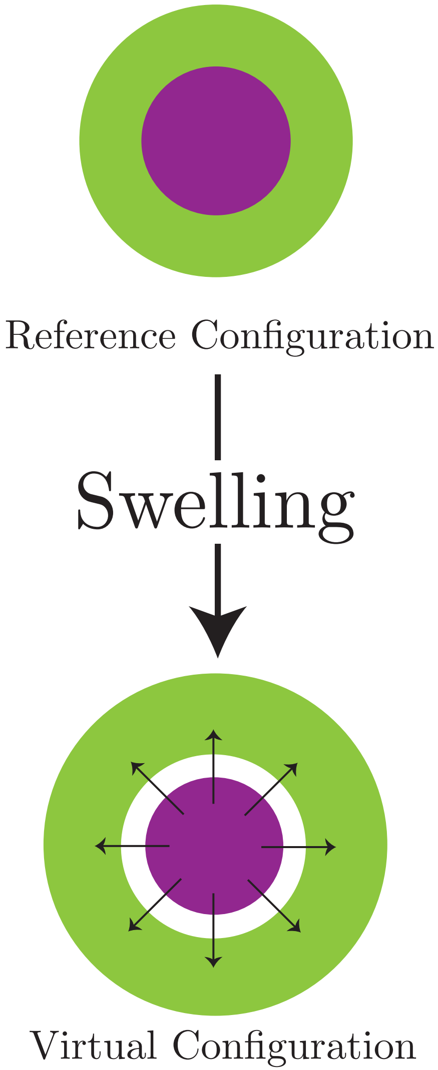
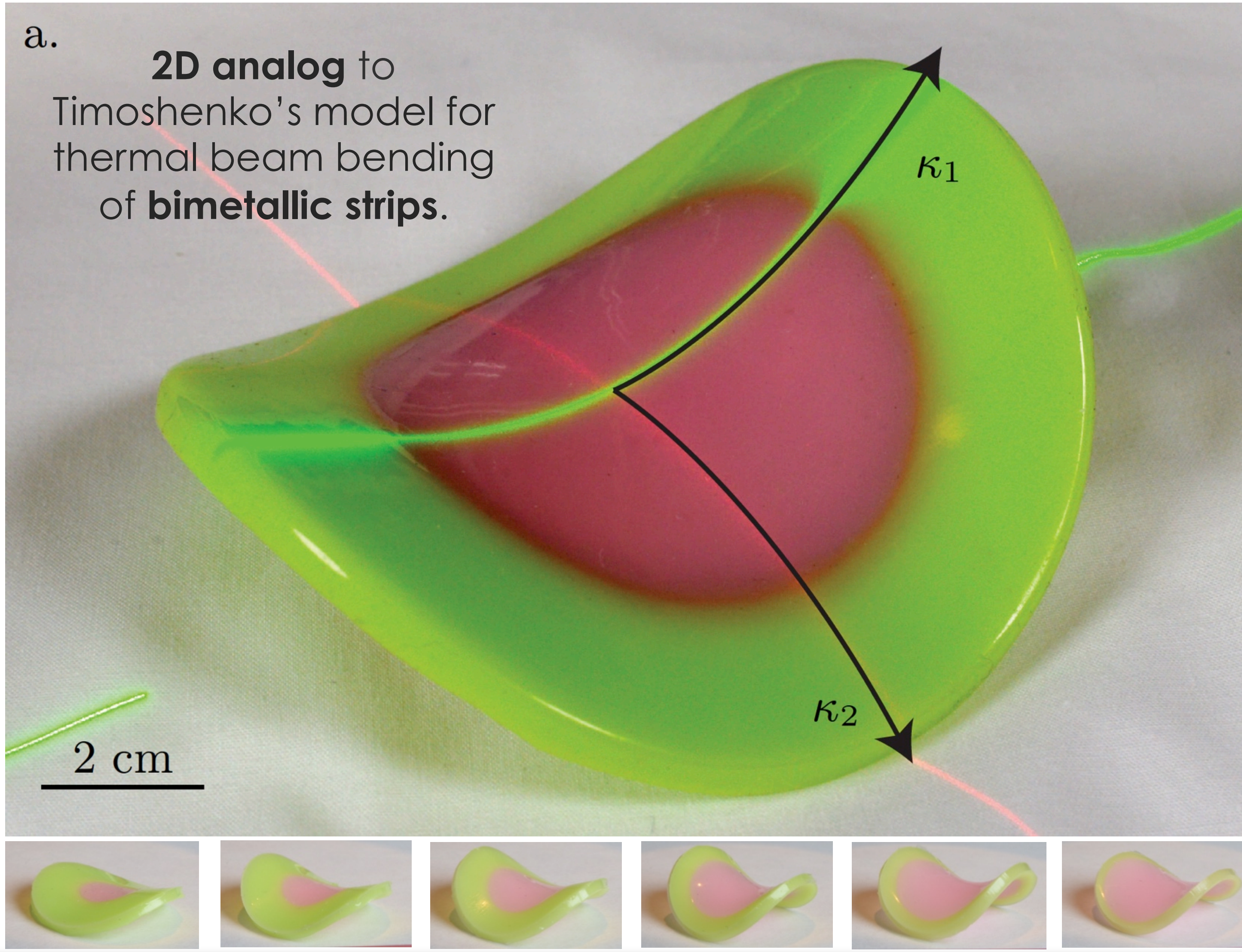
Residual Swelling





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Residual Swelling

Stretching Energy

$$\mathcal{U}_s \simeq \int_0^R \frac{(a_{\theta\theta} - \alpha^{-2} r^2)^2}{\alpha^{-2} r^3}\,\mathrm{d}r + \frac{E_a}{E_d} \int_R^{R_e} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3}\,\mathrm{d}r$$

Modulus difference

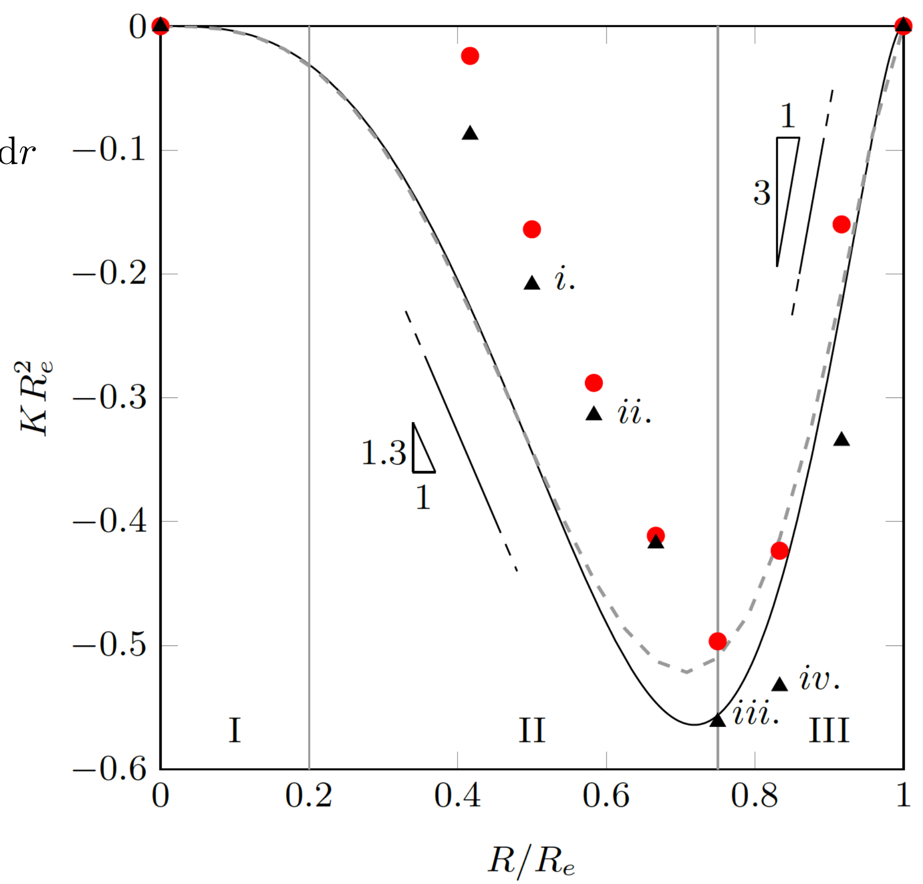
Stretching ratio

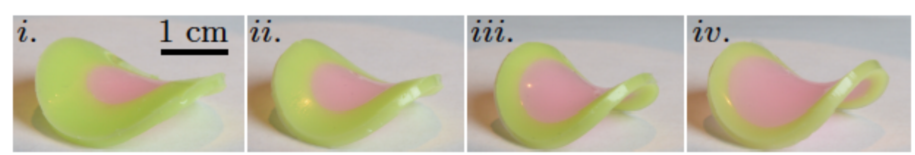
- Depends on chemical and material properties.
- Should vary with R/R_e
- Will depend on concentration gradient of free chains.

ansantz

- from conservation of mass & proportional to mass uptake in annulus.)

$$\alpha = 1 + \eta(c_d - c_a)\left(\frac{R}{R_e}\right)^2 \left[1 - \left(\frac{R}{R_e}\right)^2\right]$$





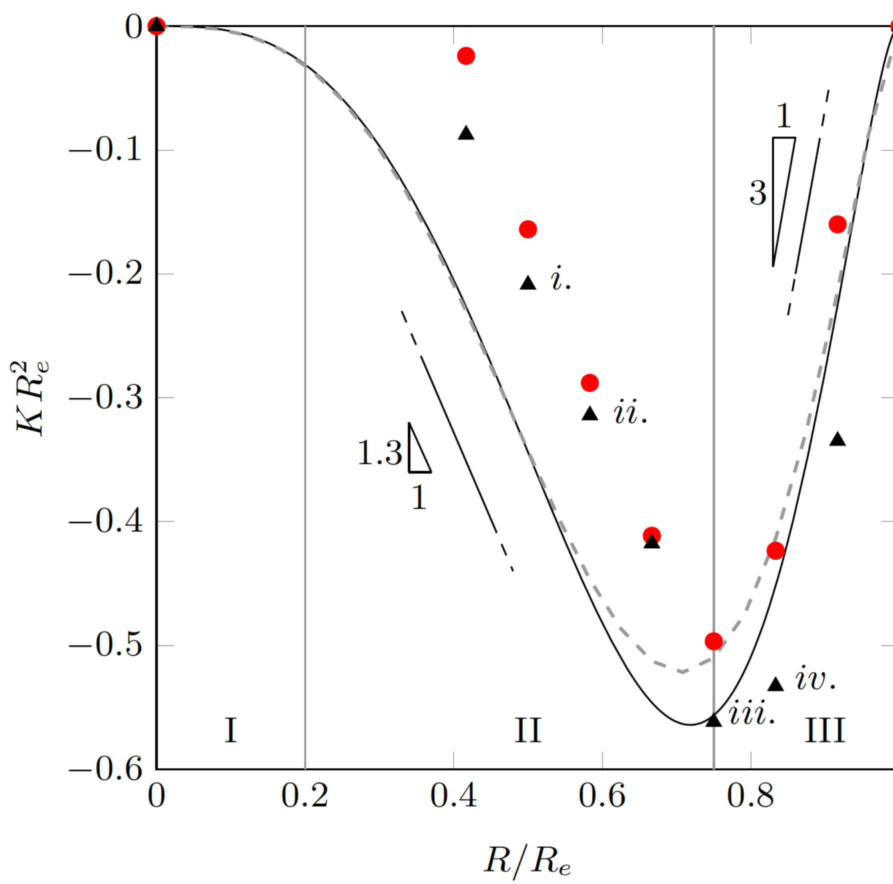
Residual Swelling

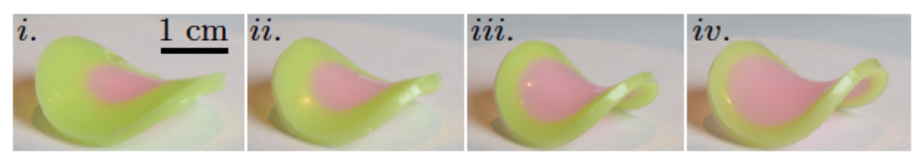
Approximate Analytical Solution

- Taylor expand about $(\alpha - 1)$
- $\overline{E} = E_a/E_d$ and $\overline{R} = R/R_e$

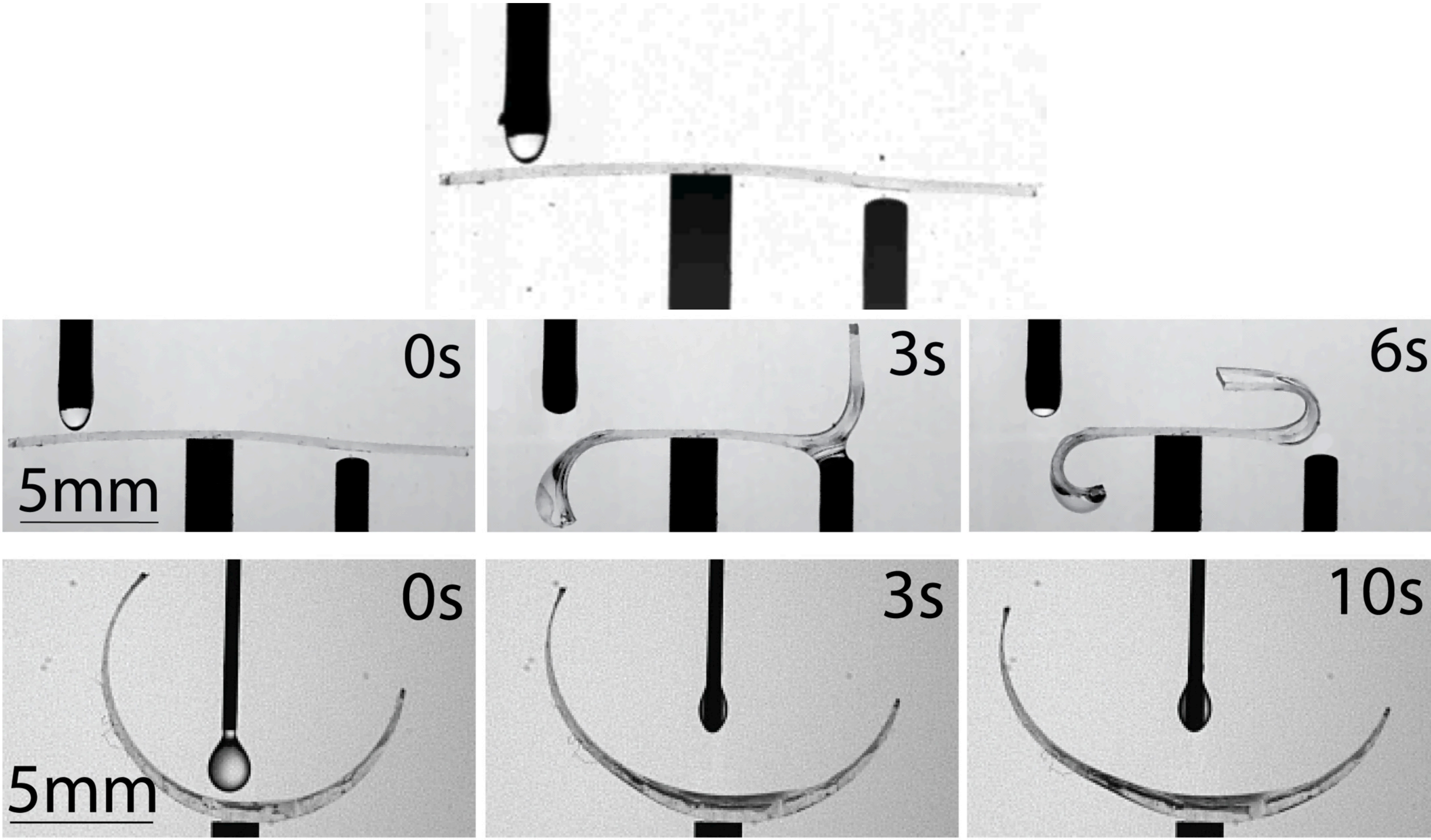
$$KR_e^2 \simeq 96(1-\alpha_{\max})\bar{E}\bar{R}^3 \frac{\left(1-\bar{R}^2\right)\left(1-\bar{R}^3\right)}{\bar{R}^6\left(1-\bar{E}\right)+\bar{E}}$$

2D analog to Timoshenko's model for thermal beam bending of **bimetallic strips**.





Controlling Shape



D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Overview

Elastic Instabilities for Form and Function
Buckling, Wrinkling, Folding, and Snapping

Geometry and Mechanics:
- Fundamental equations, geometric rigidity, morphing.

Buckling & Wrinkling:
- Stability, wavelength, flexible electronics, mechanical metamaterials, adhesion.

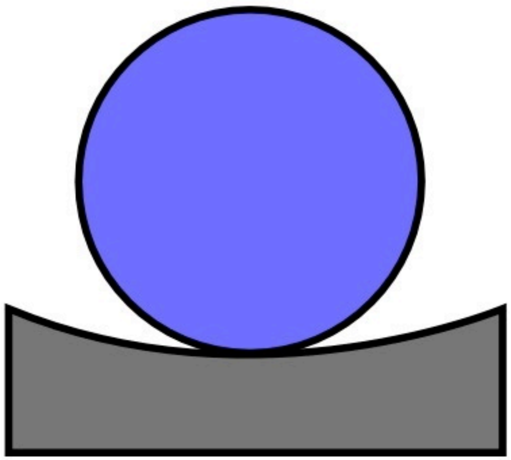
Stress Focusing – Folding & Creasing:
- Wrinkle-to-fold, origami.

Snapping:
- Snapping surfaces.

Buckling

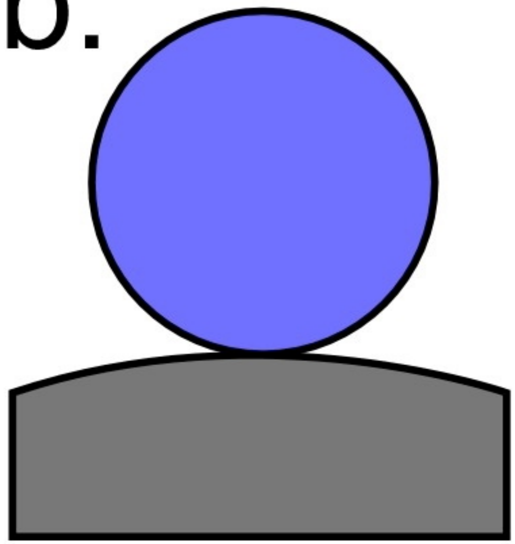
Stability

a.



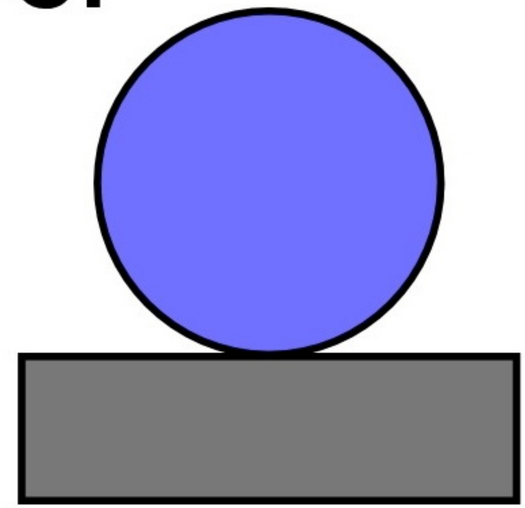
Stable Equilibrium

b.



Unstable Equilibrium

c.



Neutral Equilibrium

- Lateral displacement **raises** the ball's center of gravity.
- **Increases** the potential energy.

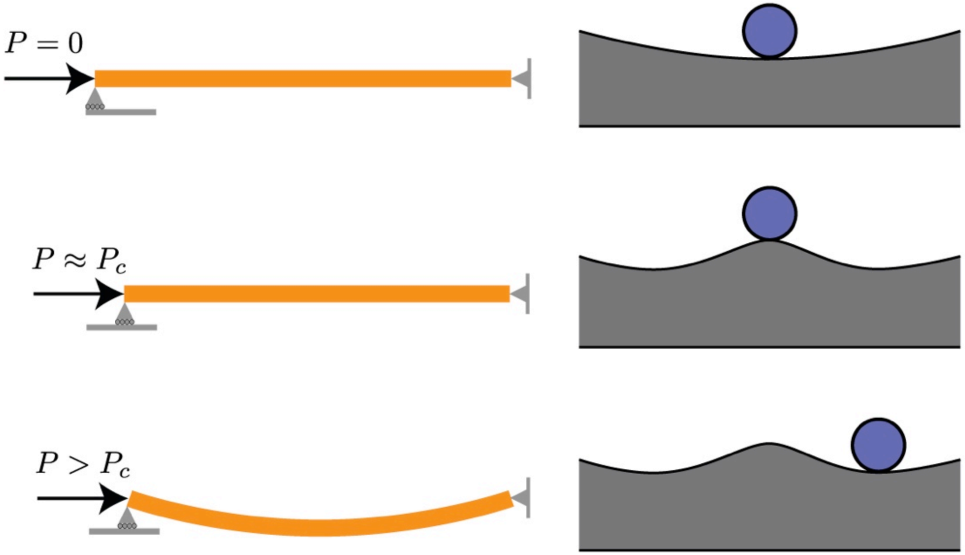
- Lateral displacement **lowers** the ball's center of gravity.
- **Decreases** the potential energy.

- Lateral displacement
  - no change in potential energy.

Stability

Euler Buckling

"Stable-Symmetric Bifurcation"



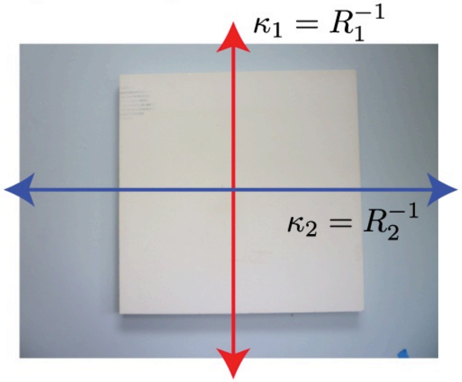
$$\overline{\nabla}_\alpha \overline{\nabla}_\beta \mu^{\alpha\beta} - \left(\bar{b}_{\alpha\beta} + \overline{\nabla}_\alpha \overline{\nabla}_\beta w\right) \sigma^{\alpha\beta} - p = 0$$

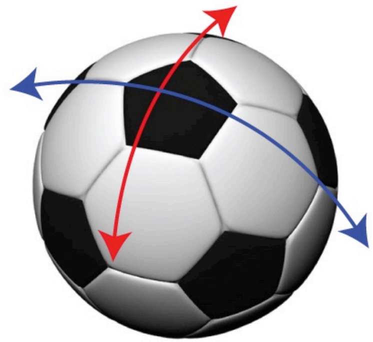
Buckling into Arches

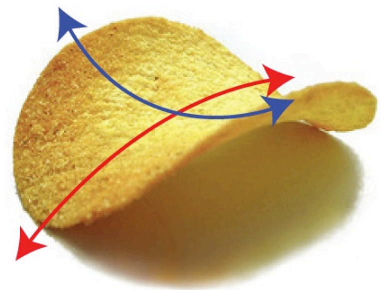
Föppl-von Kármán Plate Equations

$$\underbrace{B\nabla^4 w}_{bending} - h \underbrace{\Diamond^4[\phi, w]}_{\substack{nonlinear \\ stress\ \&\ curvature}} = 0 \qquad \underbrace{\nabla^4 \phi}_{stress} + \frac{E}{2} \boxed{\underbrace{\Diamond^4[w, w]}_{geometry}} = 0$$

Gauss Curvature

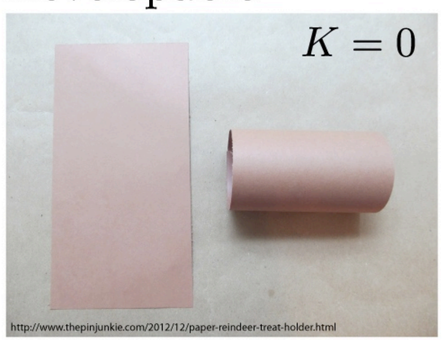


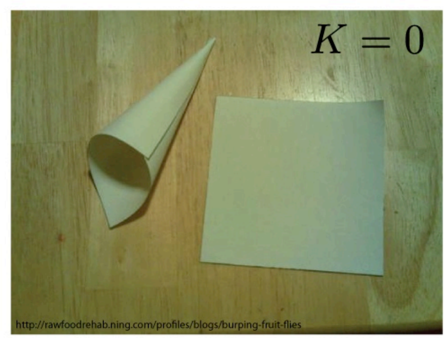




$K = \kappa_1 \kappa_2 = \Diamond^4[w, w,] = 0$ $\quad K = \kappa_1 \kappa_2 = \Diamond^4[w, w,] > 0$ $\quad K = \kappa_1 \kappa_2 = \Diamond^4[w, w,] < 0$

Developable





Buckling into Arches

Föppl-von Kármán Plate Equations

$$\underbrace{B\nabla^4 w}_{bending} - h\underbrace{\lozenge^4[\phi, w]}_{\substack{nonlinear \\ stress\ \&\ curvature}} = 0$$

$$\underbrace{\nabla^4\phi}_{stress} + \frac{E}{2}\underbrace{\cancel{\lozenge^4[w,w]}}_{geometry} = 0$$

$$\frac{\partial^4\phi}{\partial x^4} = 0$$

- Stress independent of y
- Clamped boundary conditions
- In-plane stress is constant
- Bending equation becomes ODE

$$w(x) = \frac{w_0}{2}\left(1 + \cos\frac{\pi x}{L}\right)$$

$$w_0 = \pm\frac{2h}{\sqrt{3}}\left(\frac{\sigma}{\sigma_{Eu}} - 1\right)^{1/2}$$

Euler Buckling Stress:

$$\sigma_{Eu} = \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{h}{L}\right)^2$$

B. Audoly, "Elasticity and Geometry," Oxford

mechanics of slender structures

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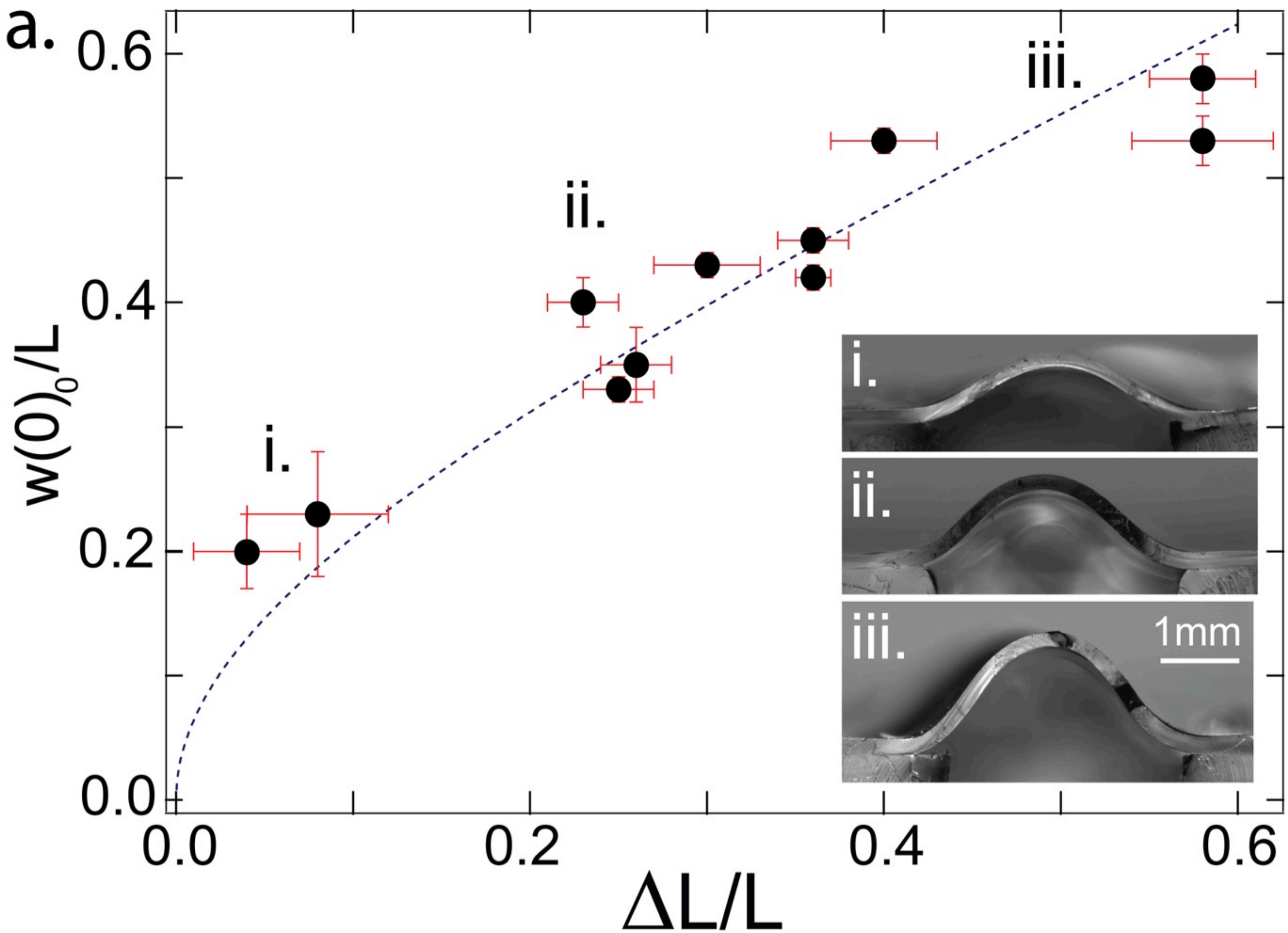
Buckling into Arches



D.P. Holmes, B. Tavakol, G. Froehlicher, and H.A. Stone, "Control and Manipulation of Microfluidic Fluid Flow via Elastic Deformations ", *Soft Matter*, **9**(29), 7049, (2013).

B. Tavakol, **D.P. Holmes**, and H.A. Stone, "Extended Lubrication Theory", *Under Review*, (2014).

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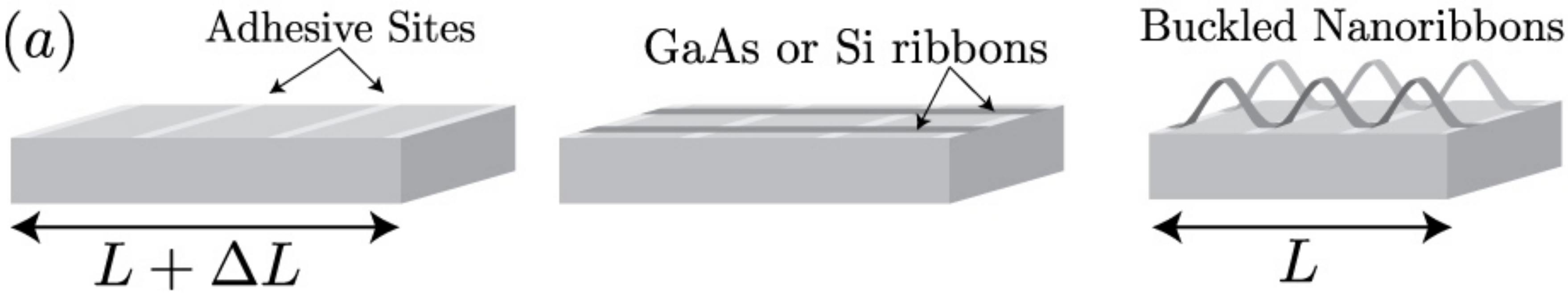
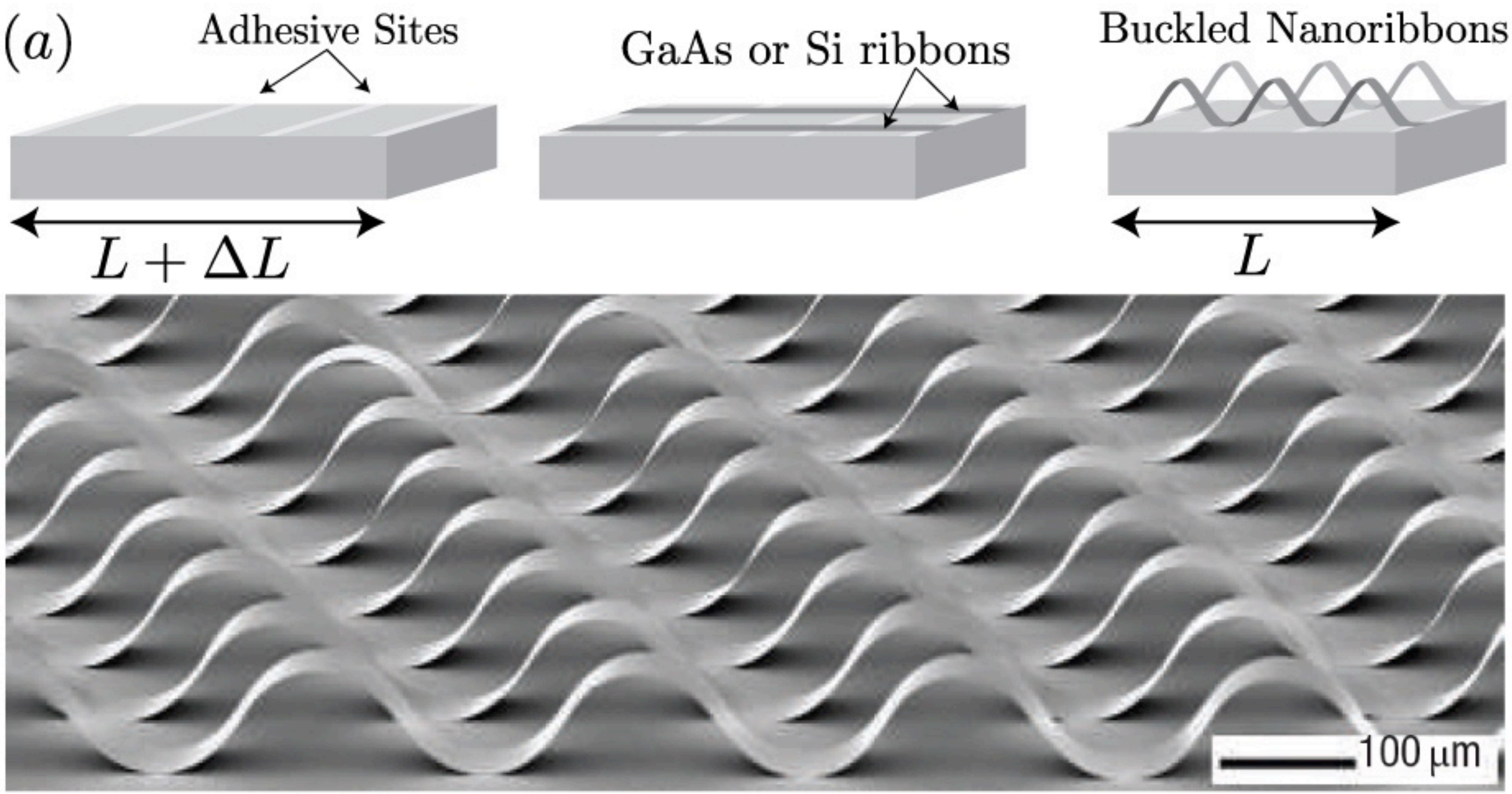
Buckling into Arches



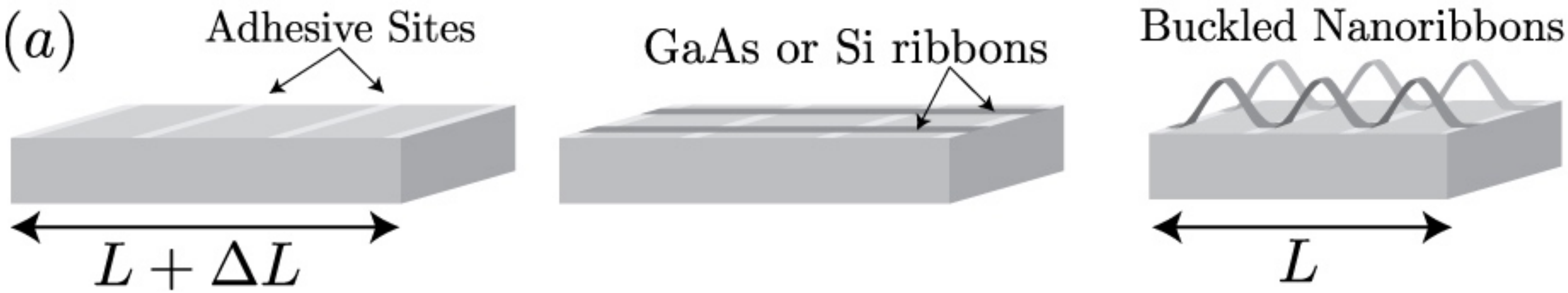
Buckling of microscale elastic **plates**

- Fabrication of semiconductor **nanoribbons** for **stretchable electronics.**
- Flexible substrate: PDMS, chemically altered with UV light.
- Uniaxially strained PDMS, strips of single-crystal **GaAs** (gallium arsenide) **bonded** to the **adhesive sites**.
- Release of the prestrain causes a uniaxial compressive stress that buckles the nanoribbons into arches.
- Adhesive boundaries (as opposed to clamped) yields a variation on the classical elastica problem – the **sticky elastica.**

Buckling into Arches



Buckling into Arches



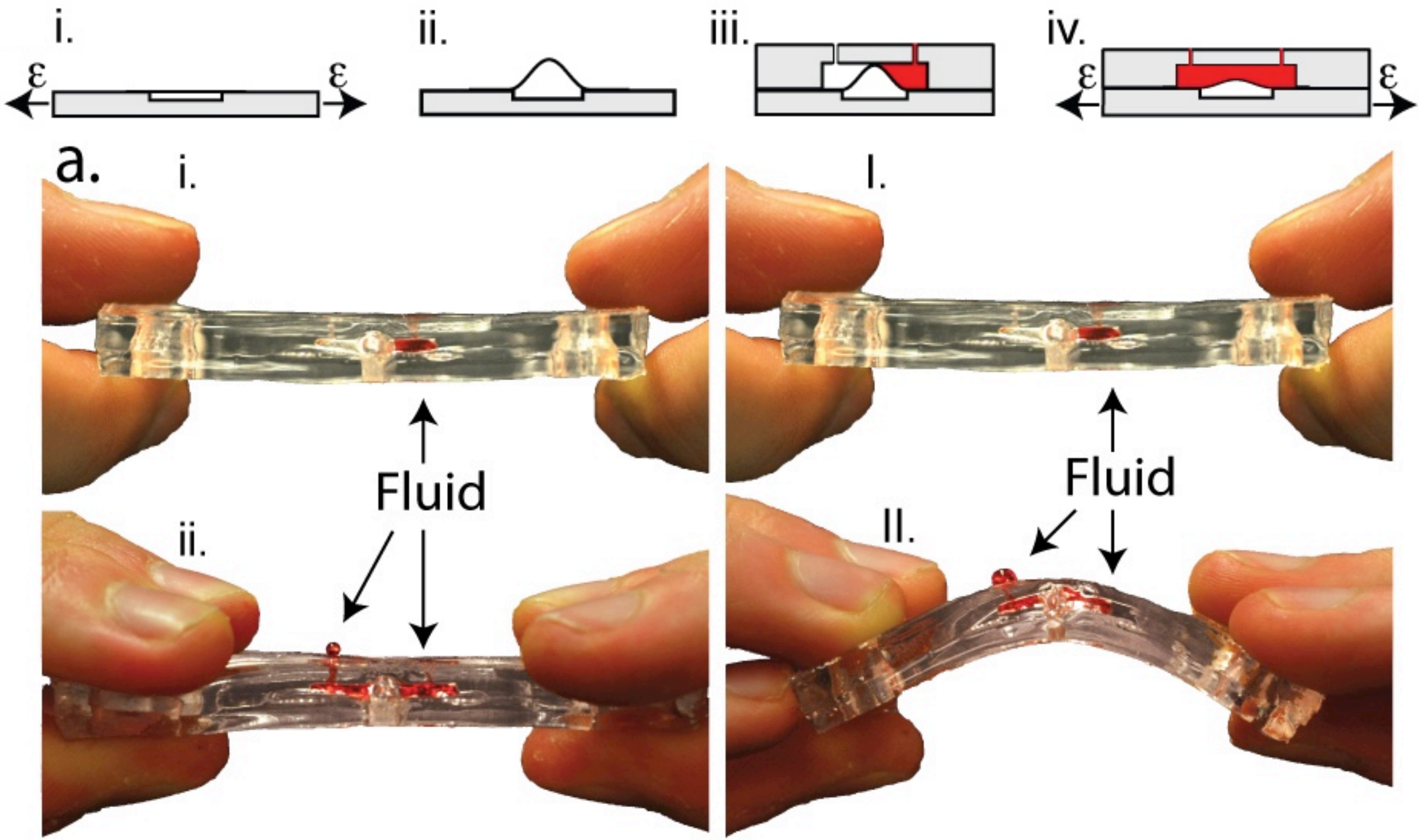
Buckling of microscale elastic **plates**

- Nanoribbons with prestrains exceeding 50% were fabricated.
- Deposition of gold onto the arches makes them functional electrodes.

- Example device: Metal-semiconductor-metal photodetectors
  - Photosensor functional up to **51.4% strain in tension** and -18.3% in compression.

- Technique has been extended to form: single-wall carbon nanotube arches, MOSFETs, piezoelectric energy harvesters.

Buckling into Arches



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B. Tavakol, **D.P. Holmes**, and H.A. Stone, "Extended Lubrication Theory", *Under Review*, (2015).

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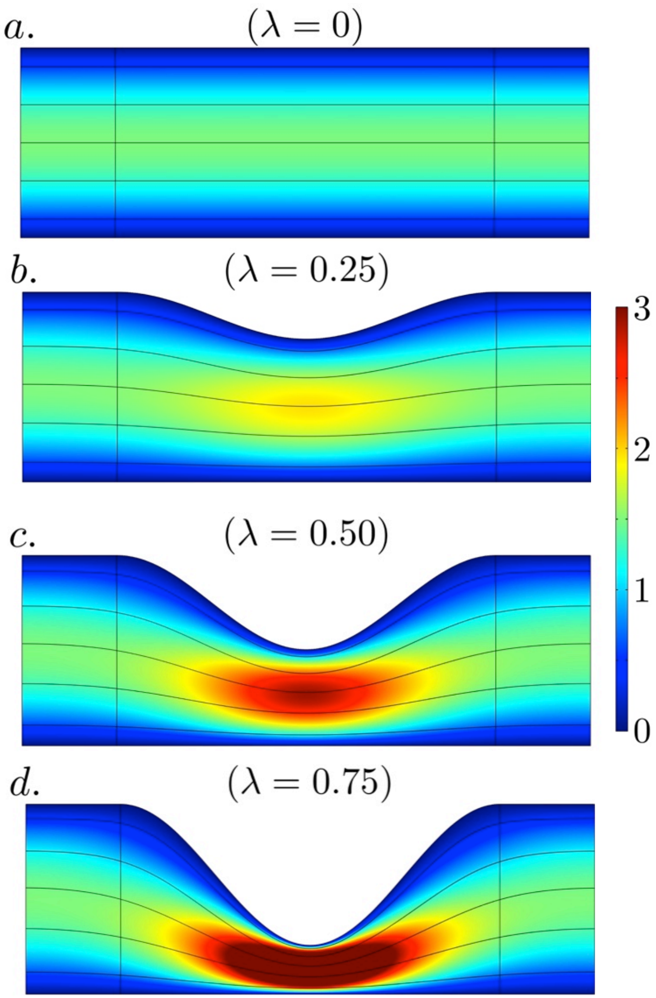
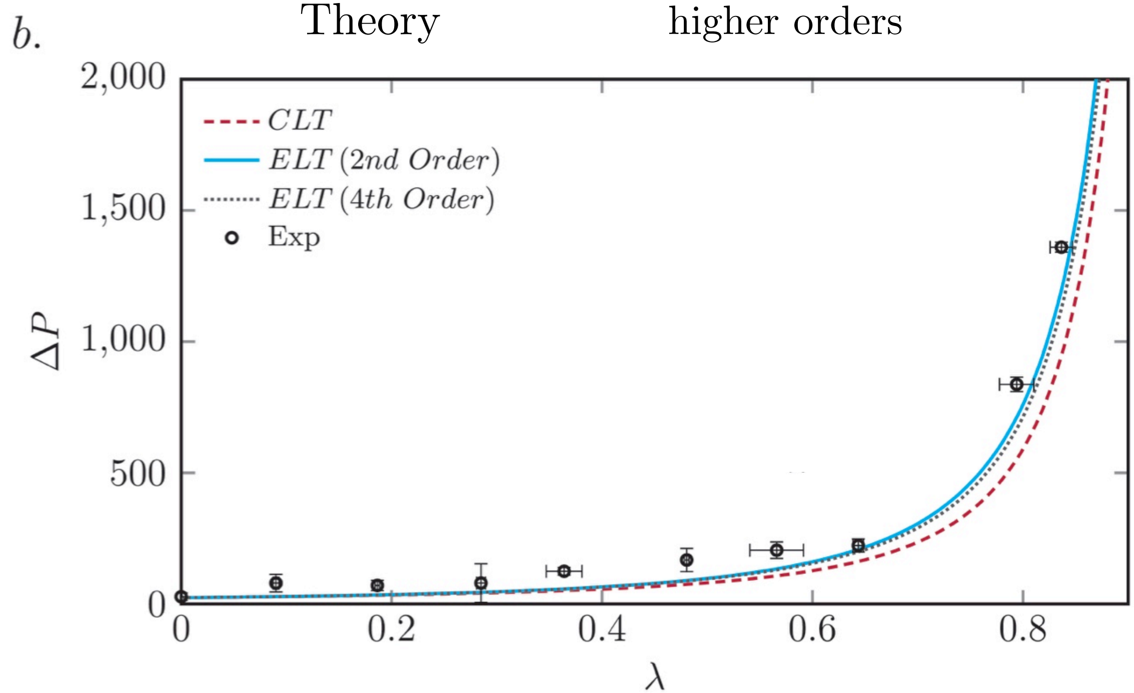
Buckling into Arches

Fluid flow through channels with variable **geometry**



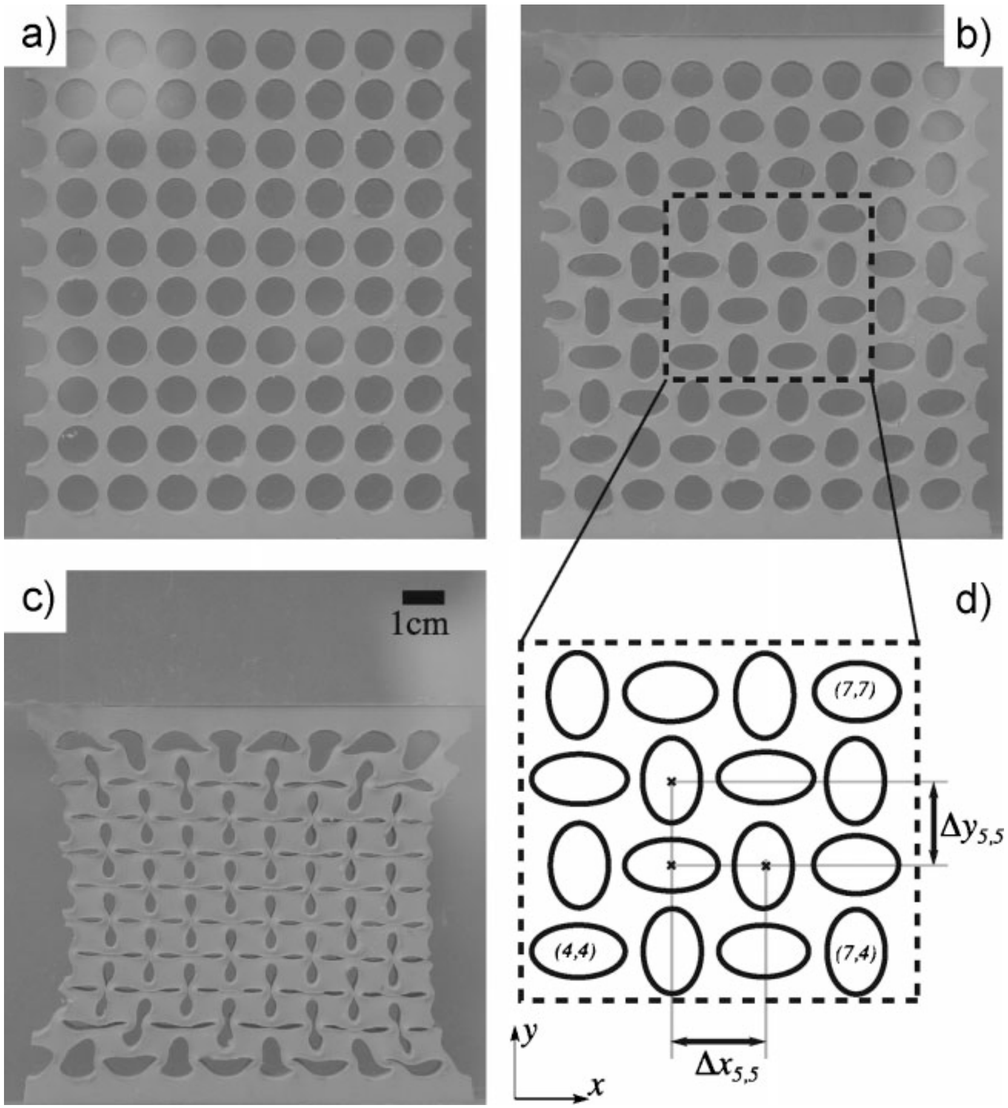
Extended Lubrication Theory

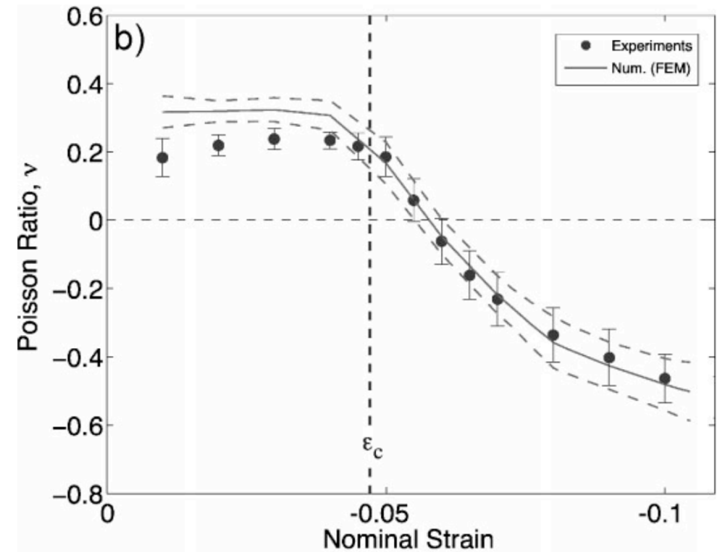
$$\Delta P = \underbrace{\Delta P_0 \Bigg(1}_{\text{Lubrication Theory}} + \underbrace{\frac{4}{5}\lambda^2\delta^2 - \frac{64}{225}\lambda^4\delta^4 + \mathcal{O}(\delta^6)}_{\text{Perturbation to higher orders}} \Bigg)$$

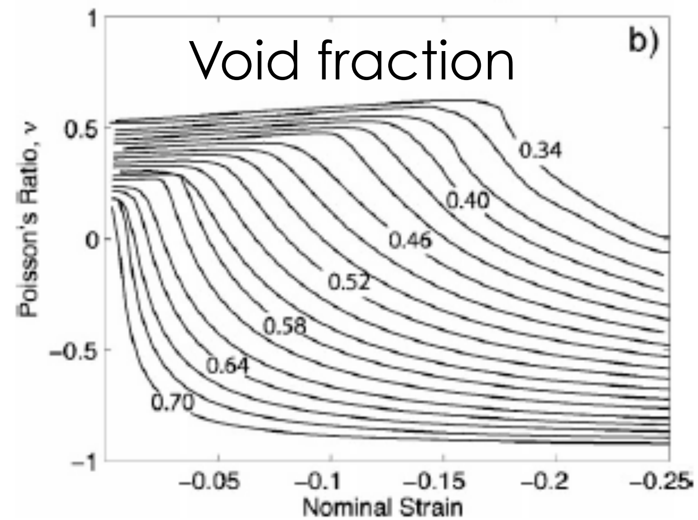


B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

Mech. Metamaterials

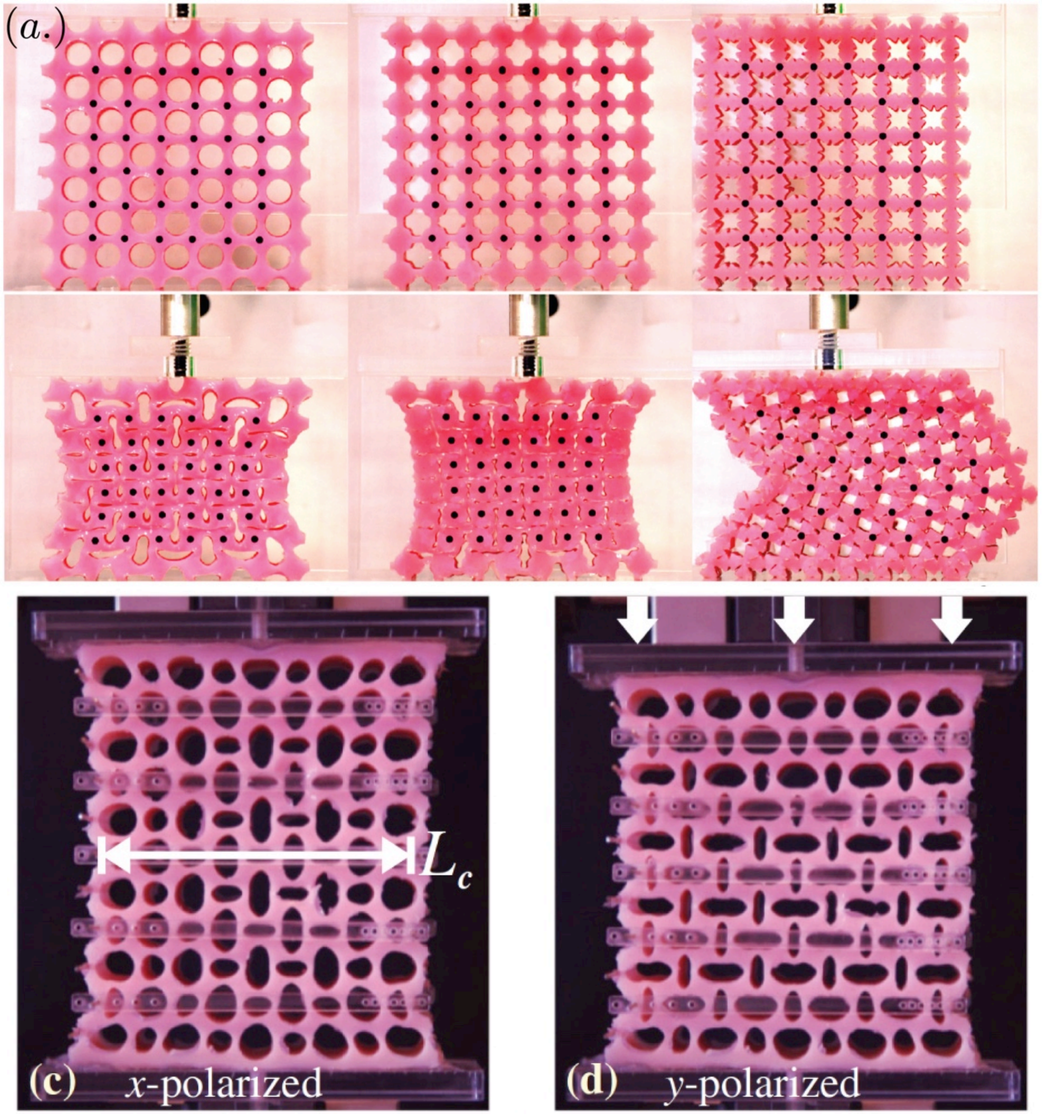


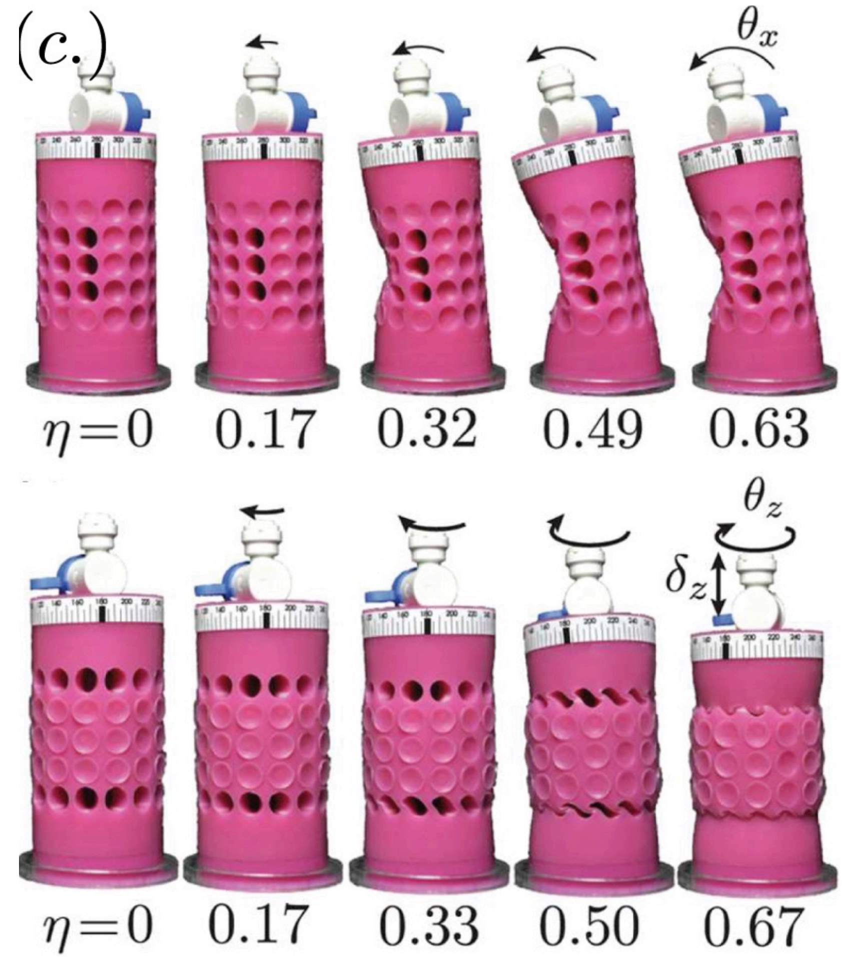




Bertoldi et a., Advanced Materials, 21, 1-6, (2009)

Mech. Metamaterials





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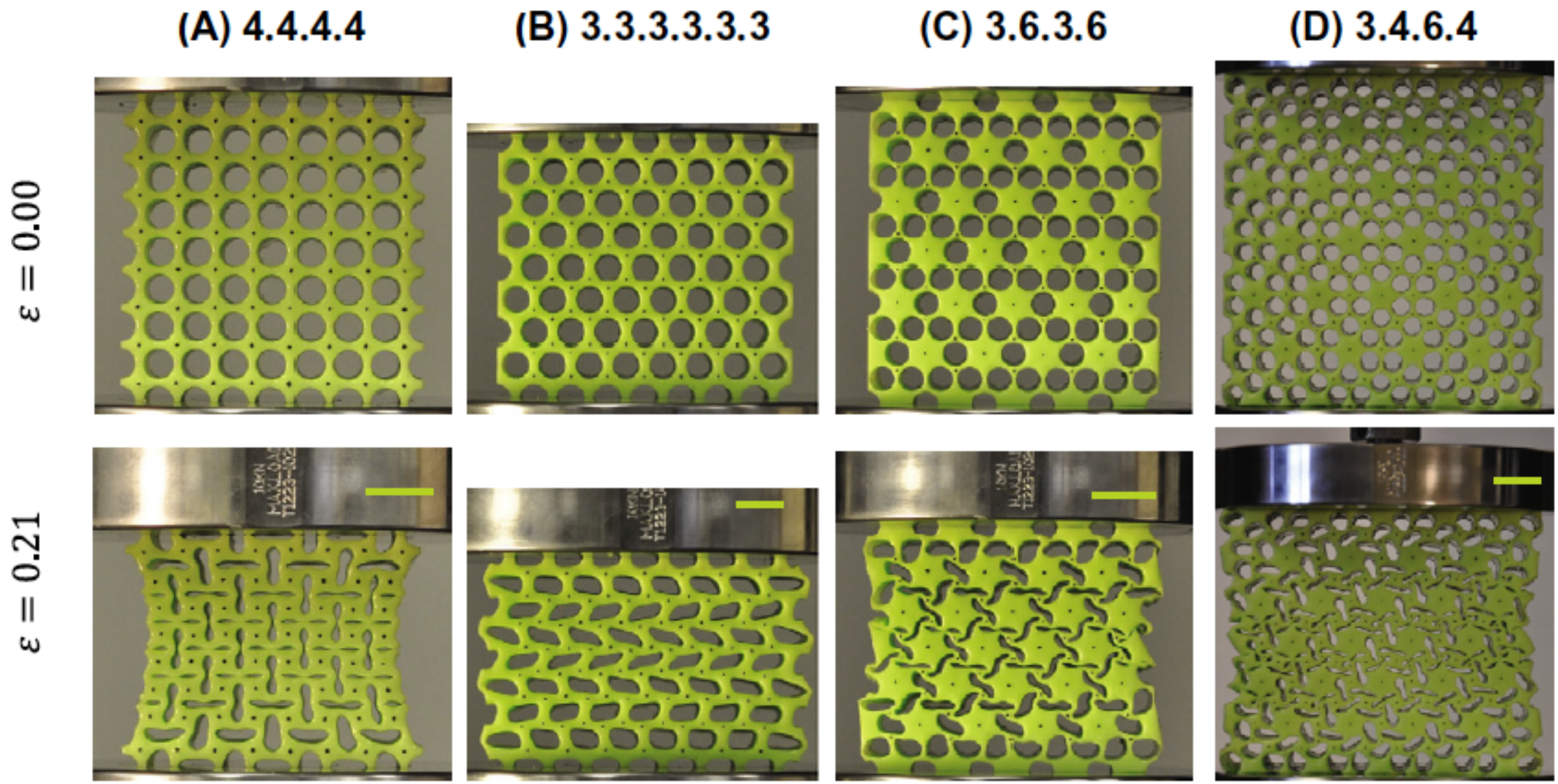
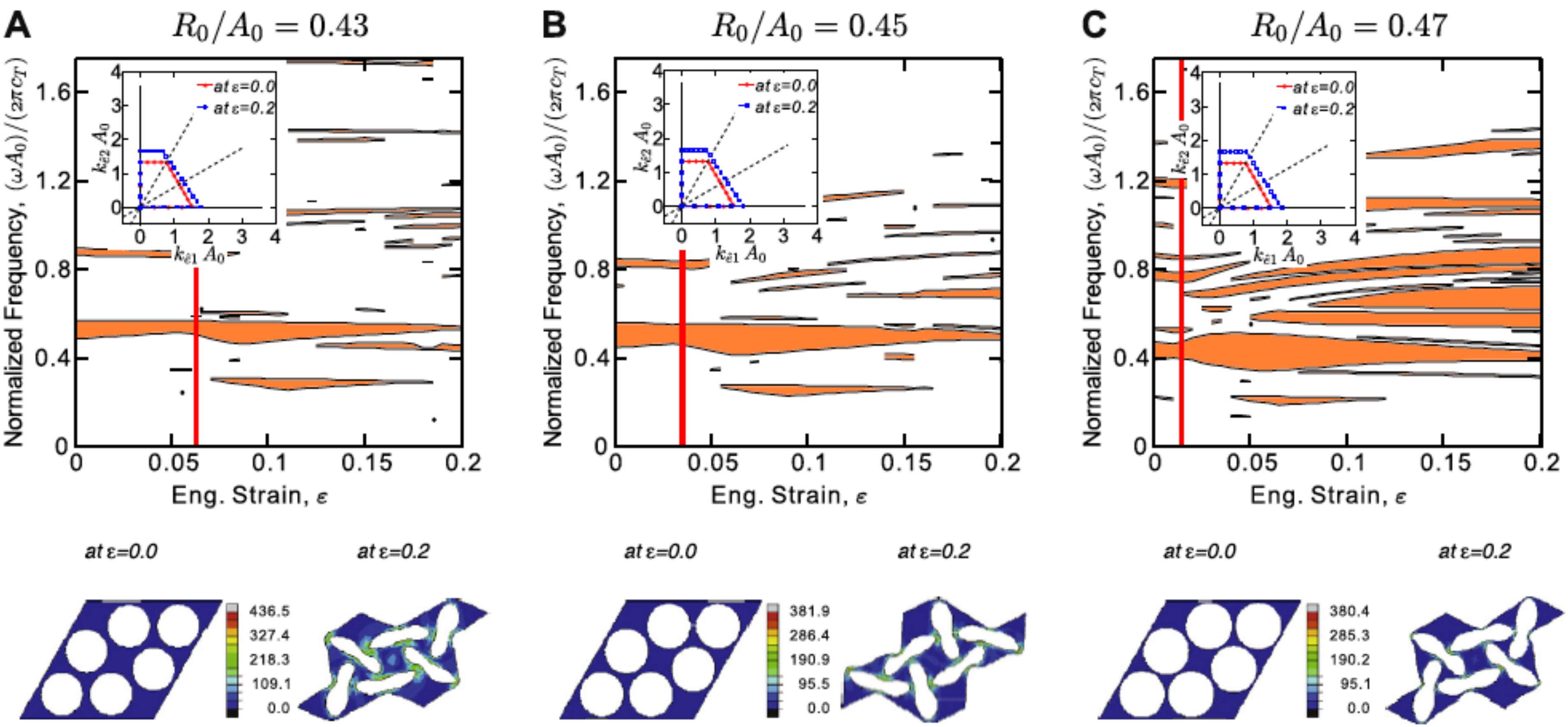
Mech. Metamaterials



J. Shim et al. IJSS, (2015).

Mech. Metamaterials



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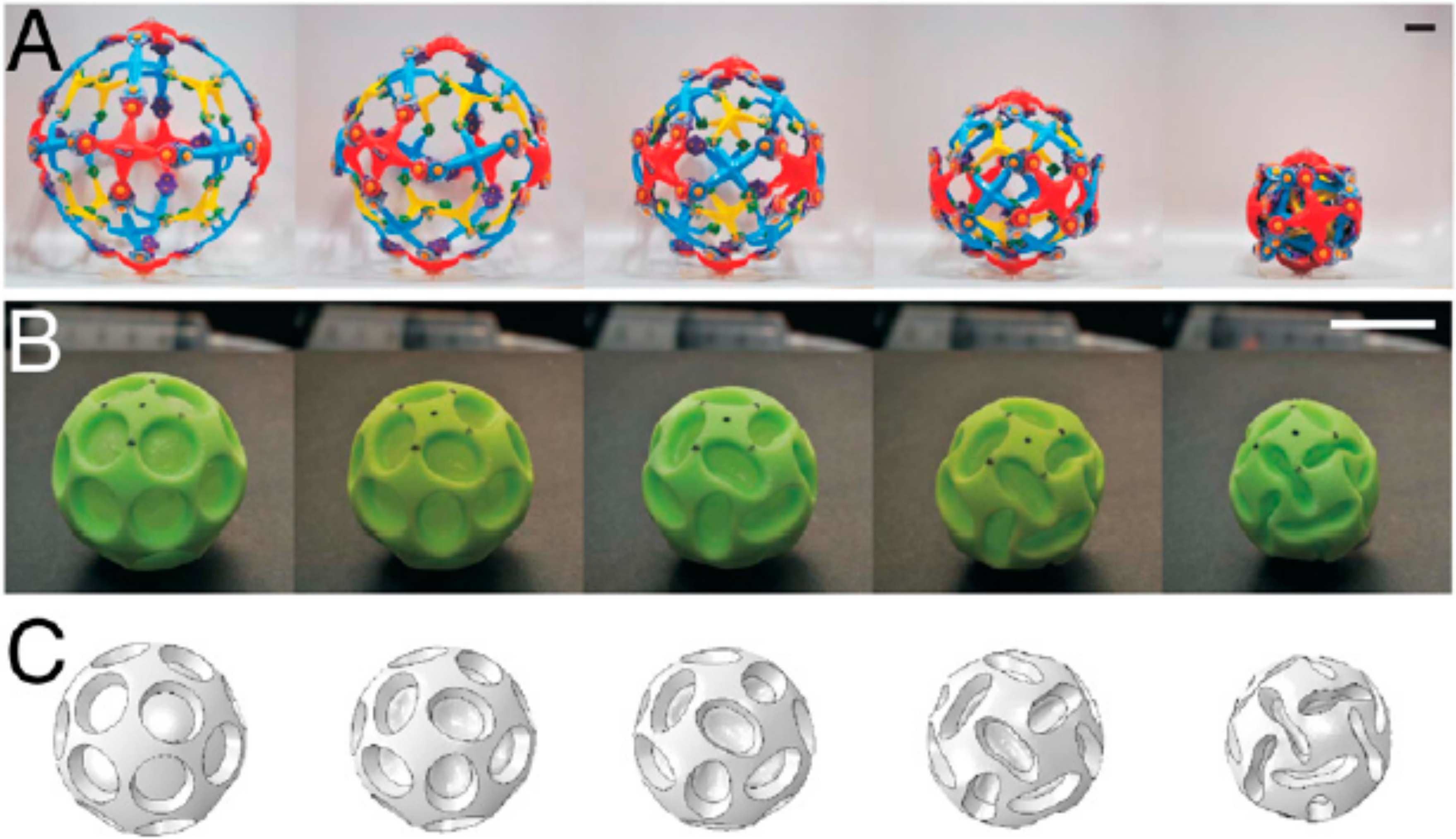
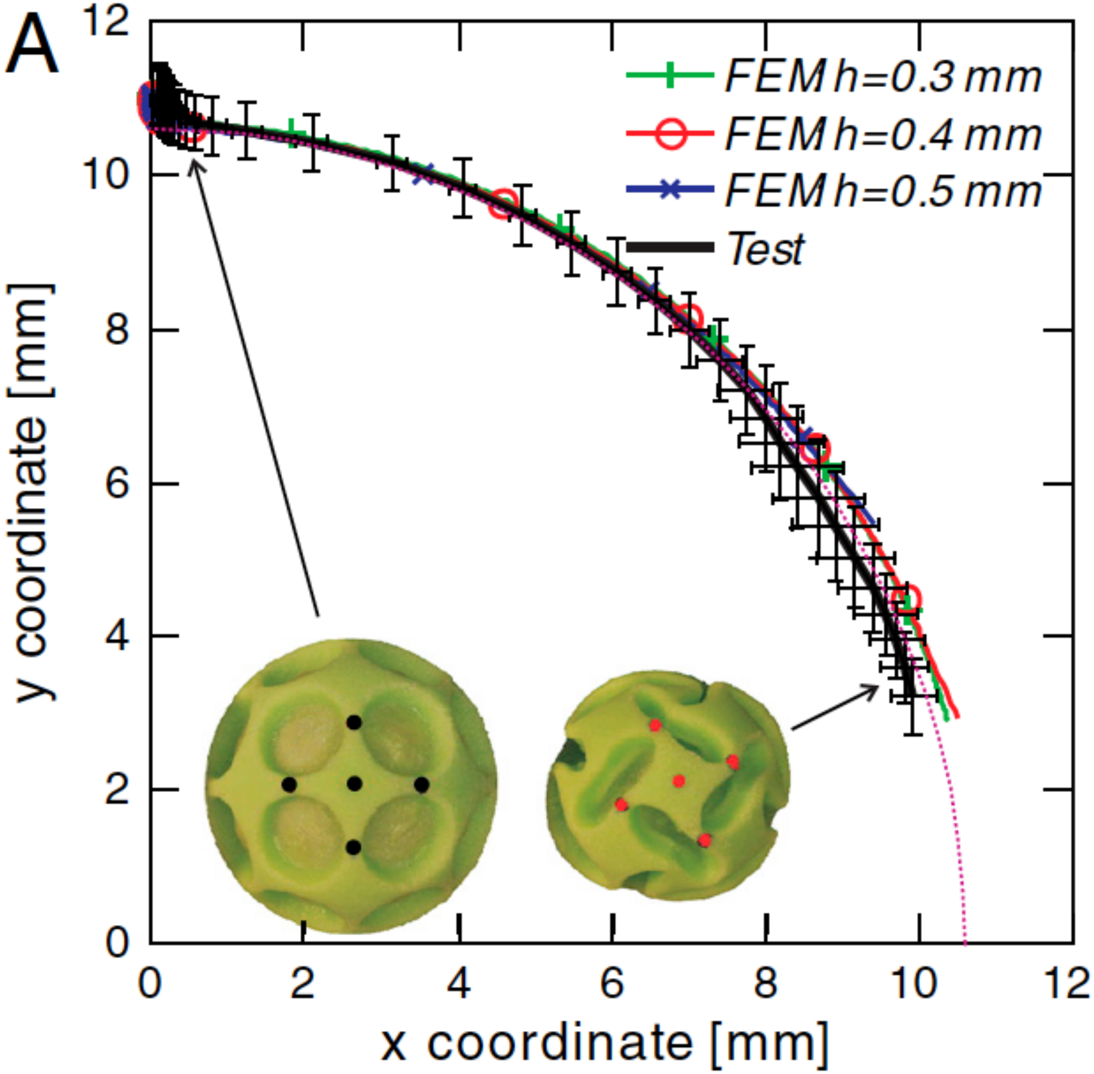
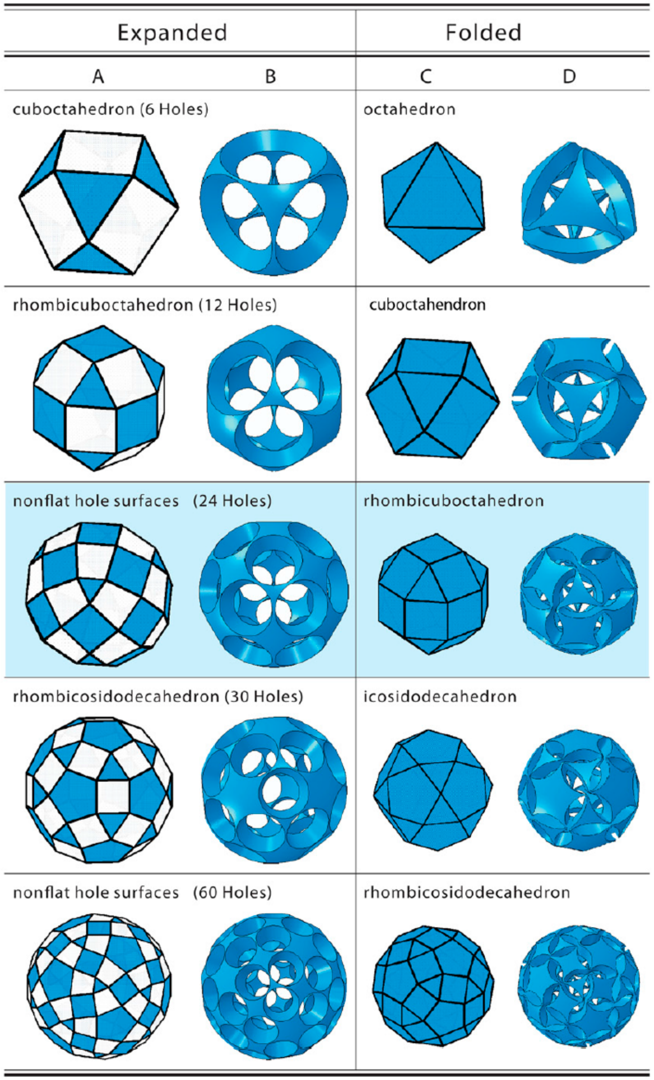


J. Shim et al., "Buckling-induced encapsulation of structured elastic shells under pressure," PNAS, (2012)

Mech. Metamaterials





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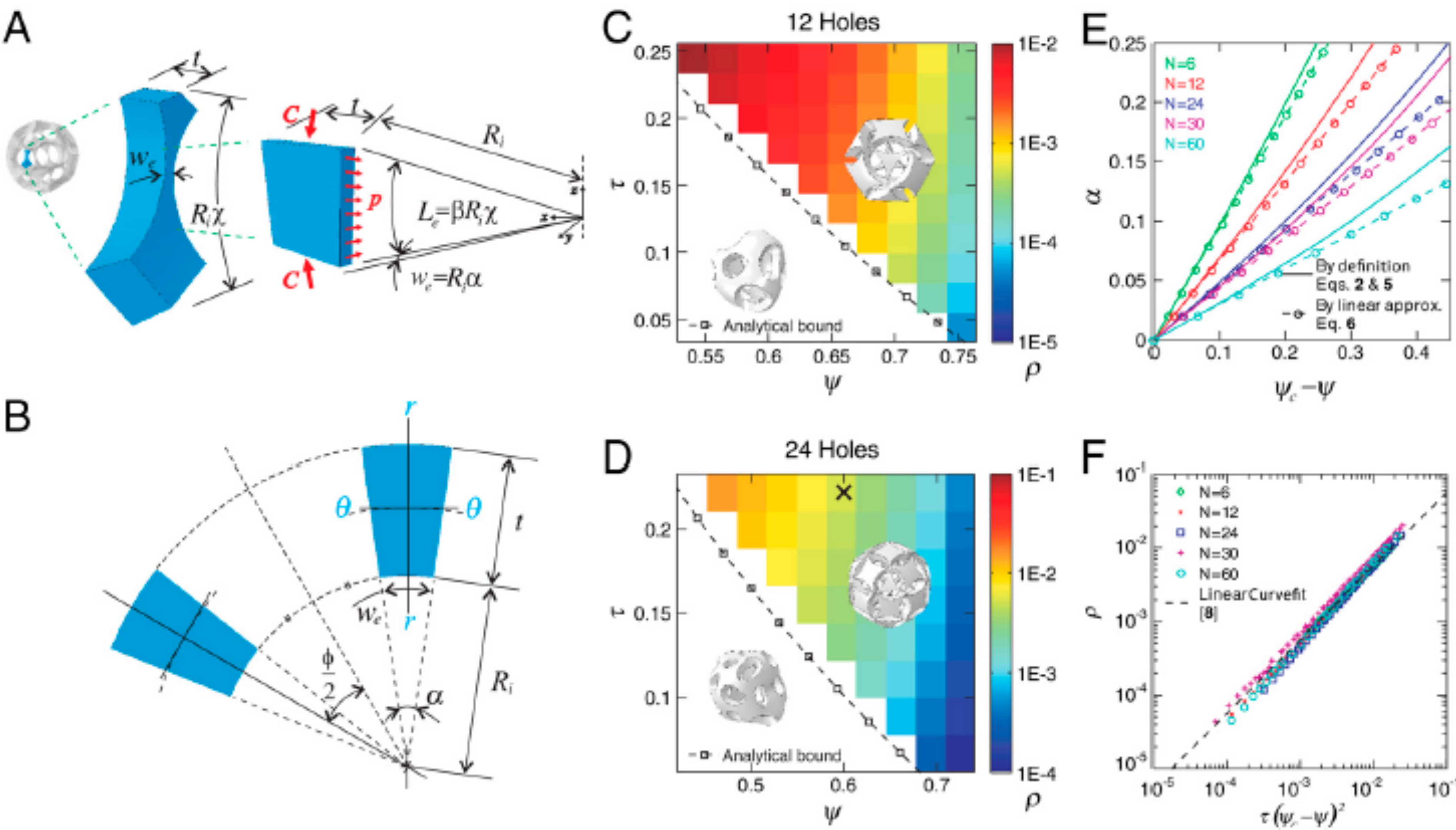


J. Shim et al., "Buckling-induced encapsulation of structured elastic shells under pressure," PNAS, (2012)

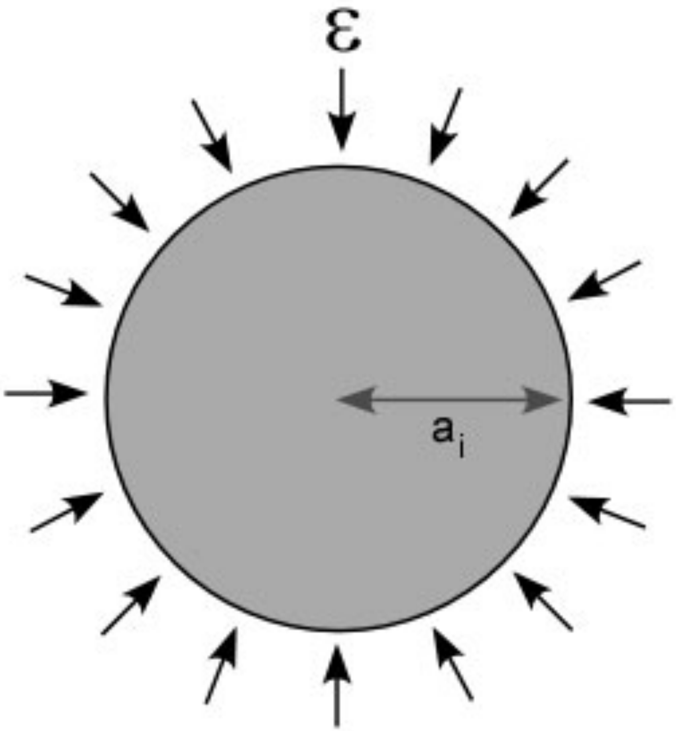
Buckling into Shells

Critical Buckling Stress

$$\sigma_c = \frac{k^2 E}{12\left(1-\nu^2\right)}\left(\frac{h}{a_i}\right)^2$$

Critical Buckling Strain

$$\varepsilon_c = \frac{k^2}{12\left(1+\nu\right)}\left(\frac{h}{a_i}\right)^2$$



Equilibrium Equation Circular Plate (cylindrical coordinates):

$$r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} + \left(\frac{P\,r^2}{D} - 1\right)\varphi = 0$$

General Solution: $\varphi = A_1 J_1(k) + A_2 Y_1(k)$

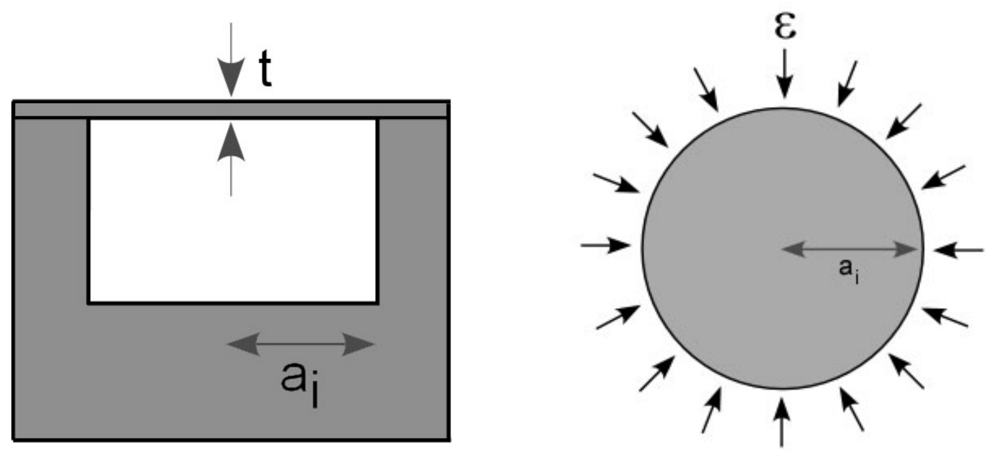
Solution:

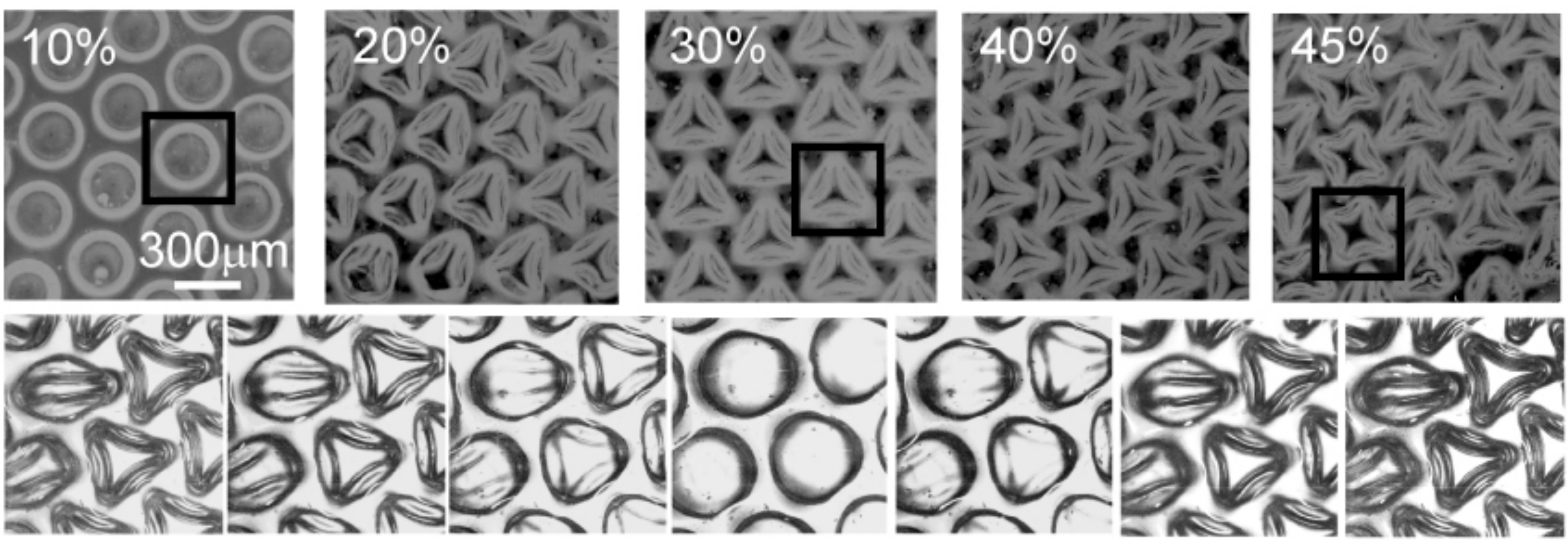
$$k\,J_0(k) - (1-\nu)J_1(k) = 0$$

First buckling mode: ***k* = 2.16**
(simply supported B.C.'s)

Buckling into Shells

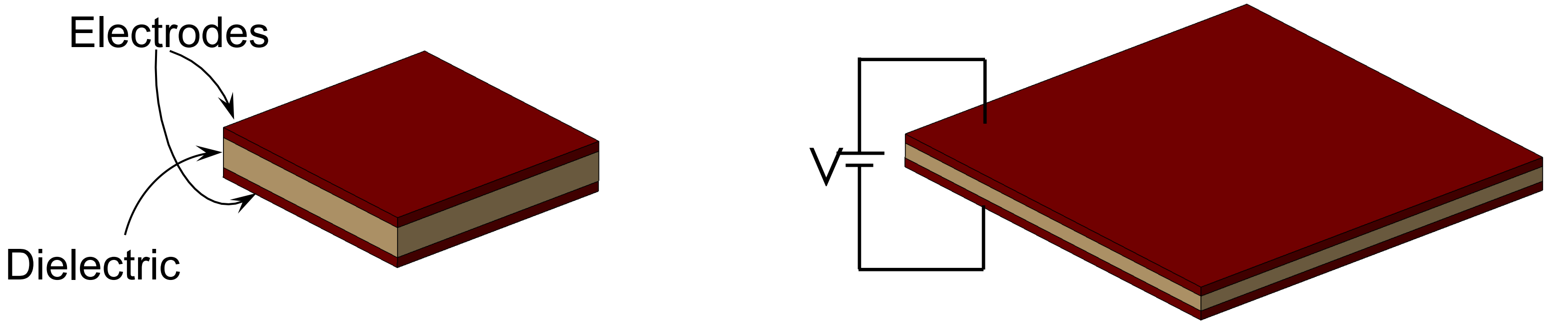
Buckling of microscale elastic **plates**

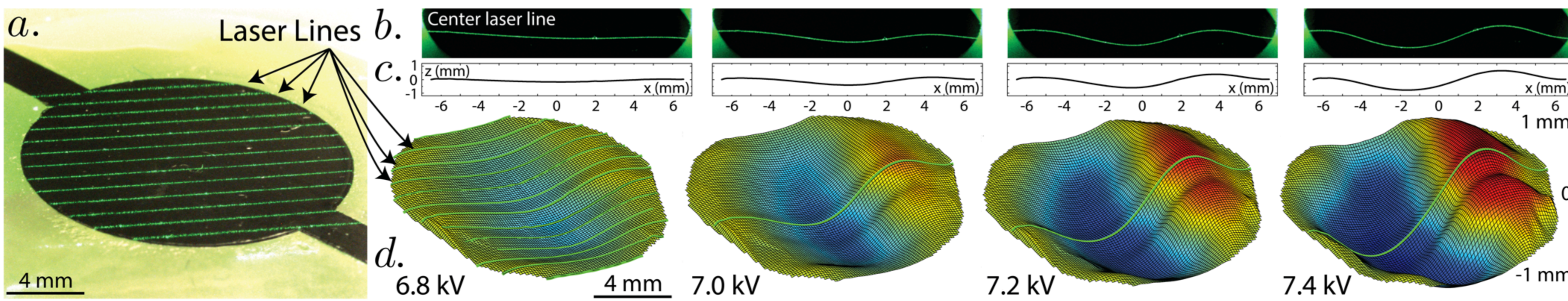


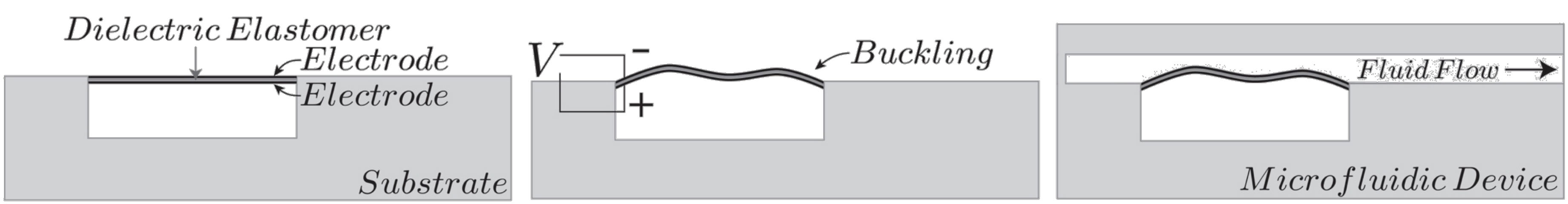


D.P. Holmes, and A.J Crosby, "Crumpled Surface Structures", *Soft Matter*, 4, 82, (2008).

Buckling into Shells

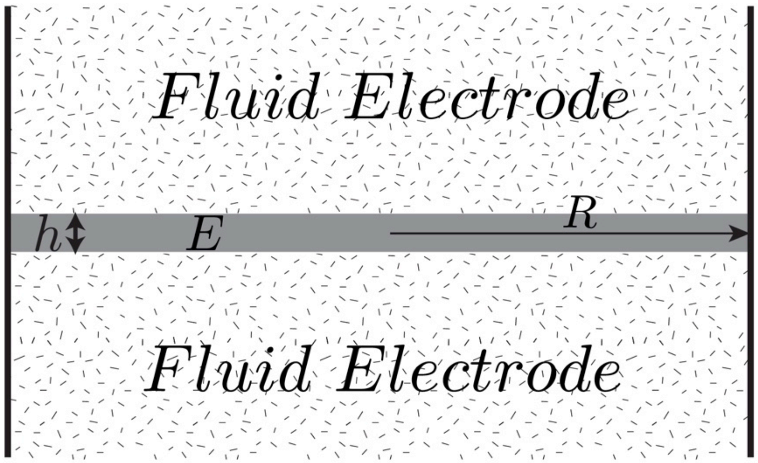


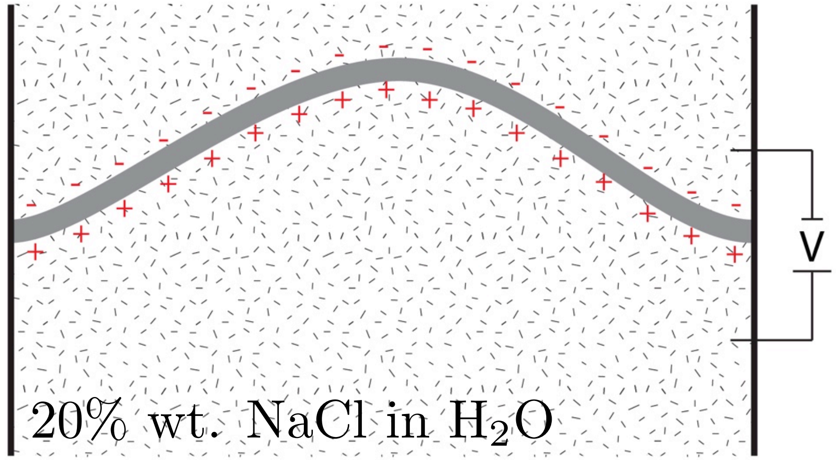




B. Tavakol, M. Bozlar, G. Froehlicher, H. A. Stone, I. Aksay, and D. P. Holmes, "Buckling Instabilities of Dielectric Elastomeric Plates for Flexible Microfluidic Pumps," Soft Matter, 10(27), 4789–4794, (2014).

Buckling into Shells





At what **voltage** will the plate **buckle**?

$$\varepsilon_r \approx \frac{\epsilon_0 \epsilon}{2E}\left(\frac{V}{h}\right)^2 \longrightarrow \sigma_r \approx \frac{\epsilon_0 \epsilon}{2(1-\nu^2)}\left(\frac{V}{h}\right)^2 \longrightarrow \sigma_c = \frac{kEh^2}{12(1-\nu^2)R^2}$$

Radial strain in a dielectric elastomer

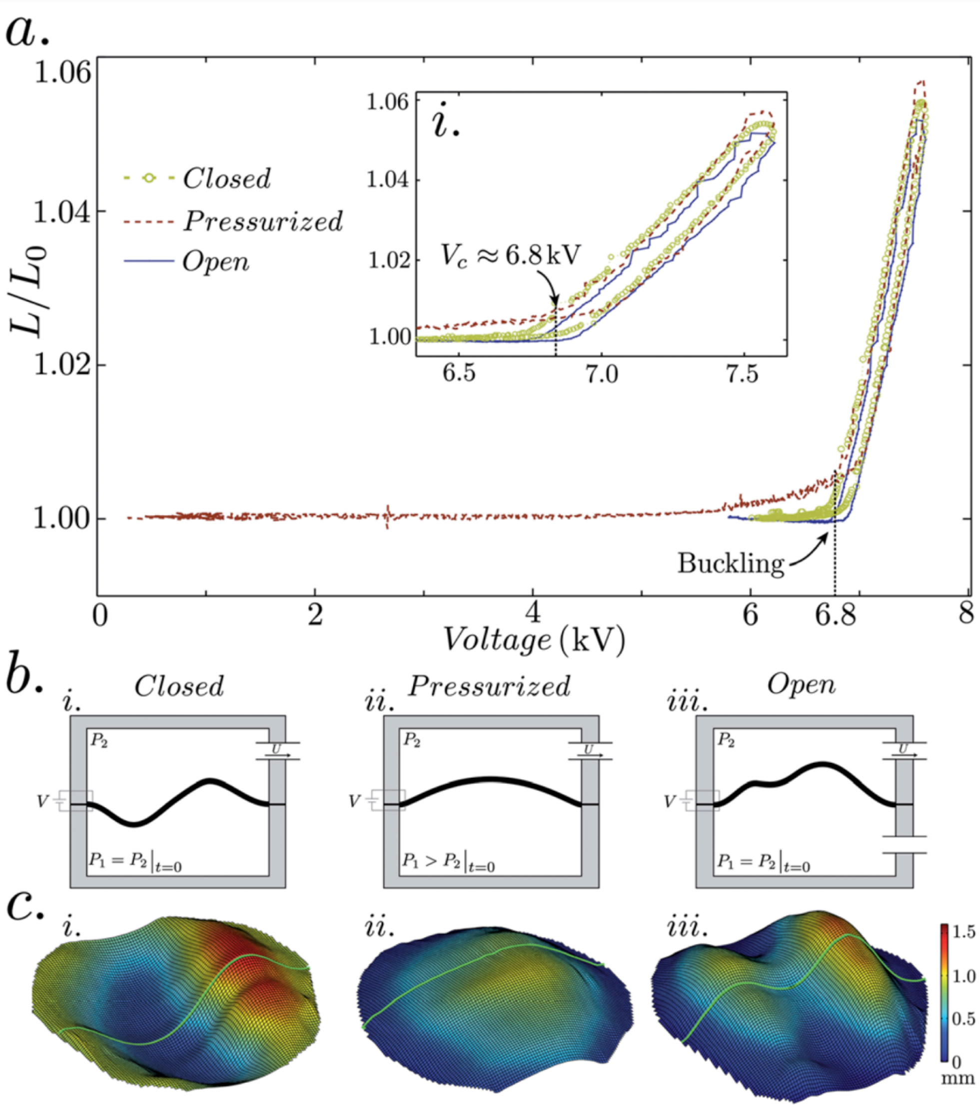
Plane stress relation

Linear stability analysis for a buckling plate

$$V_c \approx h^2\left(\frac{kE}{6\epsilon_0\epsilon R^2}\right)^{1/2}$$

B. Tavakol, M. Bozlar, G. Froehlicher, H. A. Stone, I. Aksay, and D. P. Holmes, "Buckling Instabilities of Dielectric Elastomeric Plates for Flexible Microfluidic Pumps," Soft Matter, 10(27), 4789–4794, (2014).
B. Tavakol, A. Chawan, and D. P. Holmes, "Buckling Instability of Thin Films as a Means to Control or Enhance Fluid Flow within Microchannels," in preparation, (2015).

Buckling into Shells



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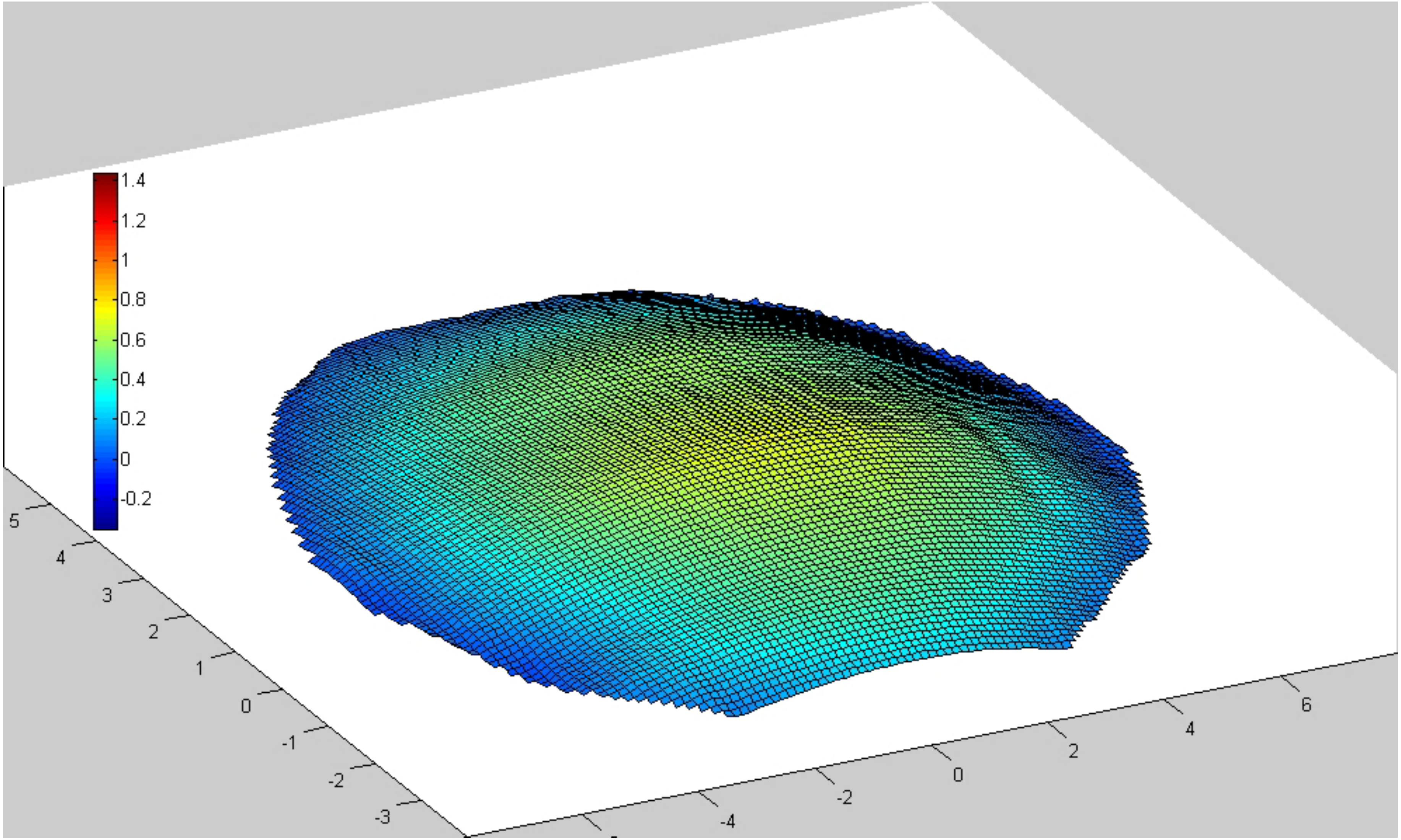

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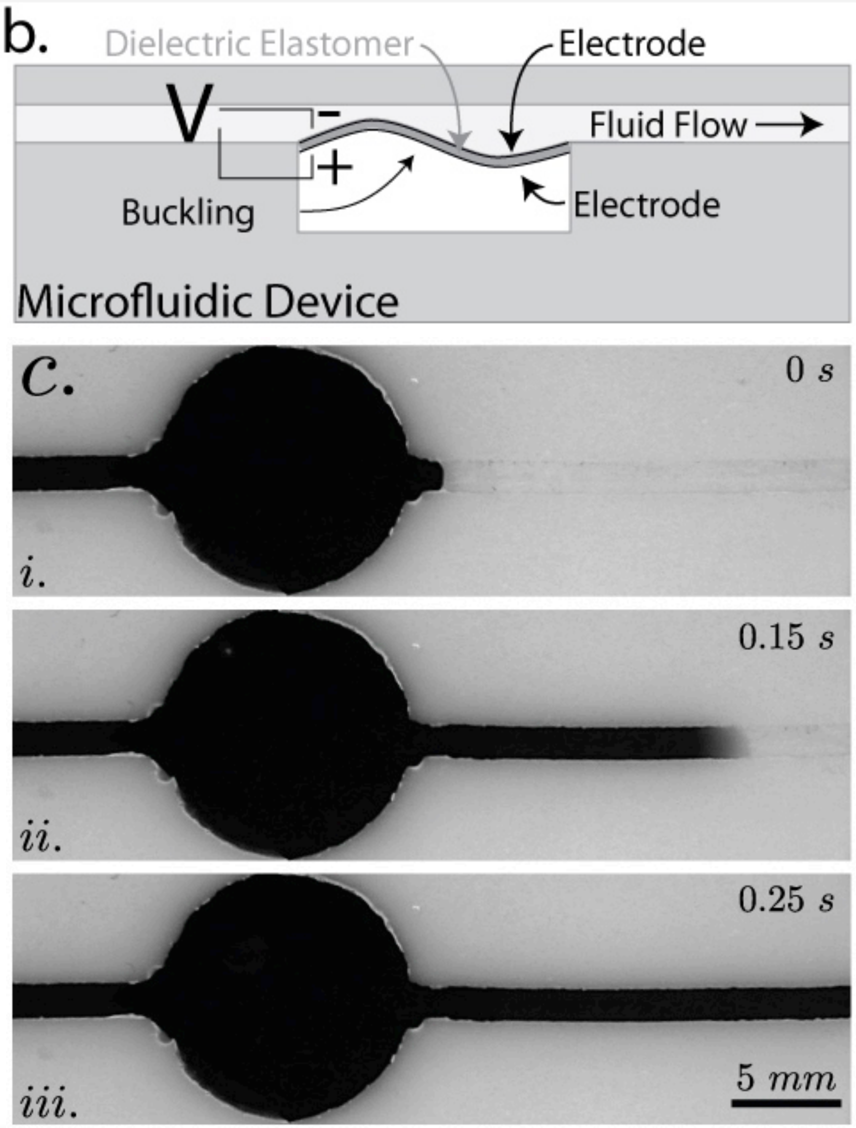

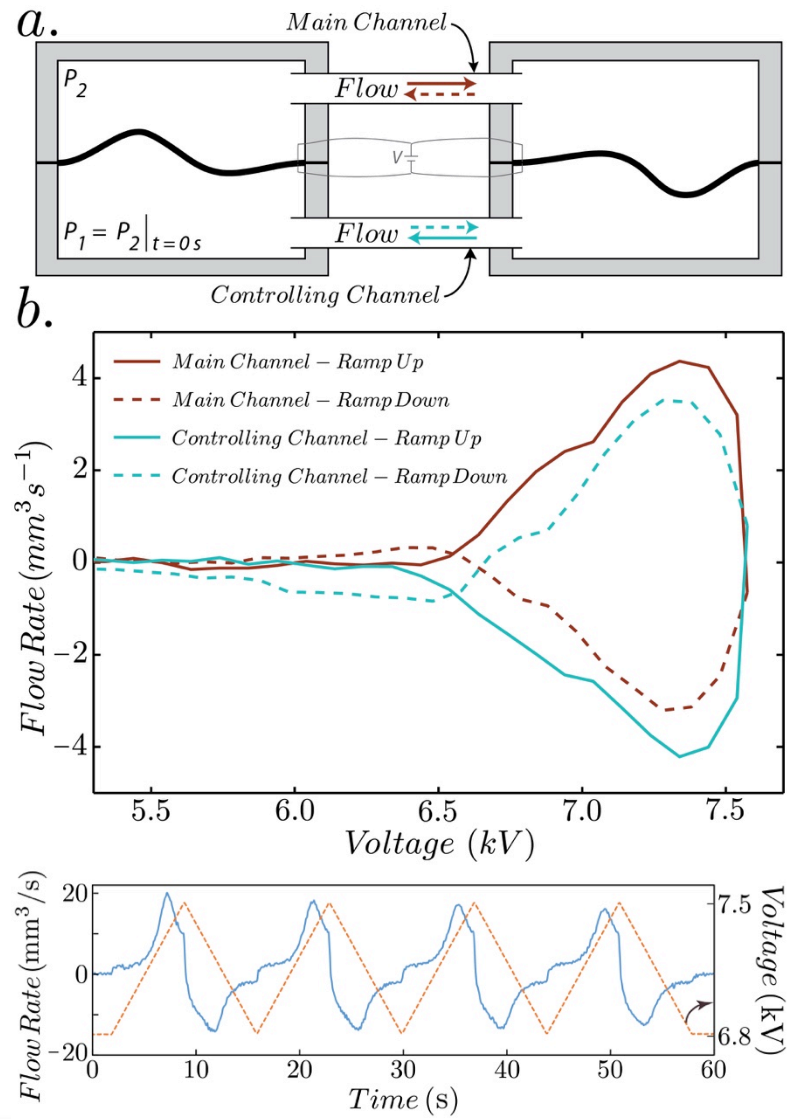

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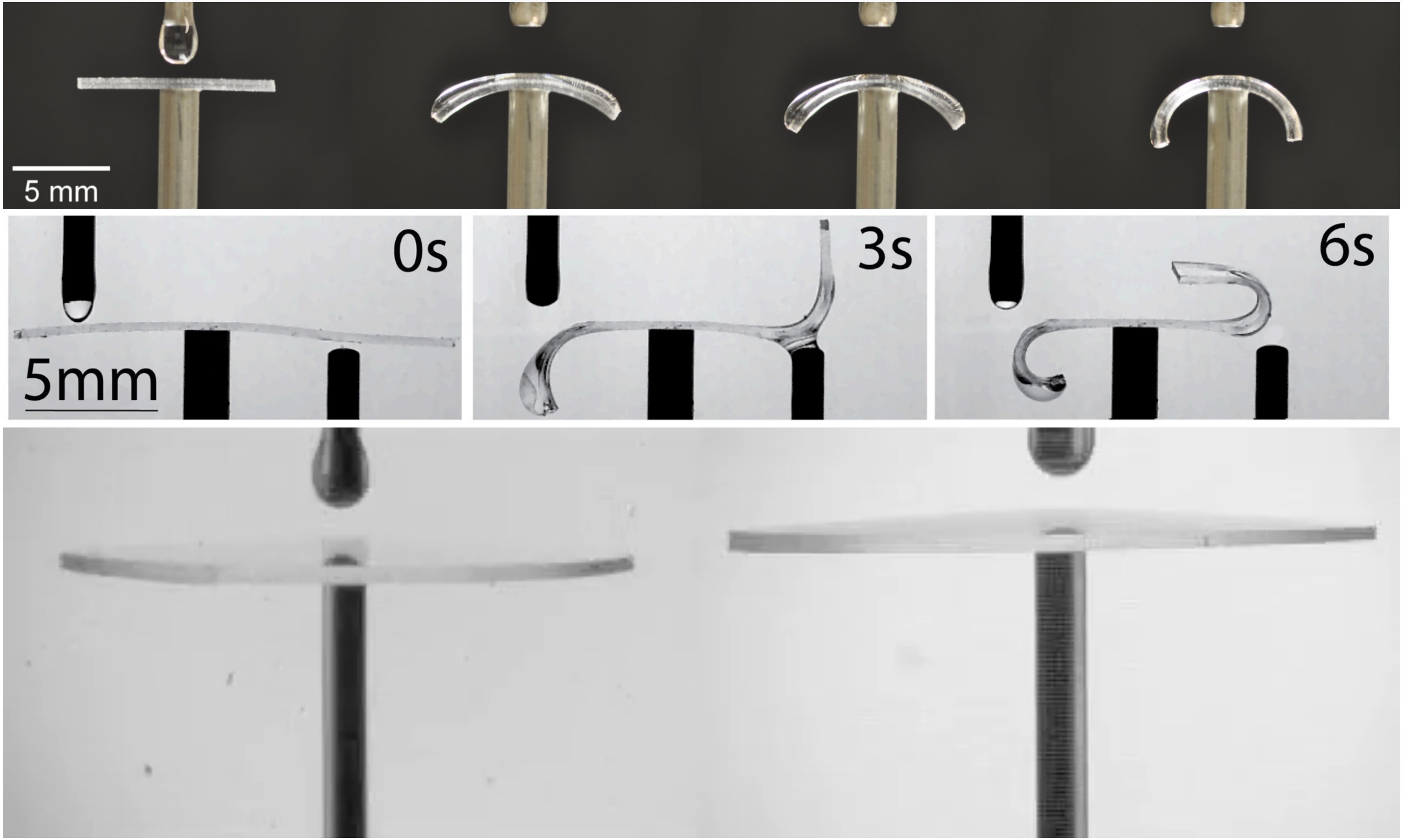
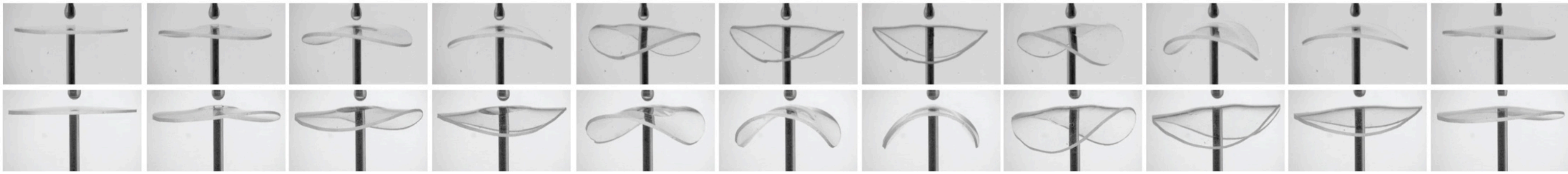
Swelling & Buckling



D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Plate Shape



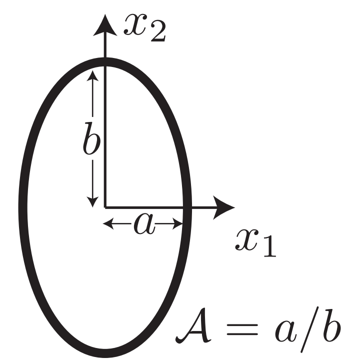
- Calculate & **minimize** the **plate's energy** as a function of time.

$$\mathcal{U}_m = \frac{h}{2E}\int_A \left[\left(\nabla^2\varphi\right)^2 - (1+\nu)\,\Diamond^4[\varphi,\varphi]\right]\mathrm{d}A,$$

$$\mathcal{U}_b = \frac{B}{2}\int_A \left[\left(\nabla^2 w\right)^2 - (1-\nu)\,\Diamond^4[w,w]\right]\mathrm{d}A$$

- Satisfy compatibility between out-of-plane **curvature** & in-plane **stretching.**

$$-\Delta K = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\ \varepsilon_{\alpha\beta,\gamma\delta} = \frac{1}{E}\nabla^4\varphi$$



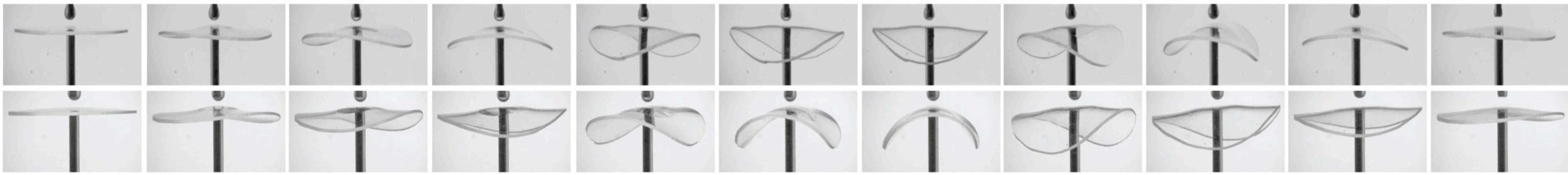
- Assume a form for the Airy stress function

$$\varphi = \frac{E\Delta K}{64}\left(x_1^2 + x_2^2\right)\left(a^2 + b^2 - x_1^2 - x_2^2\right)$$

- Minimize the energy with respect to the **principal curvatures**.

D.P. Holmes, A. Pandey, M. Pezzulla, and P. Nardinocchi,. *In Preparation* (2014).

Plate Shape



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$$\mathcal{U}_b = \frac{B}{2} \int_A \left[\left(\nabla^2 w\right)^2 - (1-\nu) \Diamond^4[w, w] \right] \mathrm{d}A$$

- Satisfy compatibility between out-of-plane **curvature** & in-plane **stretching.**

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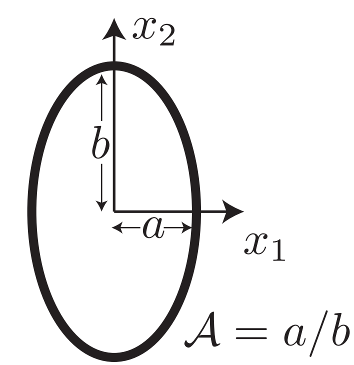

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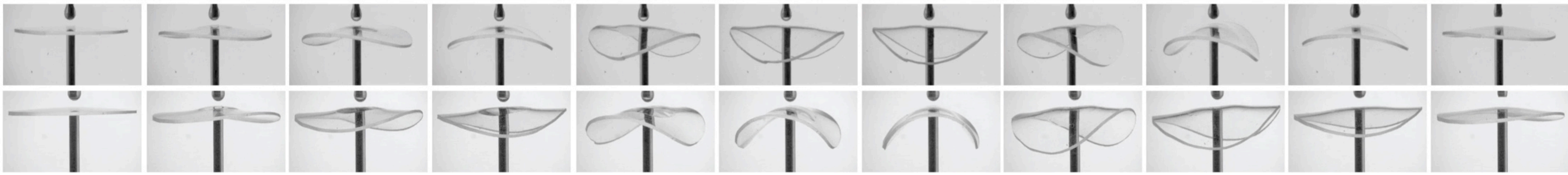
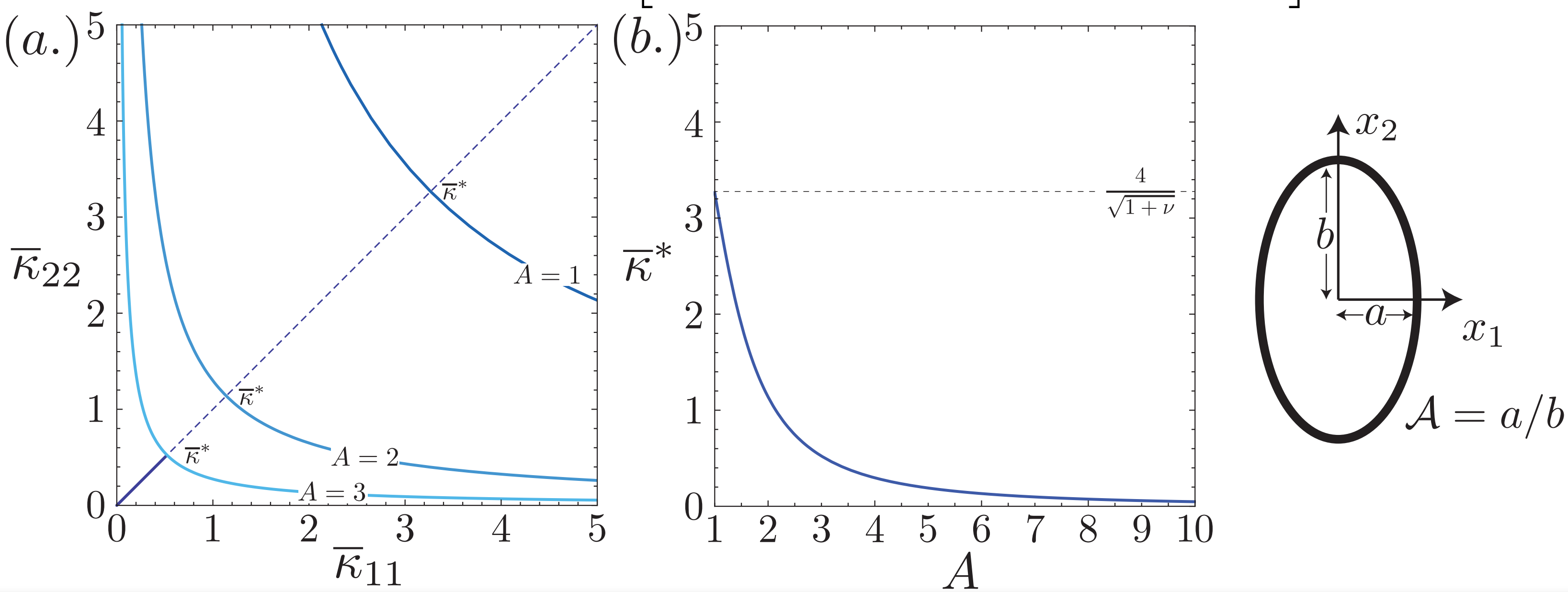
Plate Shape



Minimization of the total strain energy with respect to the unknown curvatures:

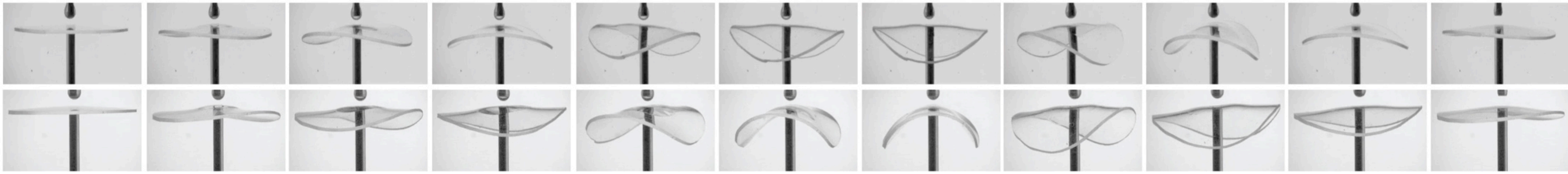
$$\overline{\kappa}_{11} = \overline{\kappa}_{22}, \text{ or}$$

$$\overline{\kappa}_{11} = -\frac{256}{\overline{\kappa}_{22}\left(1+\nu\right)\left[-9+2A^2-9A^4+3\nu\left(A^2-1\right)^2\right]}$$



D.P. Holmes, A. Pandey, M. Pezzulla, and P. Nardinocchi,. *In Preparation* (2014).

Dynamic Plate Shape



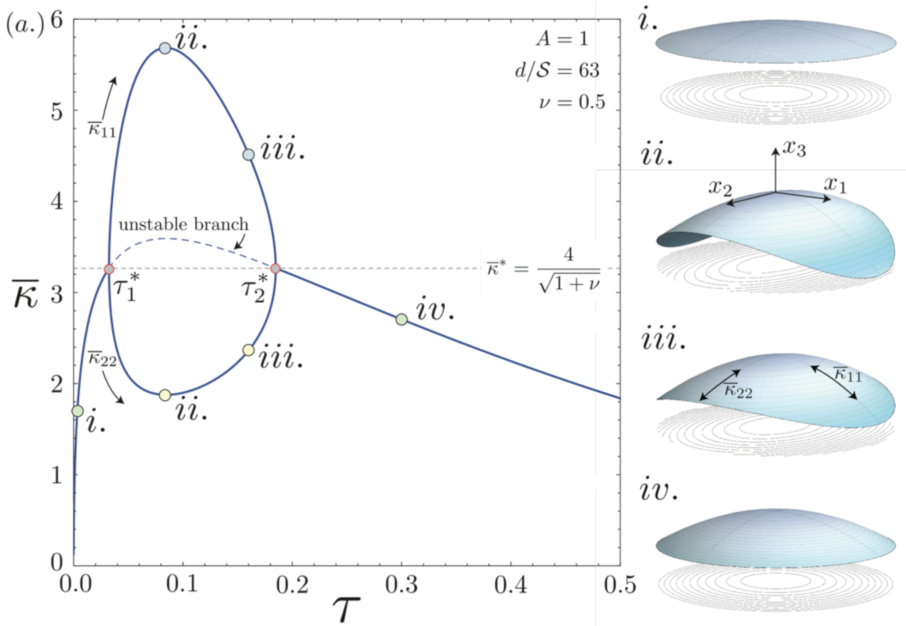
Diffusive dynamics:

$$\psi(\overline{x}_3,\tau) = 1 + \sum_{n=0}^{\infty} \mathcal{A}(n)\sin[\lambda(n)\overline{x}_3]\, e^{-\lambda(n)^2\tau}$$

Bending dynamics:

$$\overline{\kappa}_\varepsilon(\tau) = \frac{d}{\mathcal{S}}\int_{-1/2}^{1/2} \psi(\overline{x}_3,\tau)\overline{x}_3 \,\mathrm{d}\overline{x}_3$$

$d/\mathcal{S} = 12\varepsilon_{eq}L^2/h^2$



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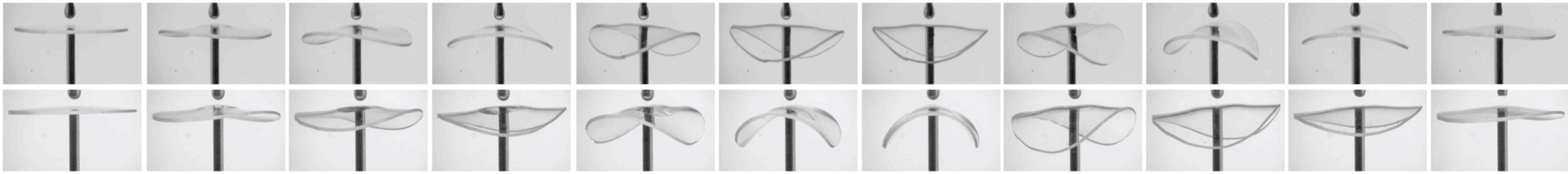
Dynamic Plate Shape



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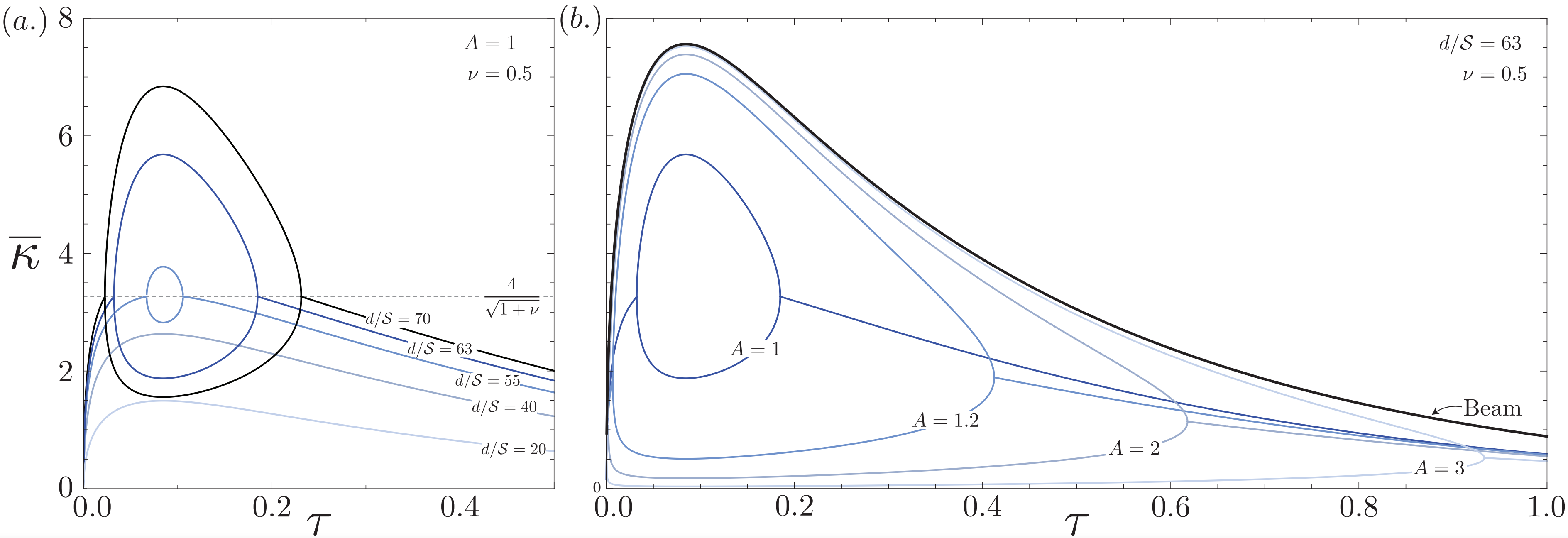
$$\psi(\overline{x}_3,\tau) = 1 + \sum_{n=0}^{\infty} \mathcal{A}(n)\sin[\lambda(n)\overline{x}_3]\, e^{-\lambda(n)^2\tau}$$

Bending dynamics:

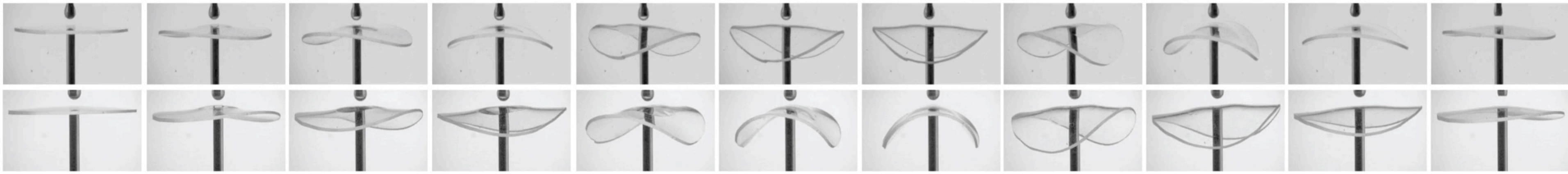
$$\overline{\kappa}_\varepsilon(\tau) = \frac{d}{\mathcal{S}}\int_{-1/2}^{1/2}\psi(\overline{x}_3,\tau)\overline{x}_3\ \mathrm{d}\overline{x}_3$$

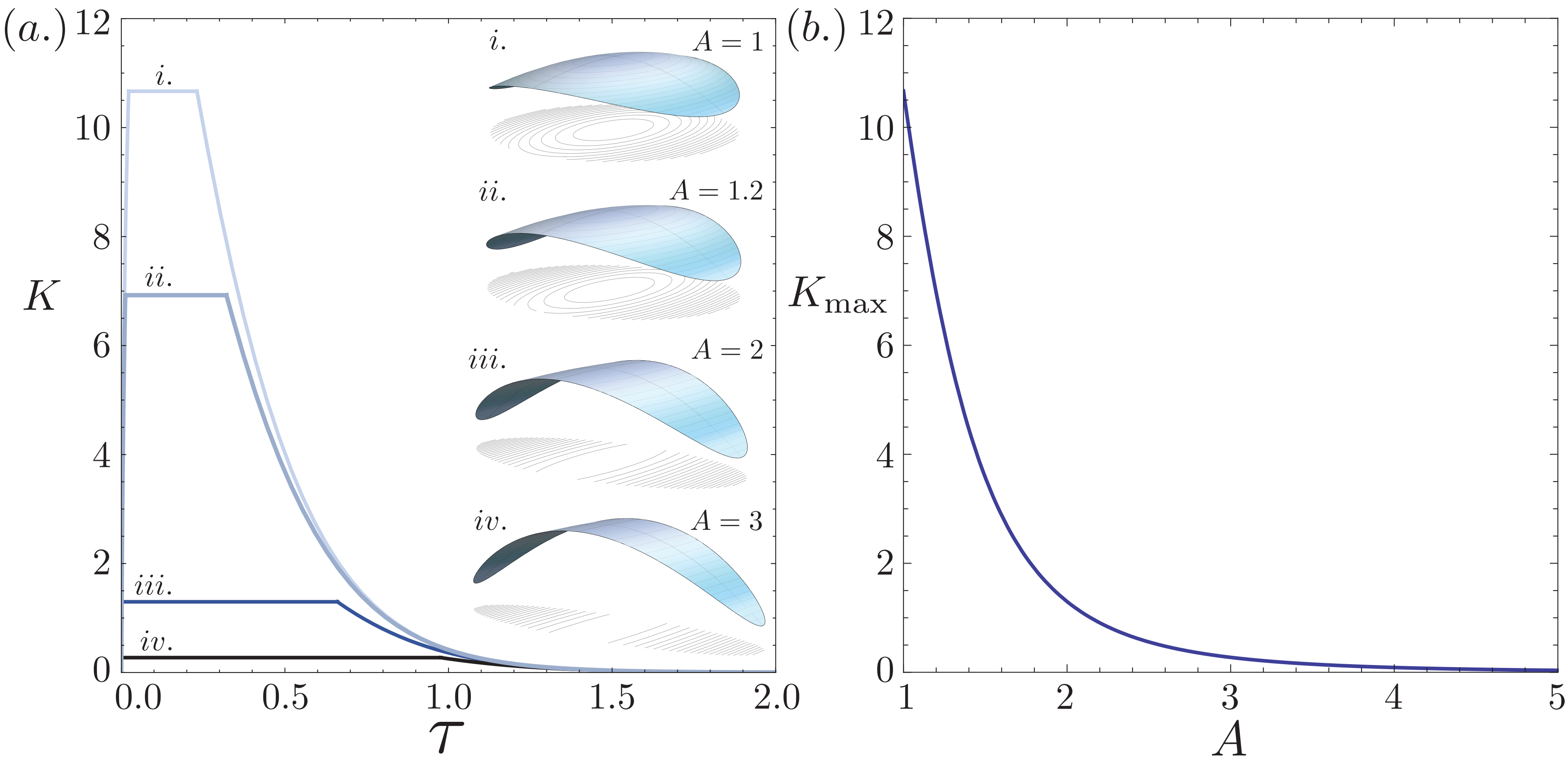
$$d/\mathcal{S} = 12\varepsilon_{eq}L^2/h^2$$



D.P. Holmes, A. Pandey, M. Pezzulla, and P. Nardinocchi,. *In Preparation* (2014).

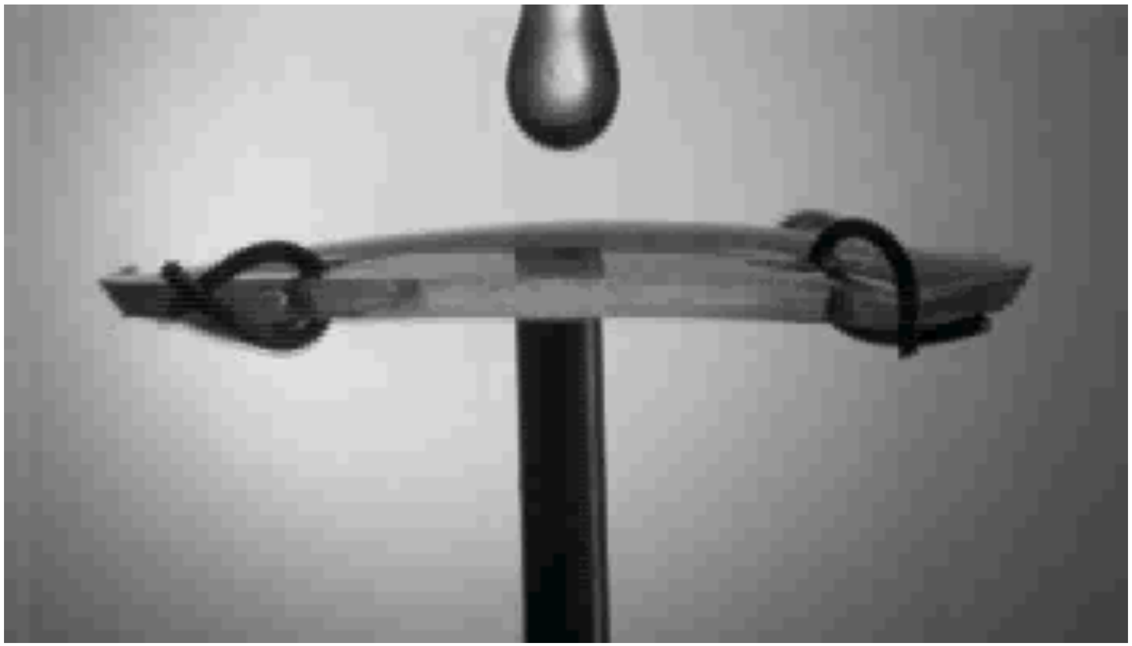
Plate Shape

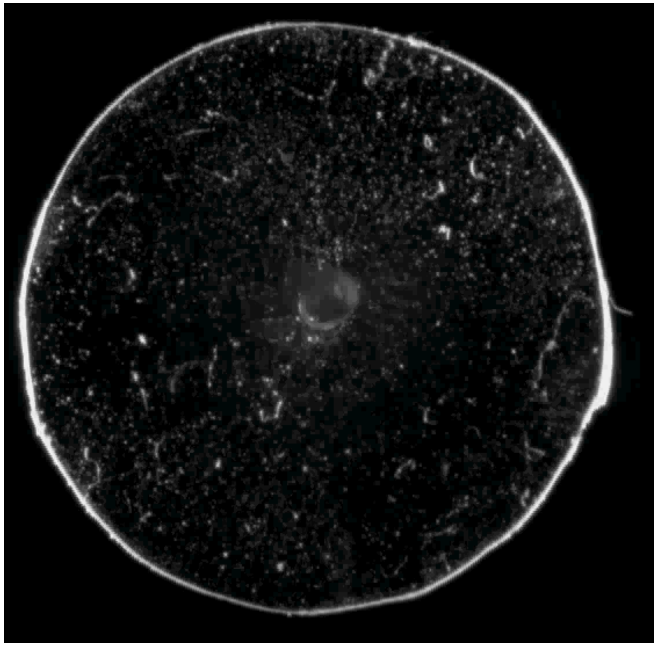




D.P. Holmes, A. Pandey, M. Pezzulla, and P. Nardinocchi,. *In Preparation* (2014).

Dynamics: Twisting

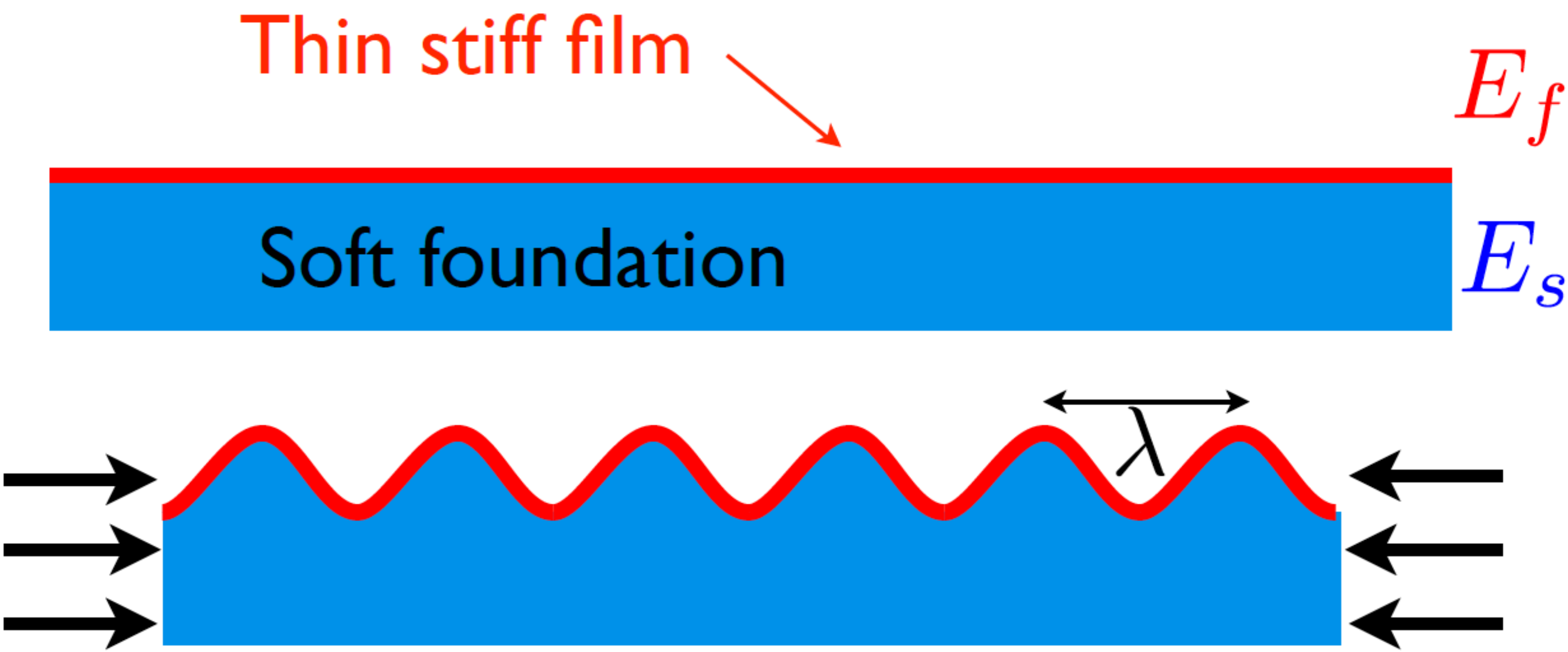




D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

Wrinkling

Uniaxial compression:



Allen, 1969.

Wrinkling

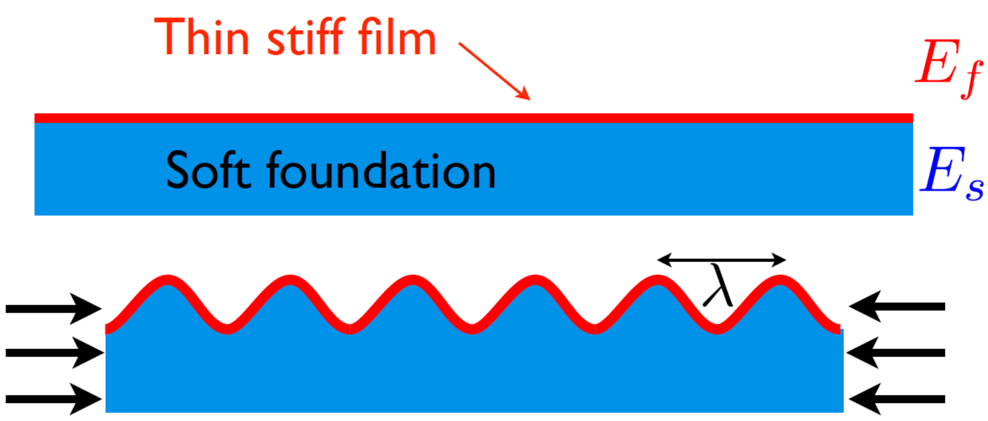
Wrinkle Wavelength:

- Balance of the **plate's bending energy**, and the energy required to **deform** the underlying **substrate**.

- **Bending** penalizes **short wavelengths**.
- Deforming the **elastic foundation** penalizes **long wavelengths.**

- **Intermediate wavelength** emerges when the reaction of the underlying layer is proportional to the deflection of the plate.

Uniaxial compression:



Equilibrium Plate Eq. (FvK)

$$B\nabla^4 w - h\sigma_{\alpha\beta}\partial^2_{\alpha\beta} w = 0$$

1D wrinkles in x-direction:

$$B\frac{\partial^4 w}{\partial x^4} - h\sigma_{xx}\frac{\partial^2 w}{\partial x^2} + Kw = 0$$

Allen, 1969.
B. Audoly, "Elasticity and Geometry", Oxford

Wrinkling

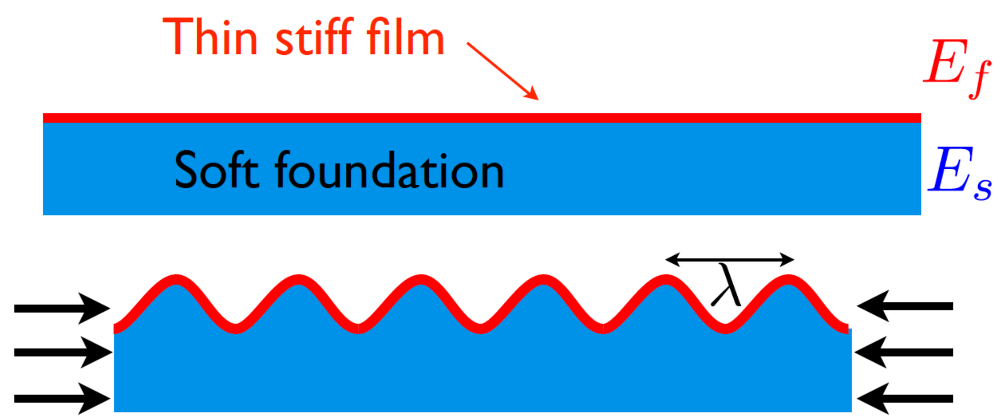
1D wrinkles in x-direction:

$$B\frac{\partial^4 w}{\partial x^4} - h\sigma_{xx}\frac{\partial^2 w}{\partial x^2} + Kw = 0$$

Linearizing this equation & disregarding any stretching of the mid-plane due to curvature, i.e. second term is zero

$$B\frac{w}{L^4} \sim Kw$$

Uniaxial compression:



Scaling sets the wavelength:

$$L \rightarrow \lambda \sim \left(\frac{B}{K}\right)^{1/4}$$

In the limit of a deep substrate:

$h \ll \lambda \ll H_s$

$$\lambda \sim h\left(\frac{E}{E_s}\right)^{1/3}$$

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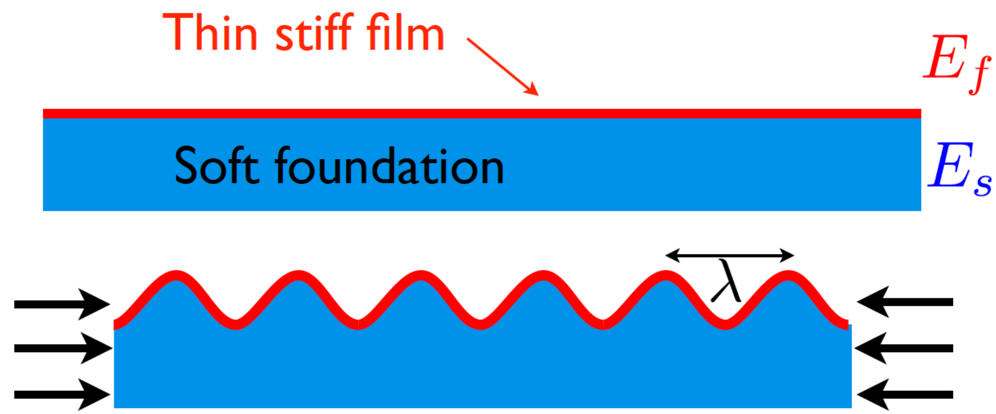
Wrinkling

1D wrinkles in x-direction:

$$B\frac{\partial^4 w}{\partial x^4} - h\sigma_{xx}\frac{\partial^2 w}{\partial x^2} + Kw = 0$$

Uniaxial compression:



Linearizing this equation about the flat, unbuckled state:

$$u_\alpha = w = 0$$

Linearized strains:

$$\varepsilon_{\alpha\beta} \approx 1/2(u_{\alpha,\beta} + u_{\beta,\alpha})$$

Perform linear stability analysis. Linearized equations below permit periodic solutions:

$$B\nabla^4 w - h\sigma_0 \nabla^2 w = -p \quad \text{and} \quad \nabla^4 \phi = 0$$

$$\sigma_c = E^*\left(\frac{3E_s^*}{2E^*}\right)^{2/3} \qquad \lambda = \pi h\left(\frac{2E^*}{3E_s^*}\right)^{1/3}$$

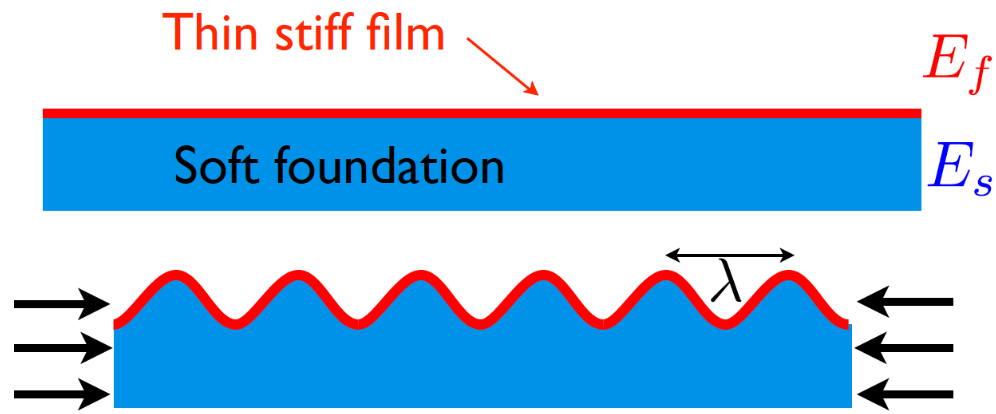
Wrinkling

1D wrinkles in x-direction:

$$\sigma_c = E^* \left(\frac{3E_s^*}{2E^*} \right)^{2/3}$$

$$\lambda = \pi h \left(\frac{2E^*}{3E_s^*} \right)^{1/3}$$

Uniaxial compression:



Effective or Reduced moduli:

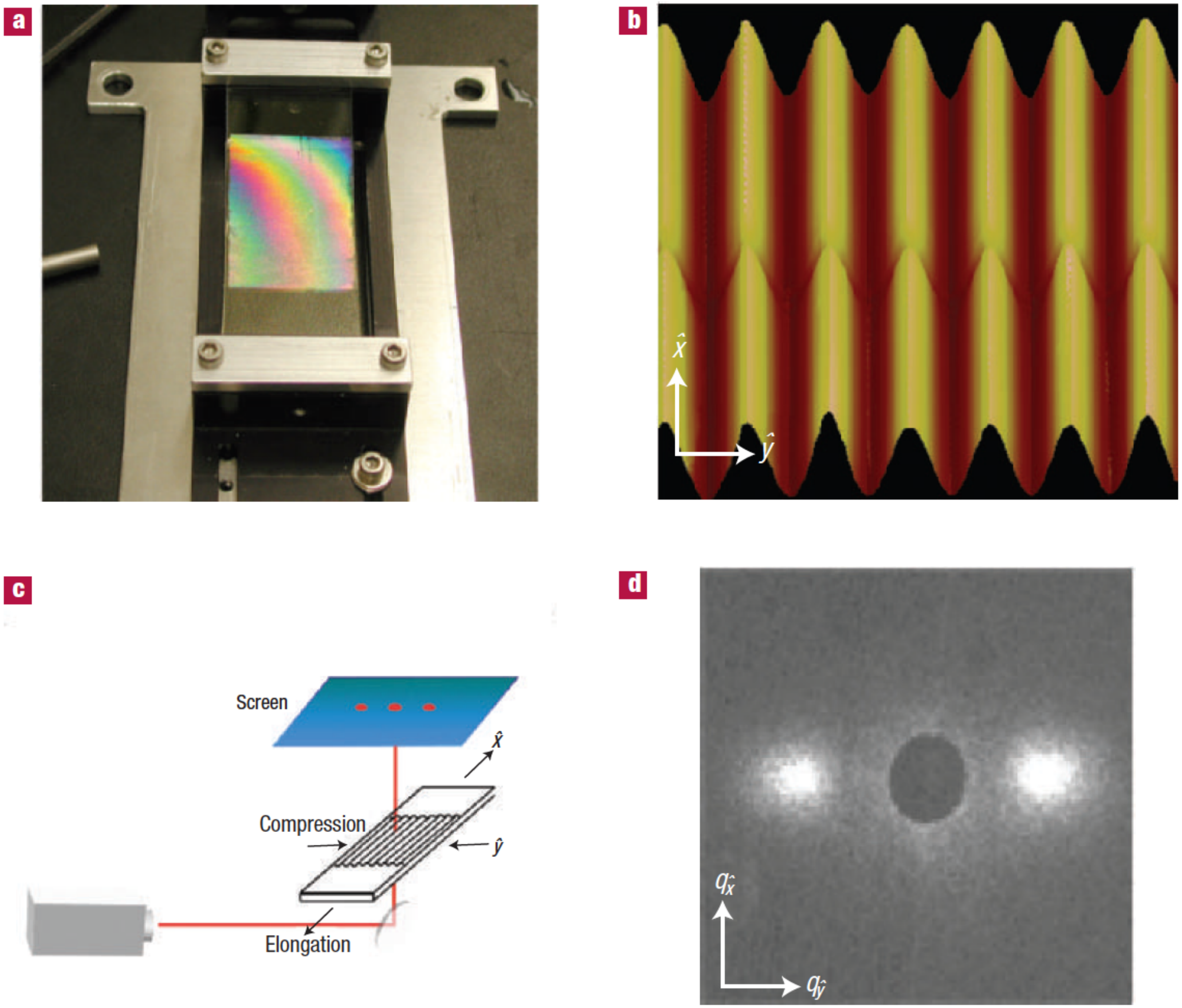
Plate:

$$E^* = \frac{E}{1-\nu^2}$$

Substrate:

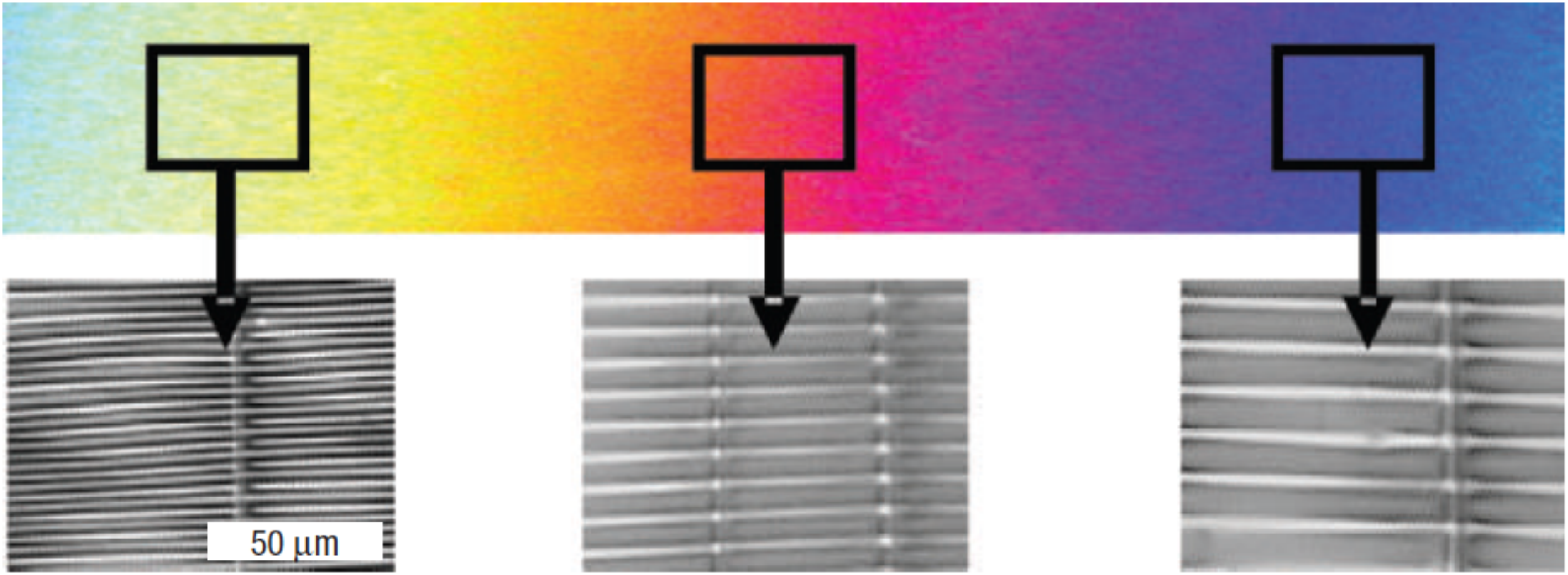
$$E_s^* = \frac{E_s(1-\nu_s)}{(1+\nu_s)(3-4\nu_s)}$$

Wrinkling


a
b
$\hat{x}$
$\hat{y}$
c
Screen
$\hat{x}$
Compression
$\hat{y}$
Elongation
d
$q_{\hat{x}}$
$q_{\hat{y}}$

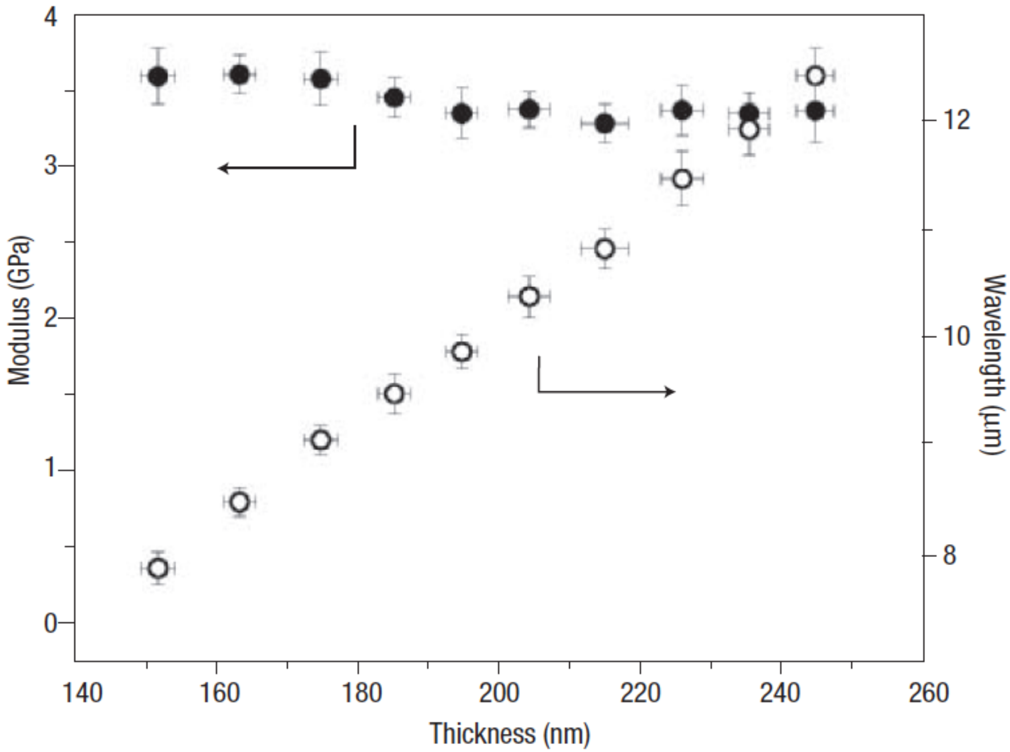
Wrinkling

Thickness gradient:

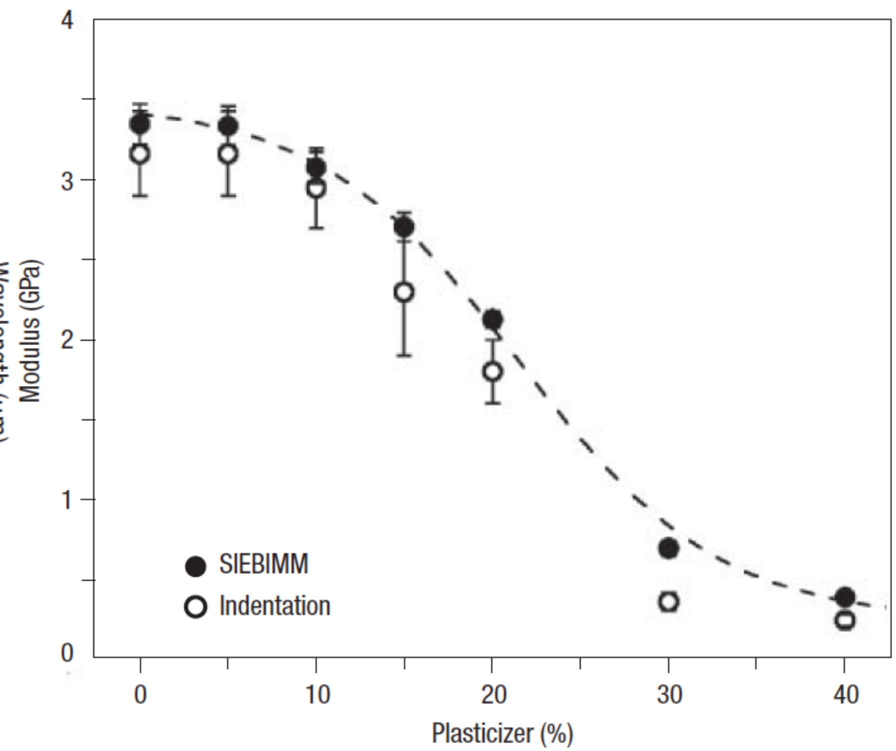


Wrinkling

Polystyrene films: E=3.4 GPa



Effect of plasticizing agent

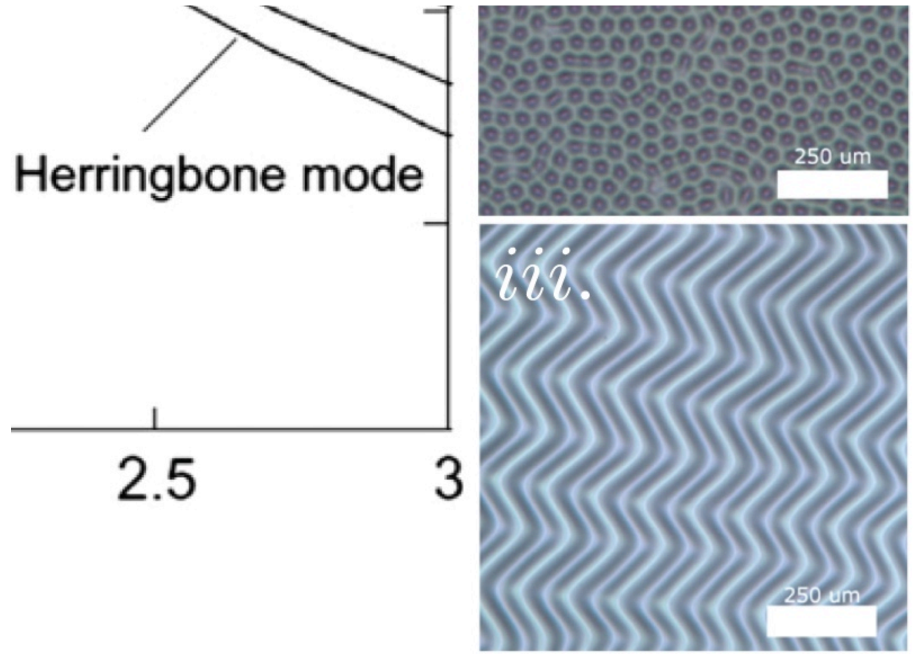


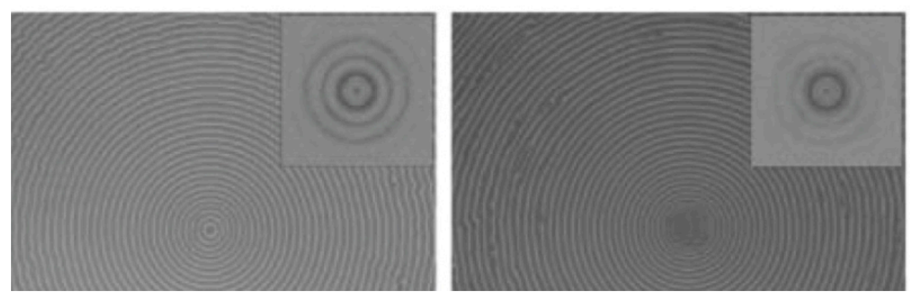
Wrinkling

Wrinkled adhesion:

- Maximum separation for a single wrinkle scales with its **perimeter** and the adhesion energy of the **interface**.
- Smaller wrinkles with a short persistence length will enhance adhesion.
- Low amplitude wrinkles enhance adhesion by increasing the contact line during separation.

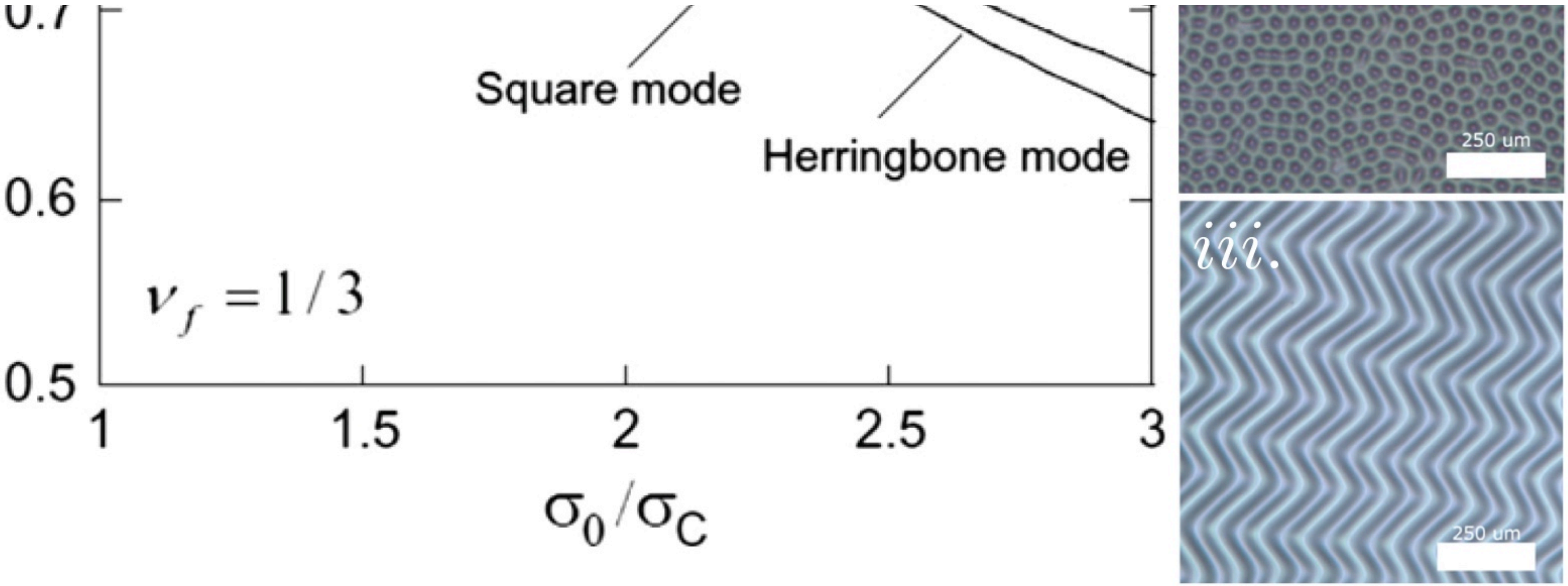
$$\frac{P_s}{P} \sim \underbrace{\left(\frac{G_c}{E^* A}\right)^{1/4}}_{materials} \underbrace{\left(\frac{R}{A}\right)^{1/4}}_{geometry}$$

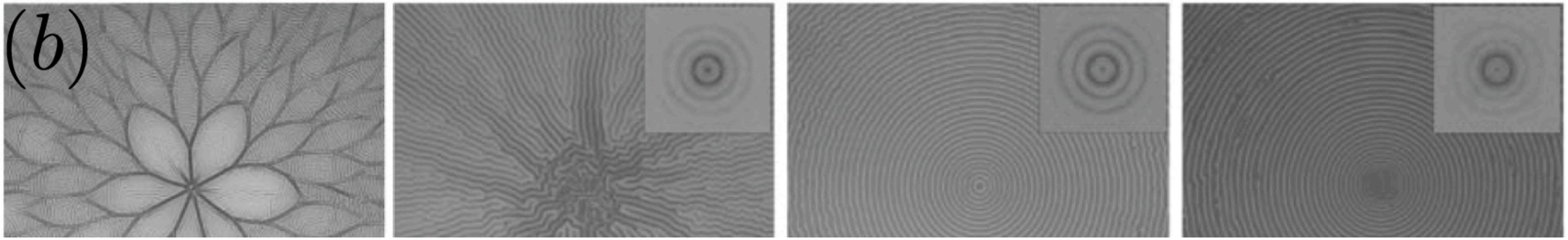




Davis et al. "Mechanics of wrinkled surface adhesion," *Soft Matter*, **7**(11), 5373, (2011).

Wrinkling





Davis et al. "Mechanics of wrinkled surface adhesion," *Soft Matter*, **7**(11), 5373, (2011).

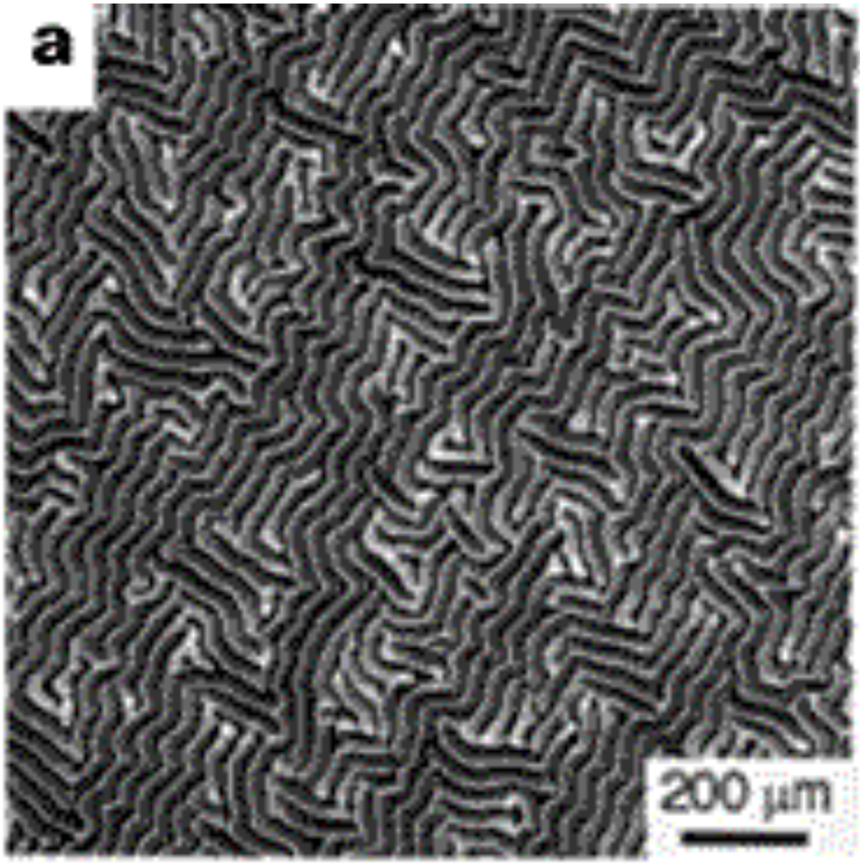
Wrinkling

Biaxial Compression

- Early examples of wrinkling appeared when **thin metallic films** (50 nm) were **deposited** by electron beam evaporation **onto PDMS**.

- Deposition locally heats the PDMS, upon cooling the compressive stress buckles the bilayer into a herringbone patter.

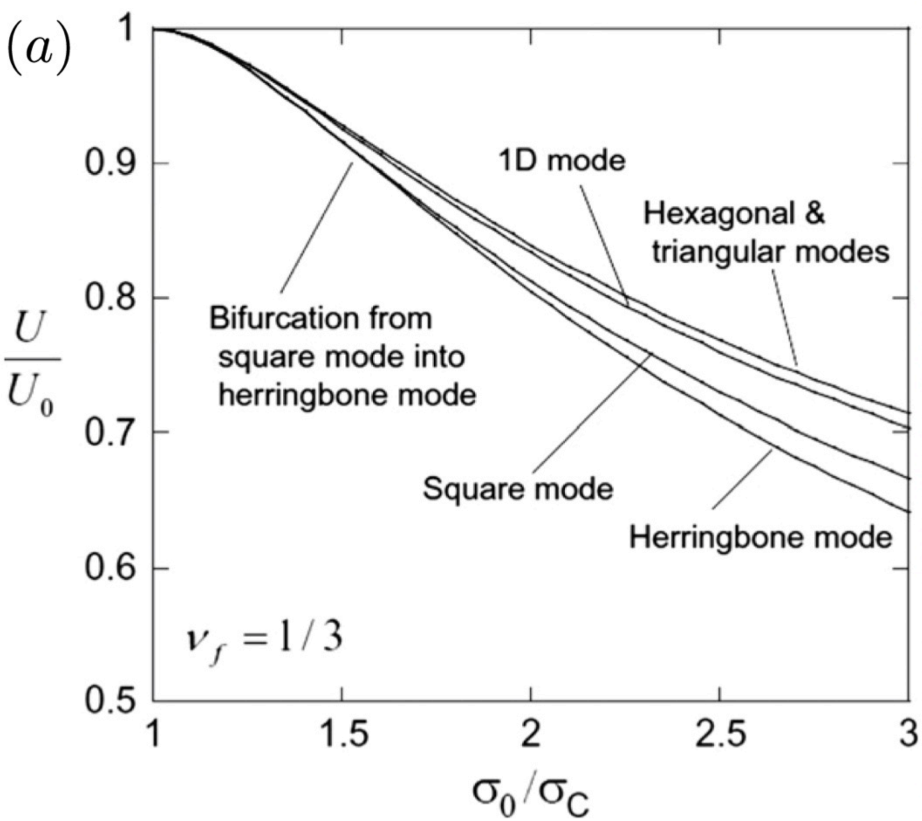
$$20\mu m \leq \lambda \leq 50\mu m$$

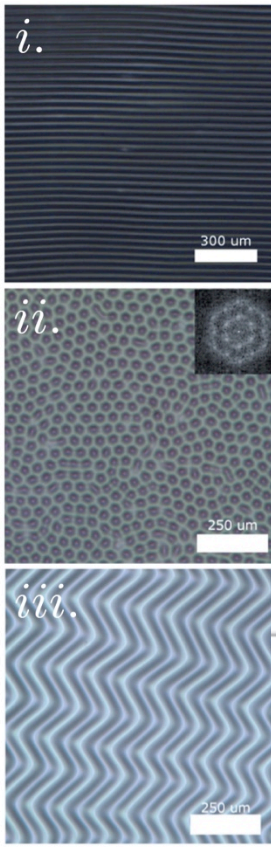


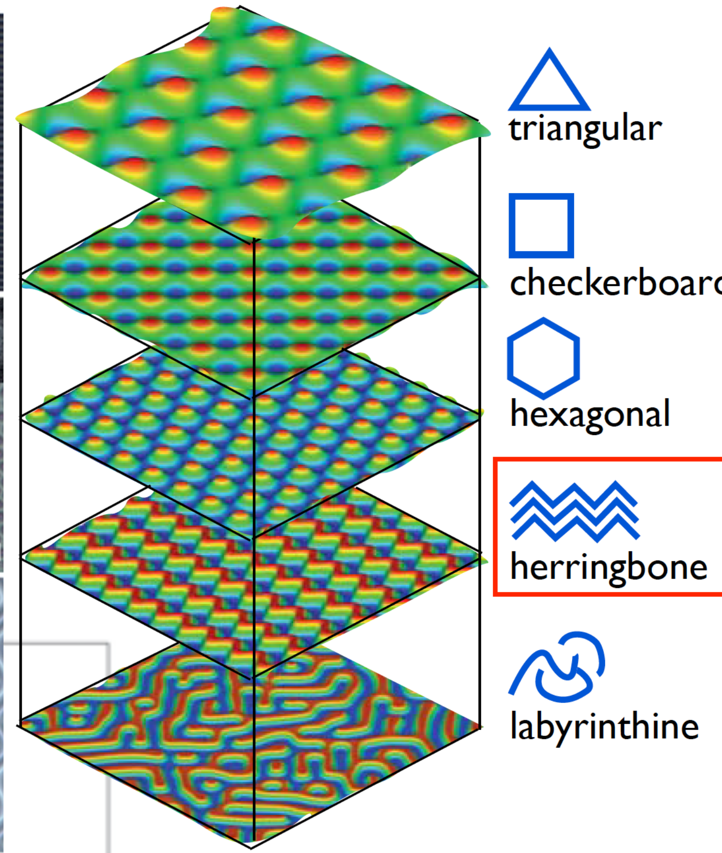
Bowden et al., *Nature*, 1998

Wrinkling

Biaxial Compression



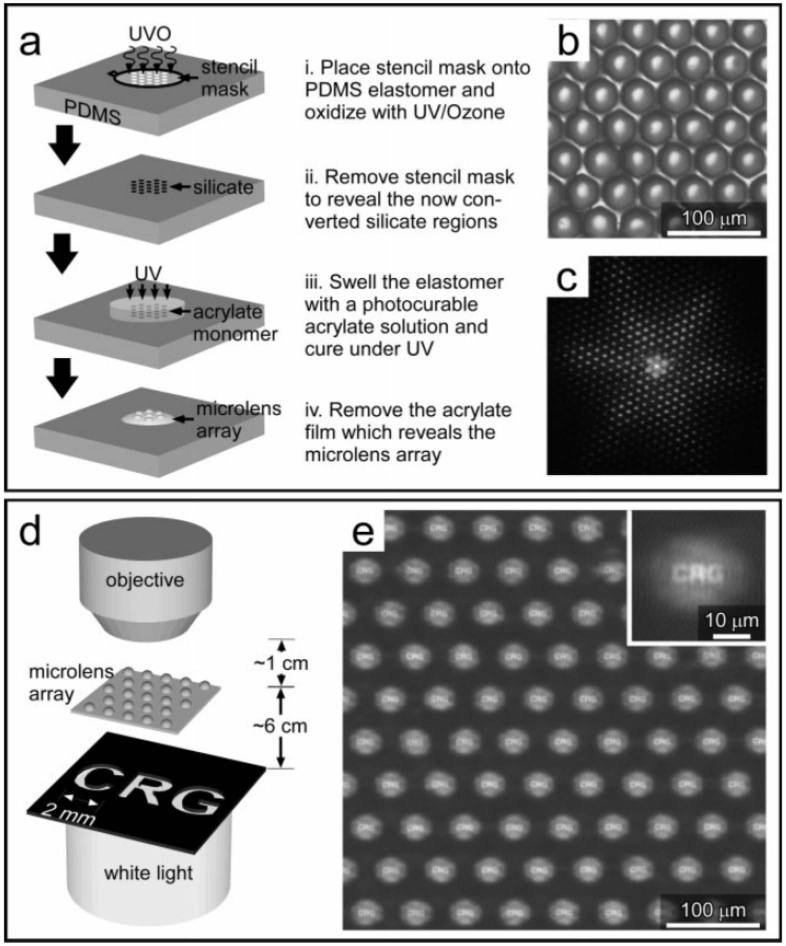


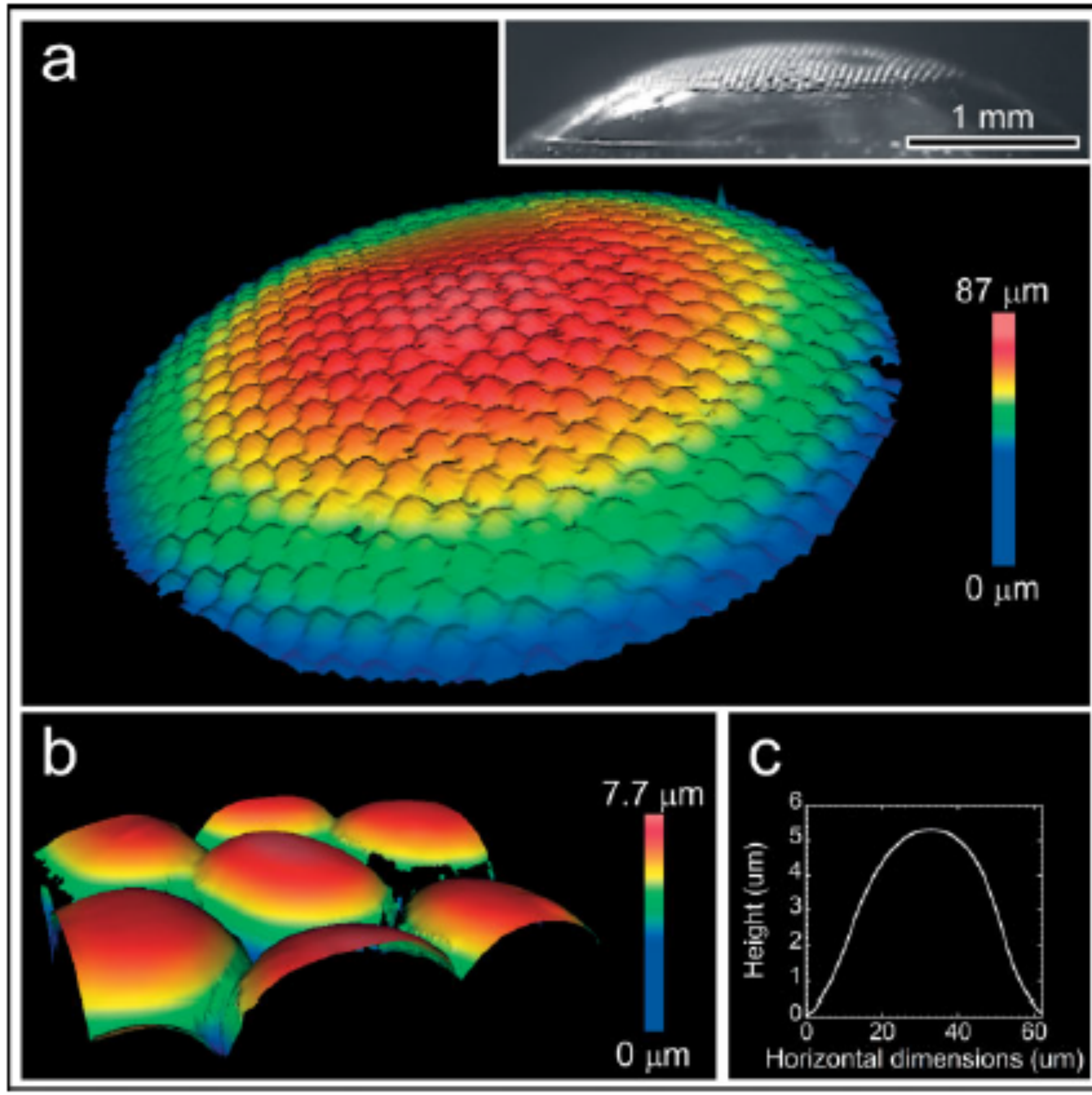


Obtained by linear stability of the cylindrical wrinkling pattern

Wrinkling

Biaxial Compression





E.P. Chan and A.J. Crosby, "Fabricating Microlens Arrays by Surface Wrinkling", Advanced Materials, (2006)

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Overview

Elastic Instabilities for Form and Function
Buckling, Wrinkling, Folding, and Snapping

Geometry and Mechanics:

- Fundamental equations, geometric rigidity, morphing.

Buckling & Wrinkling:

- Stability, wavelength, flexible electronics, mechanical metamaterials, adhesion.

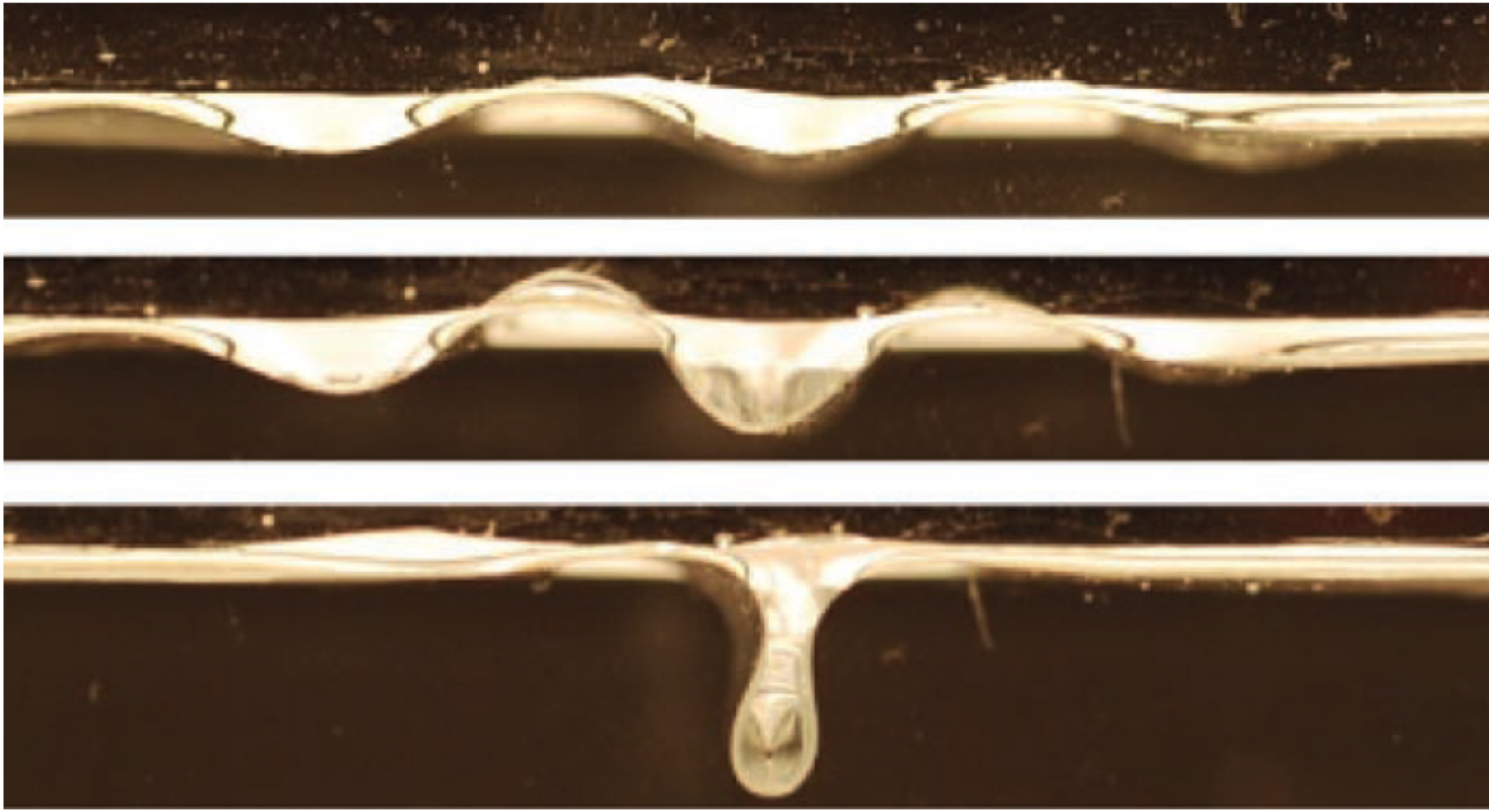
Stress Focusing – Folding & Creasing:

- Wrinkle-to-fold, origami.

Snapping:

- Snapping surfaces.

Folding



Folding

Wrinkling to Fold

Bending

$$\mathcal{U}_b \approx \frac{B}{2}\int_0^L \mathrm{d}l\ \partial_x^2 w \sim BL\left(\frac{A}{\lambda^2}\right)^2$$

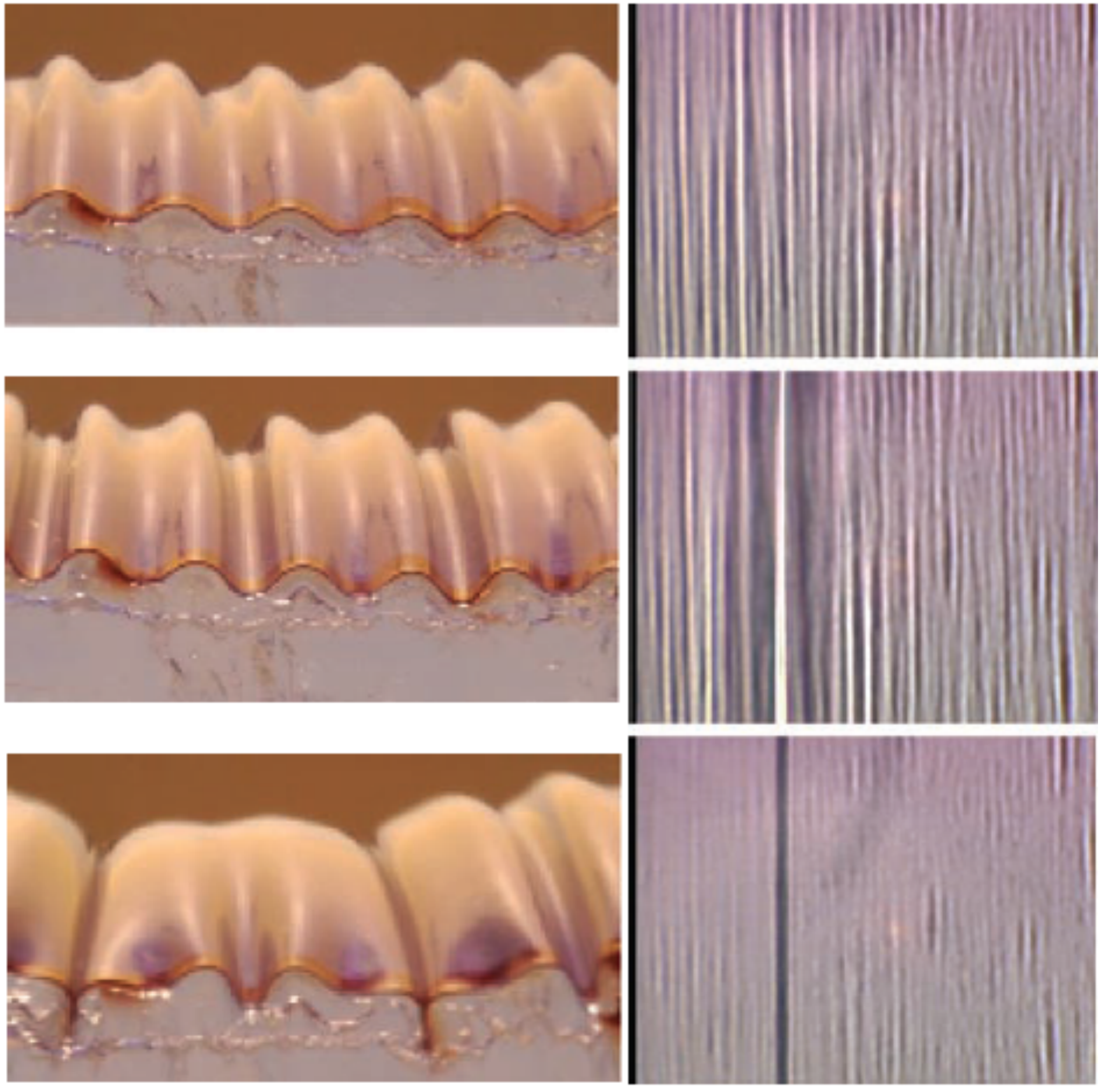
Resistance of substrate

$$\mathcal{U}_K \approx \frac{K}{2}\int_0^L \mathrm{d}l\ \partial_x^2 w \sim KLA^2$$

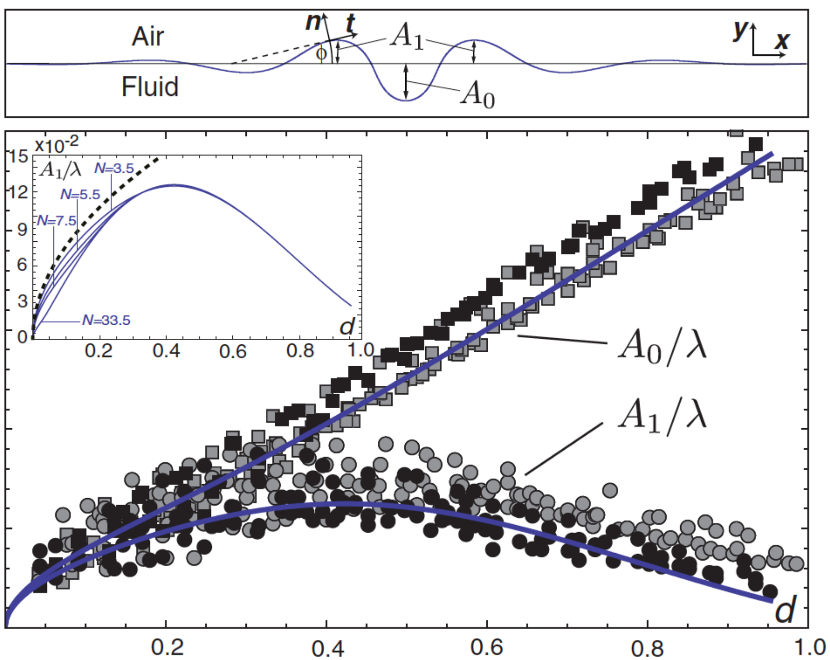
Inextensibility

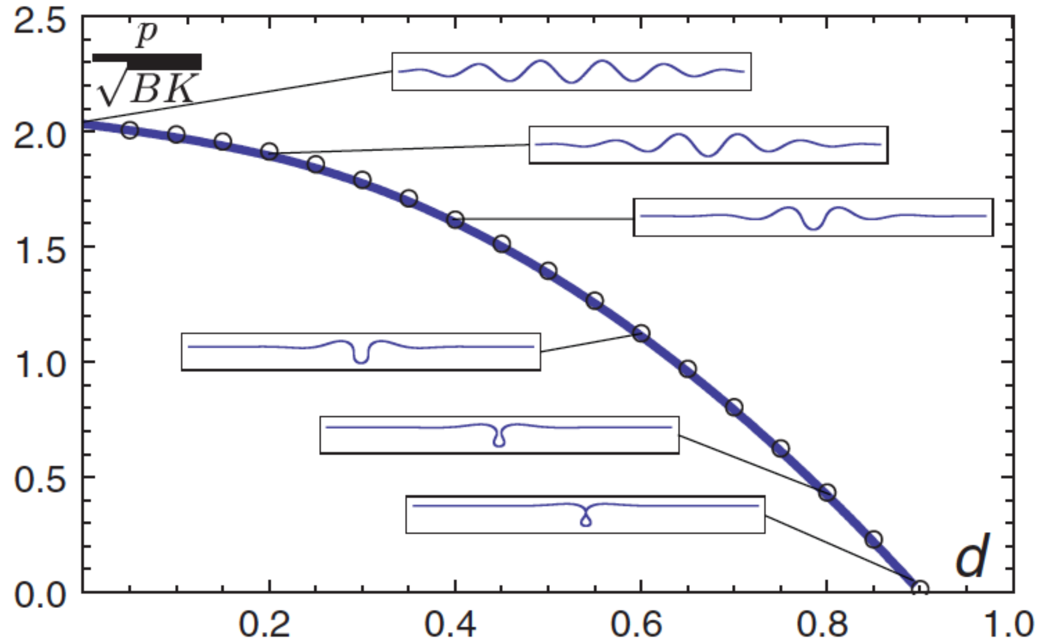
$$\Delta \approx \frac{K}{2}\int_0^L \mathrm{d}l\ \partial_x^2 w \sim L\left(\frac{A}{\lambda}\right)^2$$

Fold: $\dfrac{\Delta}{\lambda} \approx 0.3$

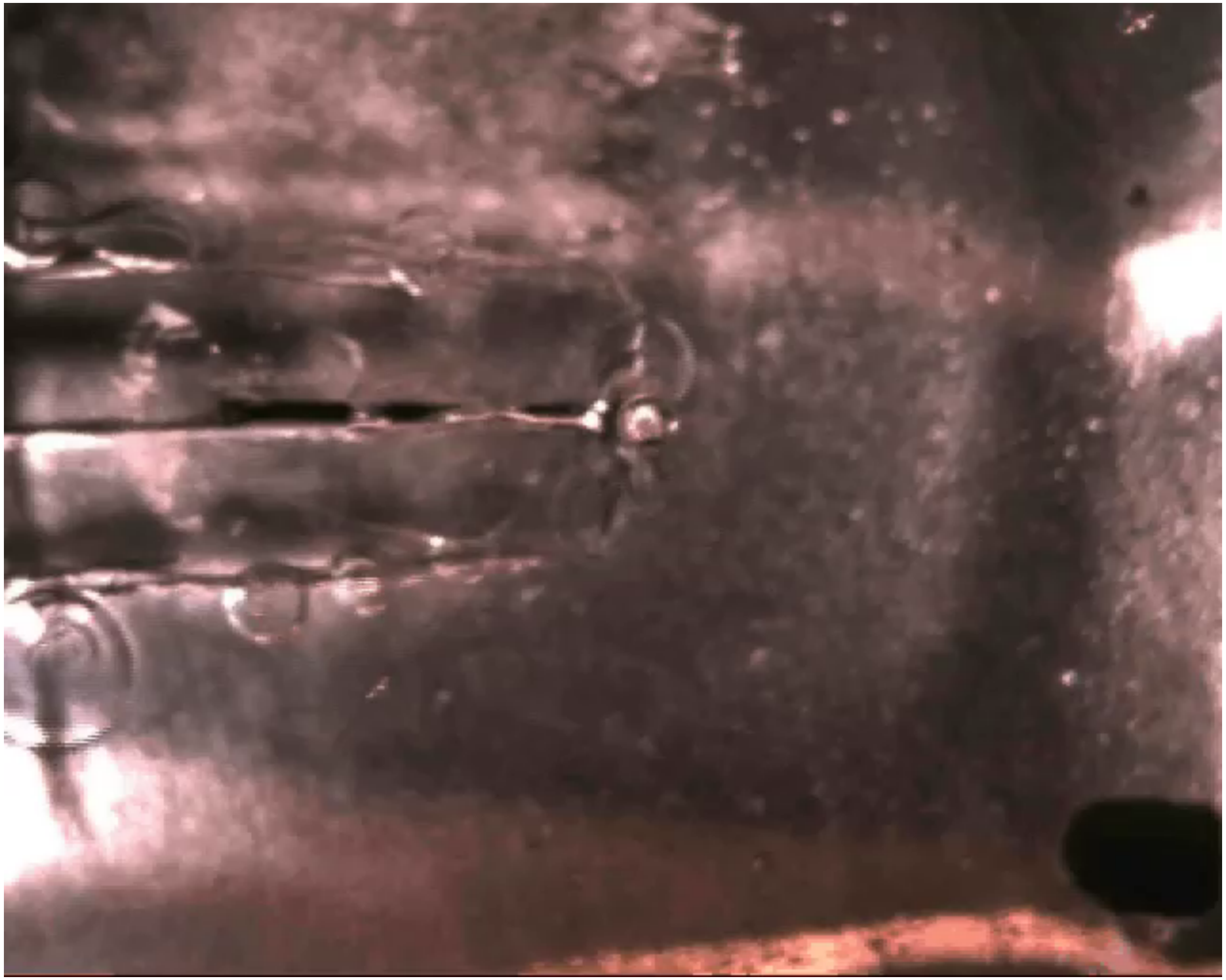


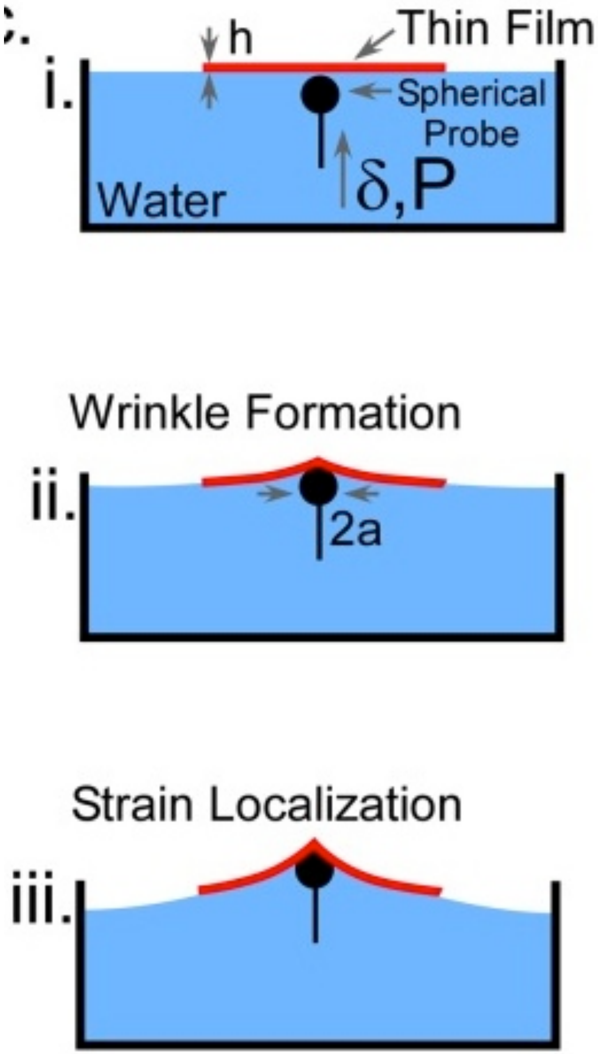
Wrinkling to Fold





Wrinkling & Folding





D.P. **Holmes** and A.J. Crosby, "Draping Films: A Wrinkle to Fold Transition" *Physical Review Letters*, **105**, 038303, (2010).

Wrinkling & Folding

Wrinkling – Point Load

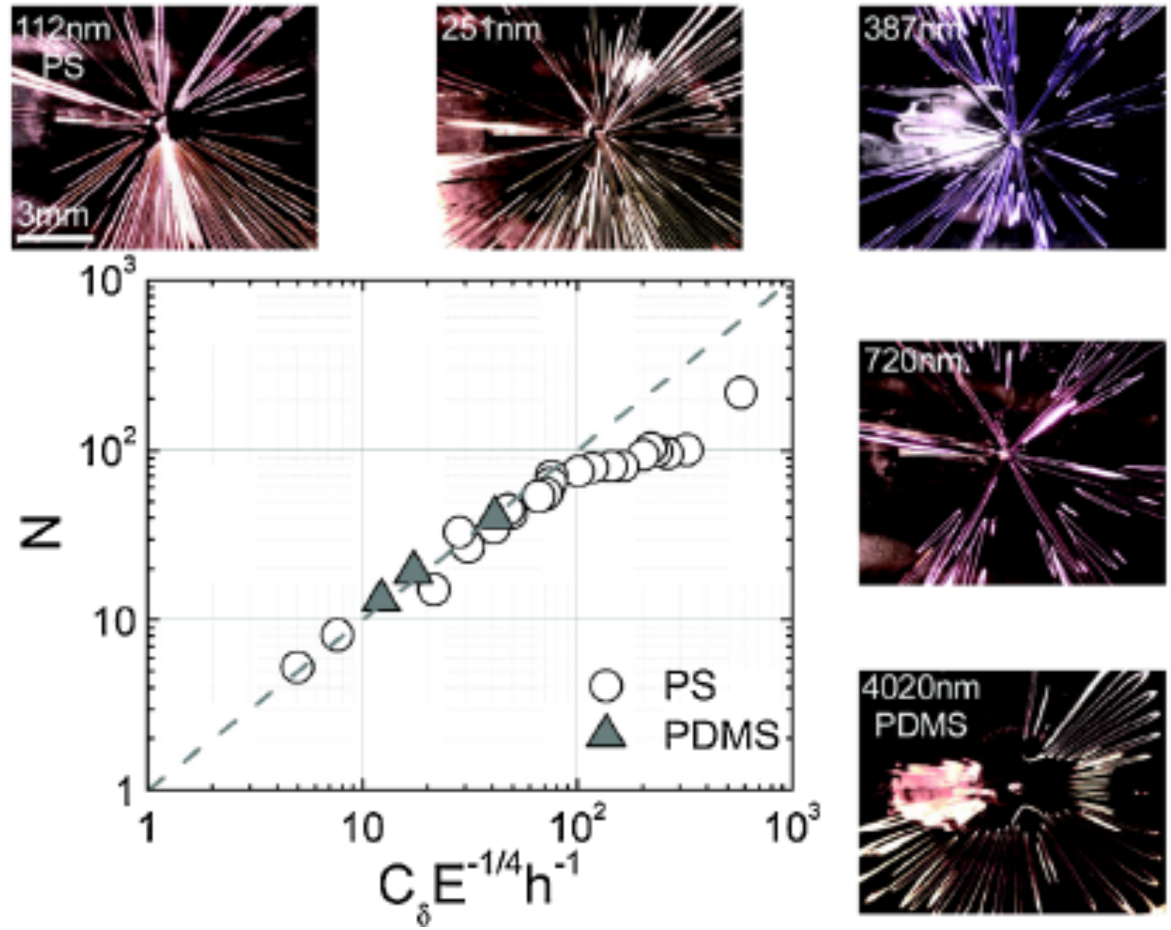
- Stress distribution:

$$\sigma_{rr}^{w} \sim \rho g \delta \left(\frac{L}{h}\right)^2$$

- Resistance of fluid substrate proportional to ρg

- **Number of radial wrinkles:**

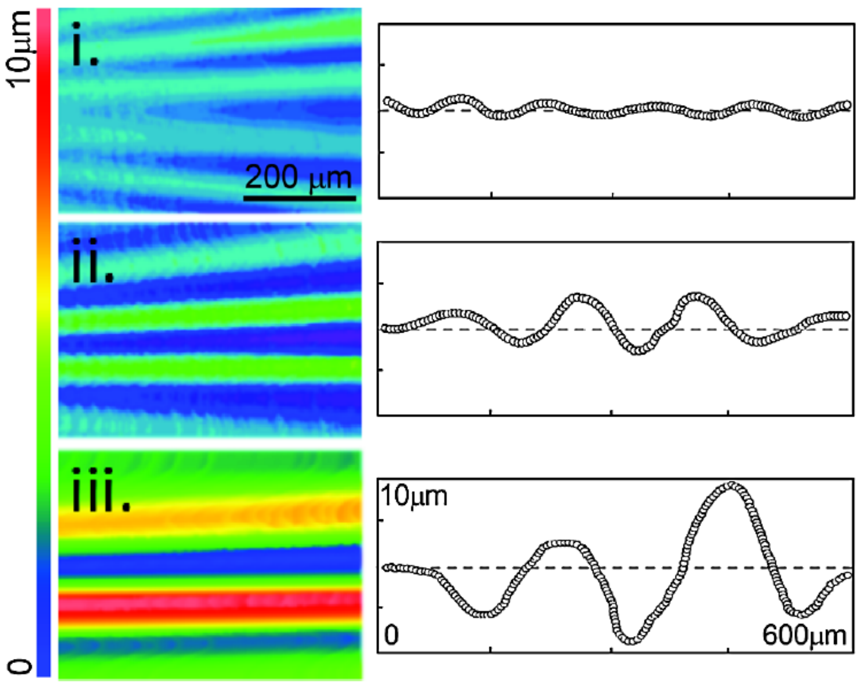
$$N \sim \left(\frac{\sigma_{rr} a^2}{\overline{E} h^2}\right)^{1/4} \sim \frac{1}{h}\left(\frac{\rho g \delta L^2 a^2}{\overline{E}}\right)^{1/4}$$



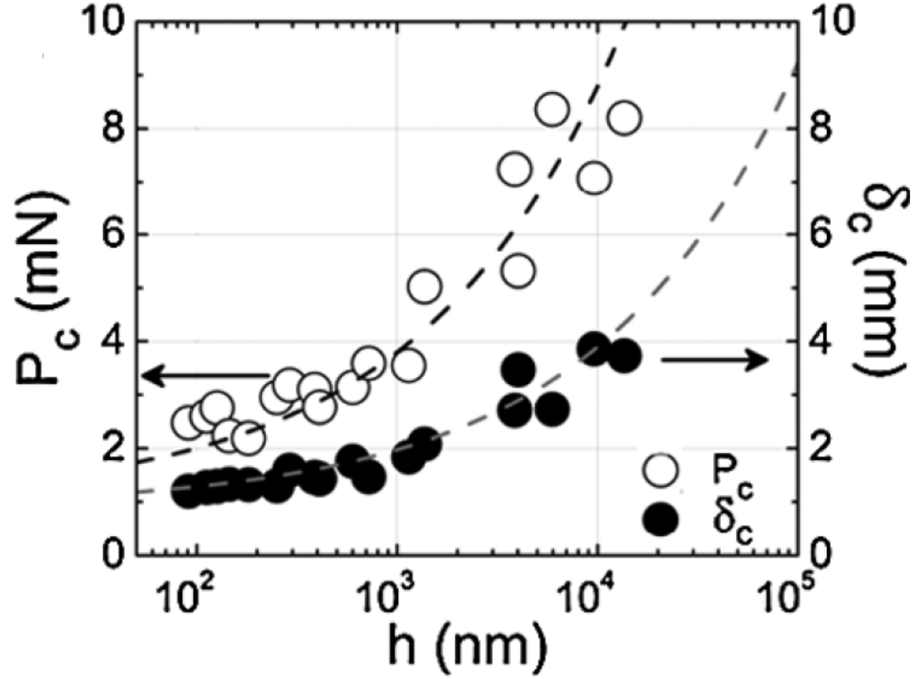
D.P. Holmes and A.J. Crosby, "Draping Films: A Wrinkle to Fold Transition" *Physical Review Letters*, **105**, 038303, (2010).

Wrinkling & Folding

Wrinkling – Point Load



$$\delta_c \sim P_c \sim h^{1/2}$$



Folding at large deformation – **asymptotic isometry**

- Thin elastic film is forced into a curved, nondevelopable shape

D.P. Holmes and A.J. Crosby, "Draping Films: A Wrinkle to Fold Transition" *Physical Review Letters*, **105**, 038303, (2010).
D. Vella et al. "Indentation of ultrathin elastic films and the emergence of aymptotic isometry," arXiv 1410.2795

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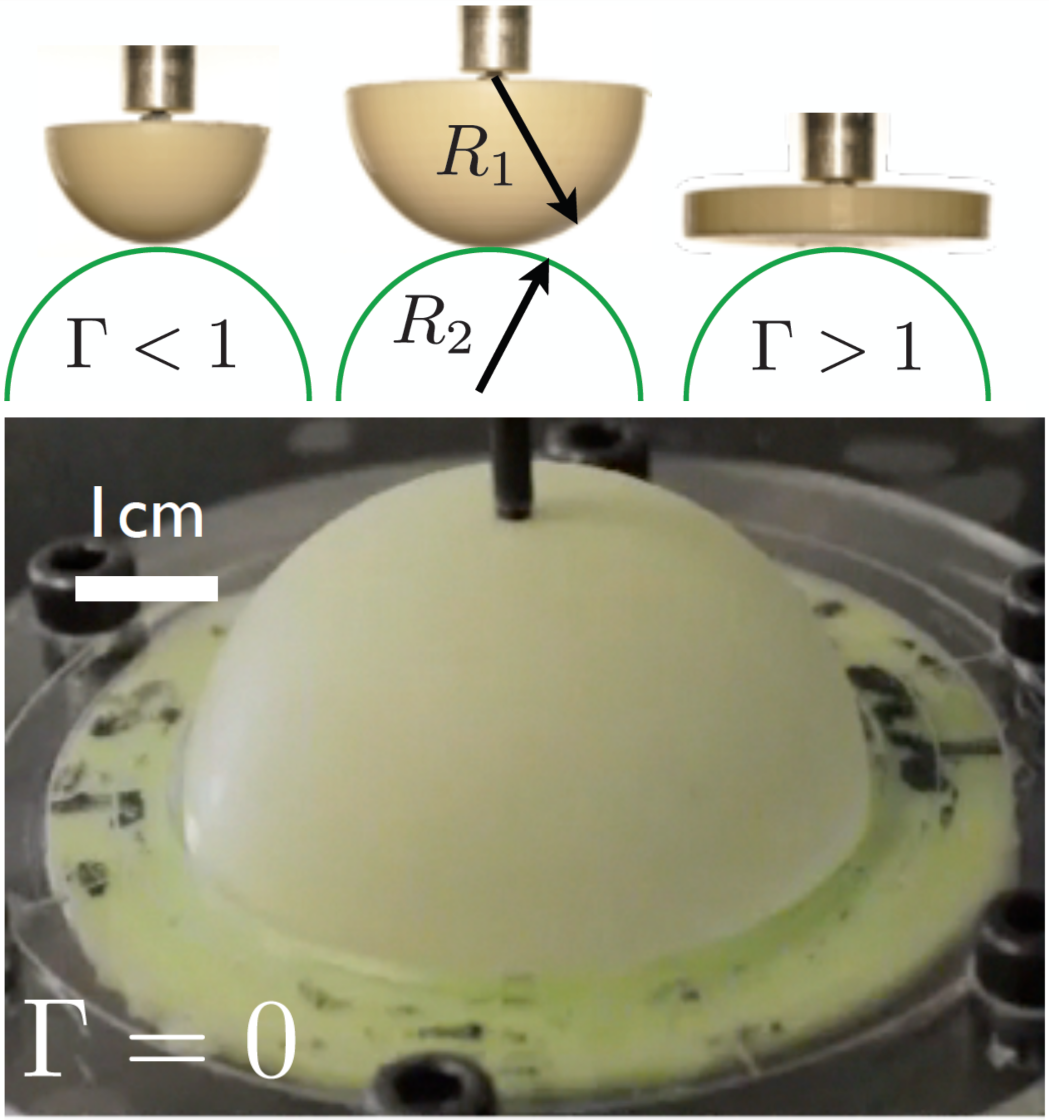
Localization

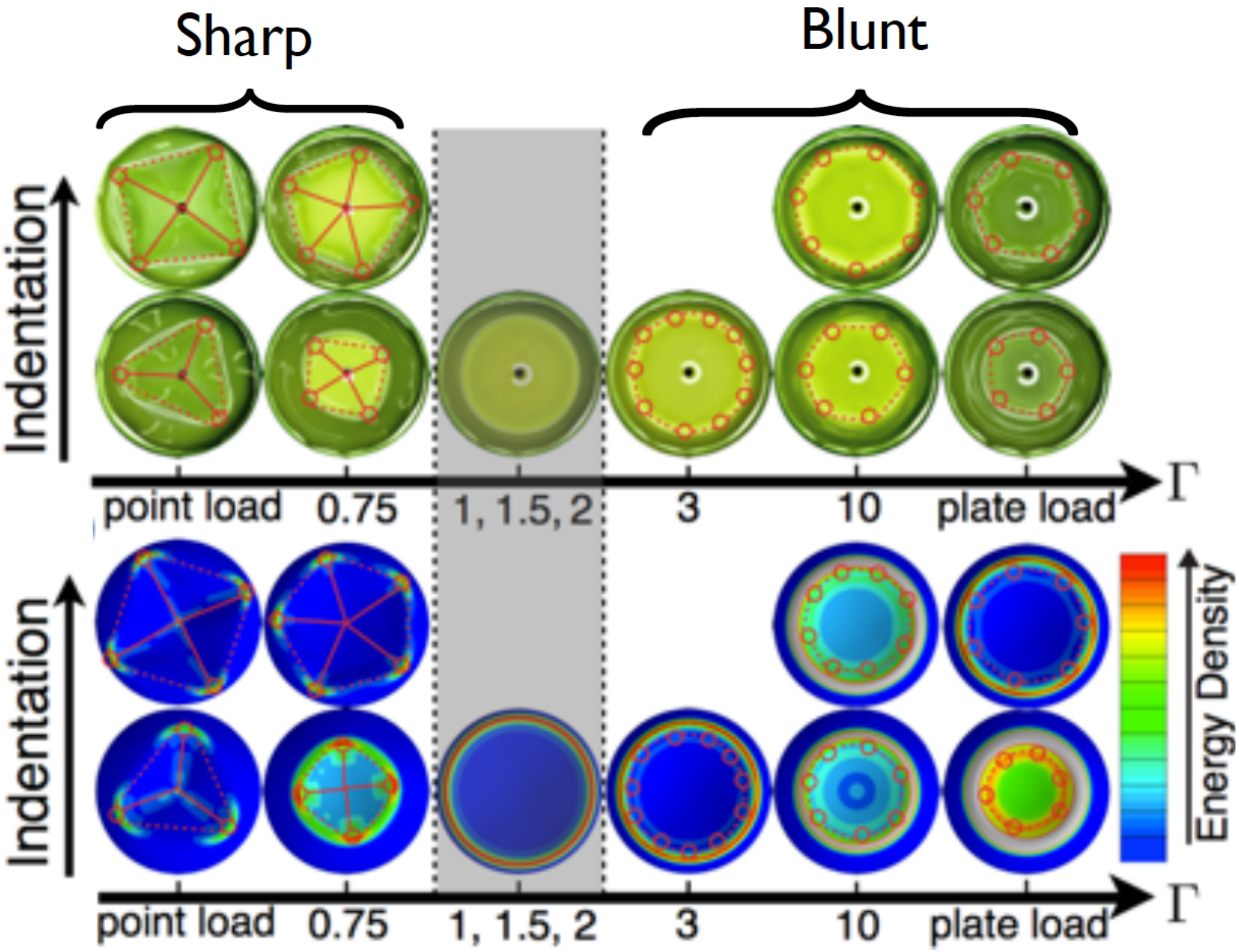


Indenter-shell aspect ratio:

$$\Gamma = \frac{R_1}{R_2}$$

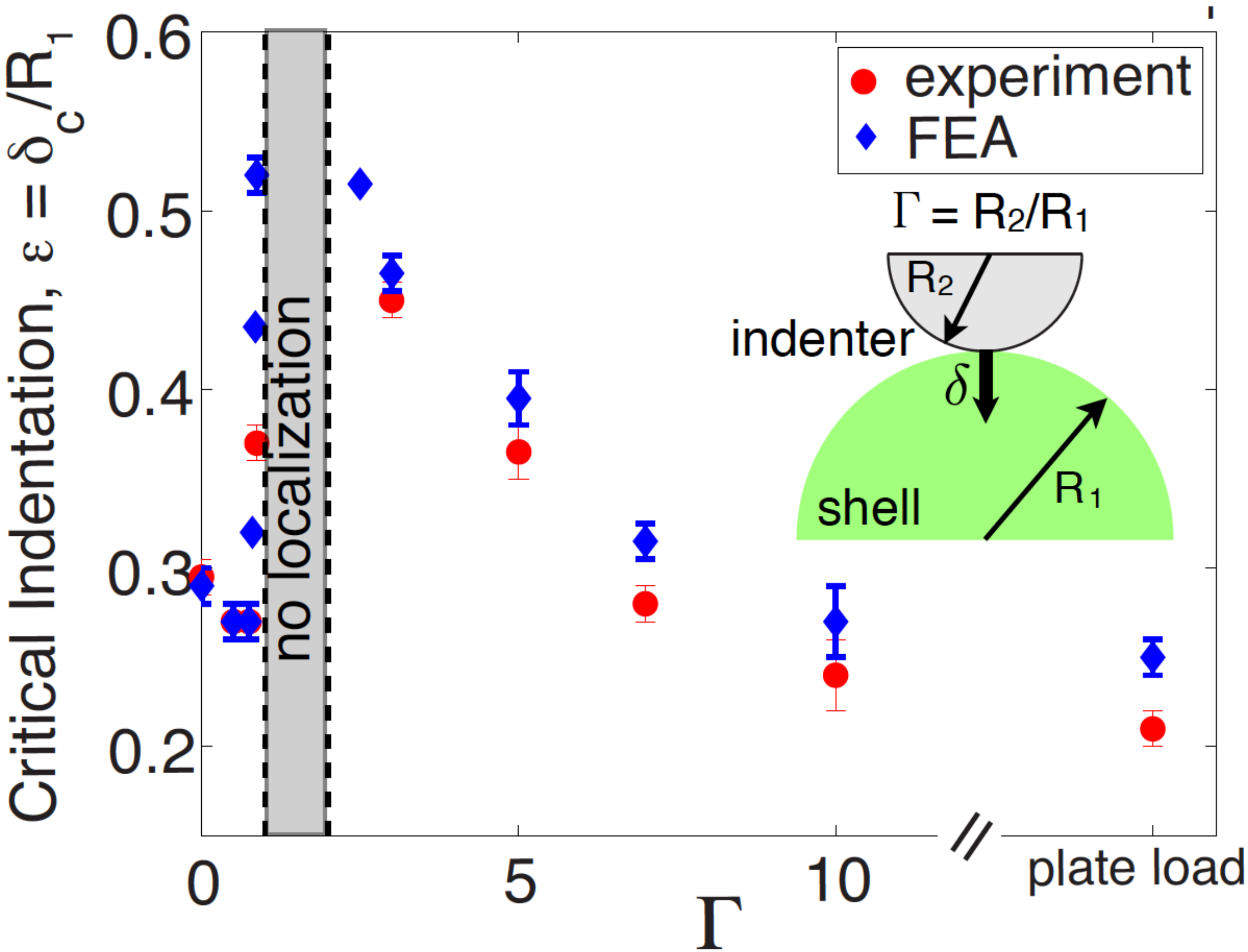
Lazarus, et al. Soft Matter, (2013).

Localization



Lazarus, et al. Soft Matter, (2013).

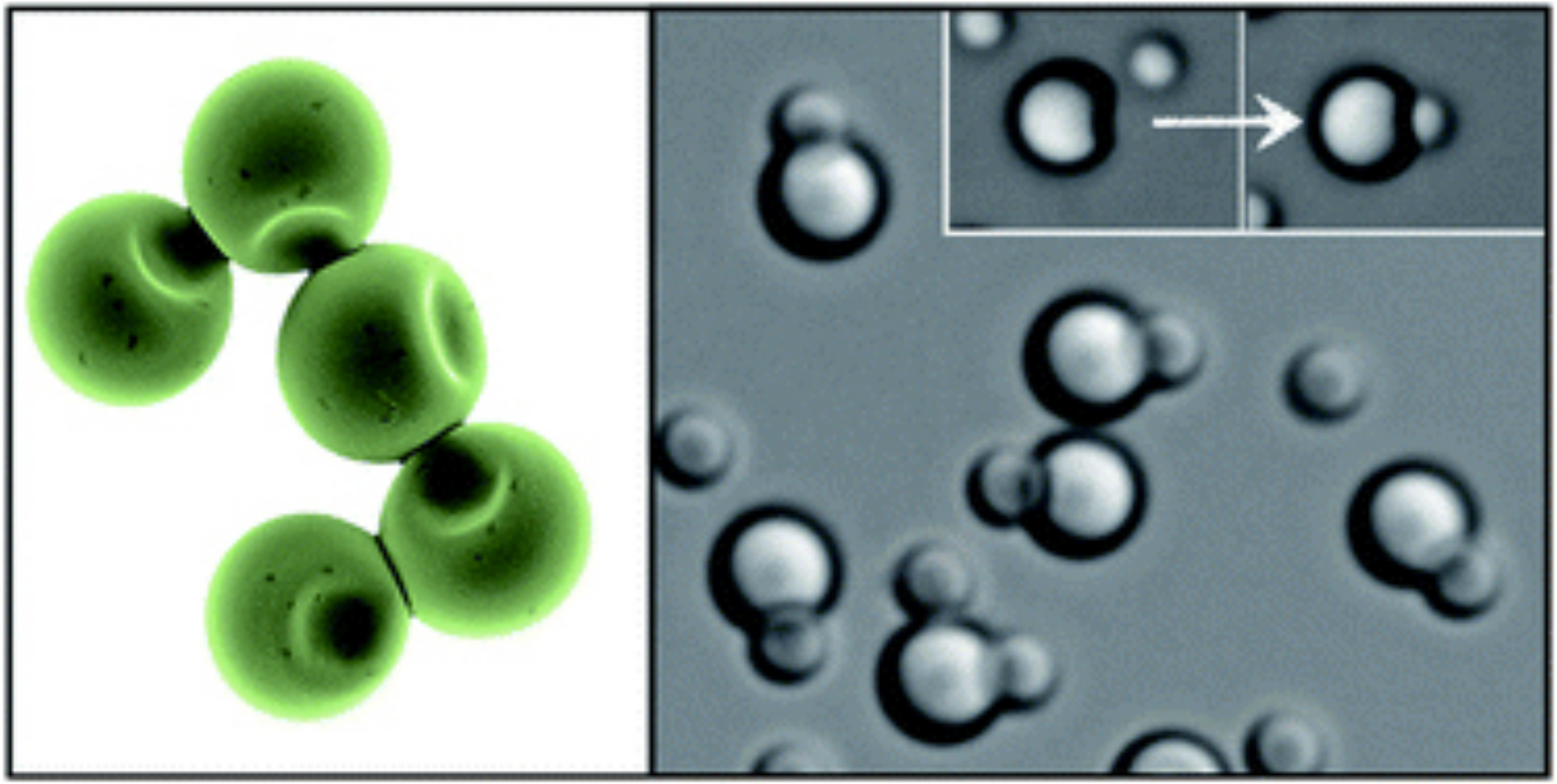
Localization



Lazarus, et al. Soft Matter, (2013).

Localization

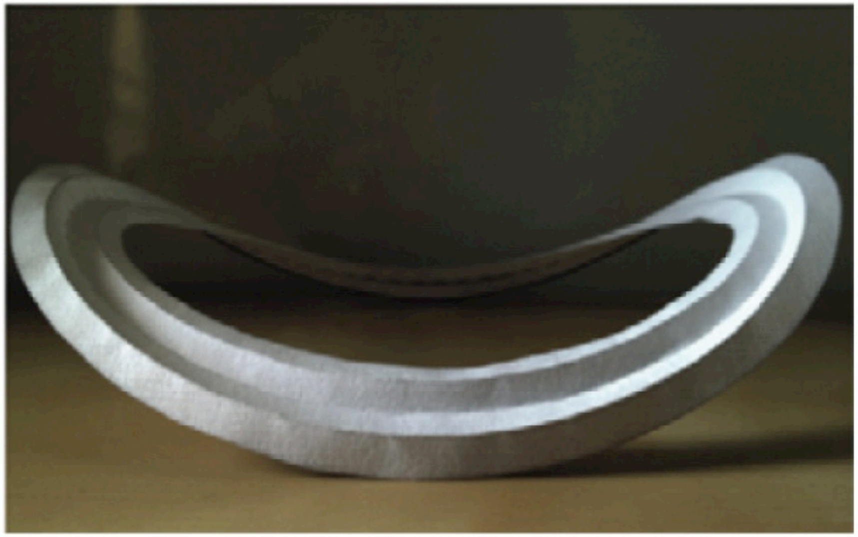
Lock & Key Colloids – Polymerization induced buckling.



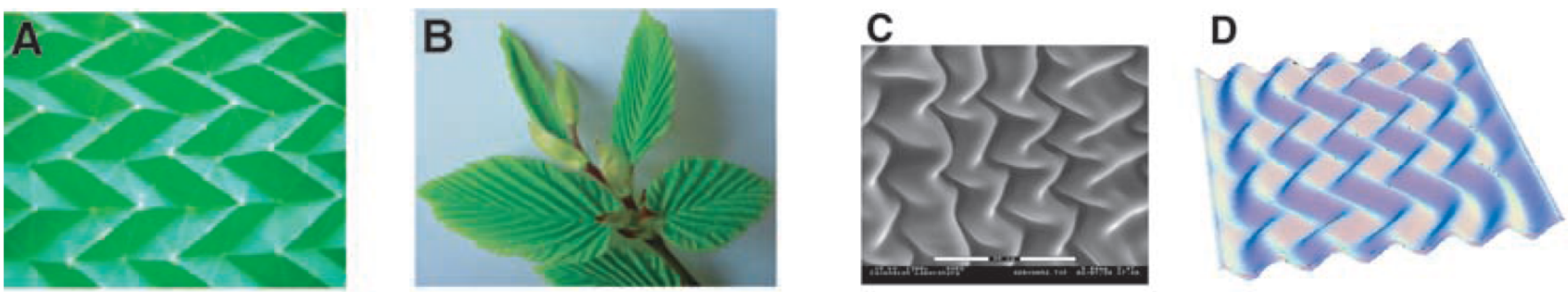
Stefano Sacanna, William T. M. Irvine, Laura Rossi and David J. Pine, Soft Matter, 2011, DOI:10.1039/C0SM01125H

Folding

Curved Crease



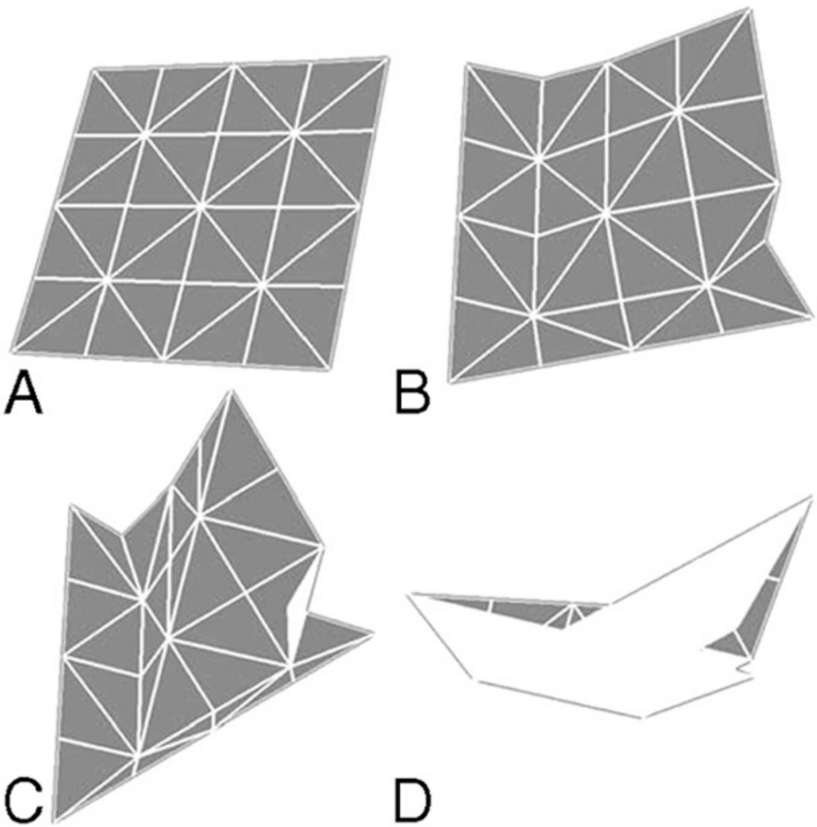
Origami – Miura Ori

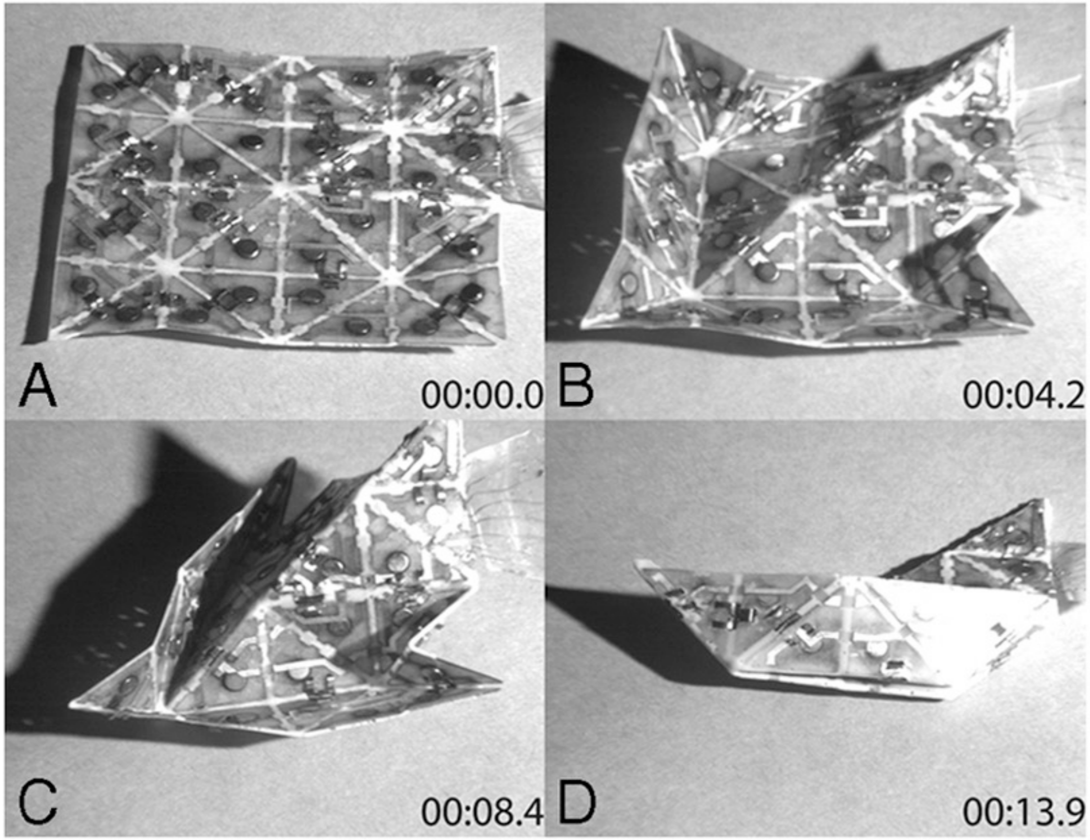


Dias, Marcelo A., and Christian D. Santangelo. "The shape and mechanics of curved-fold origami structures." EPL (Europhysics Letters) 100.5 (2012): 54005.
Mahadevan, Science, 2005.

Folding

Nitinol Shape Memory Alloy

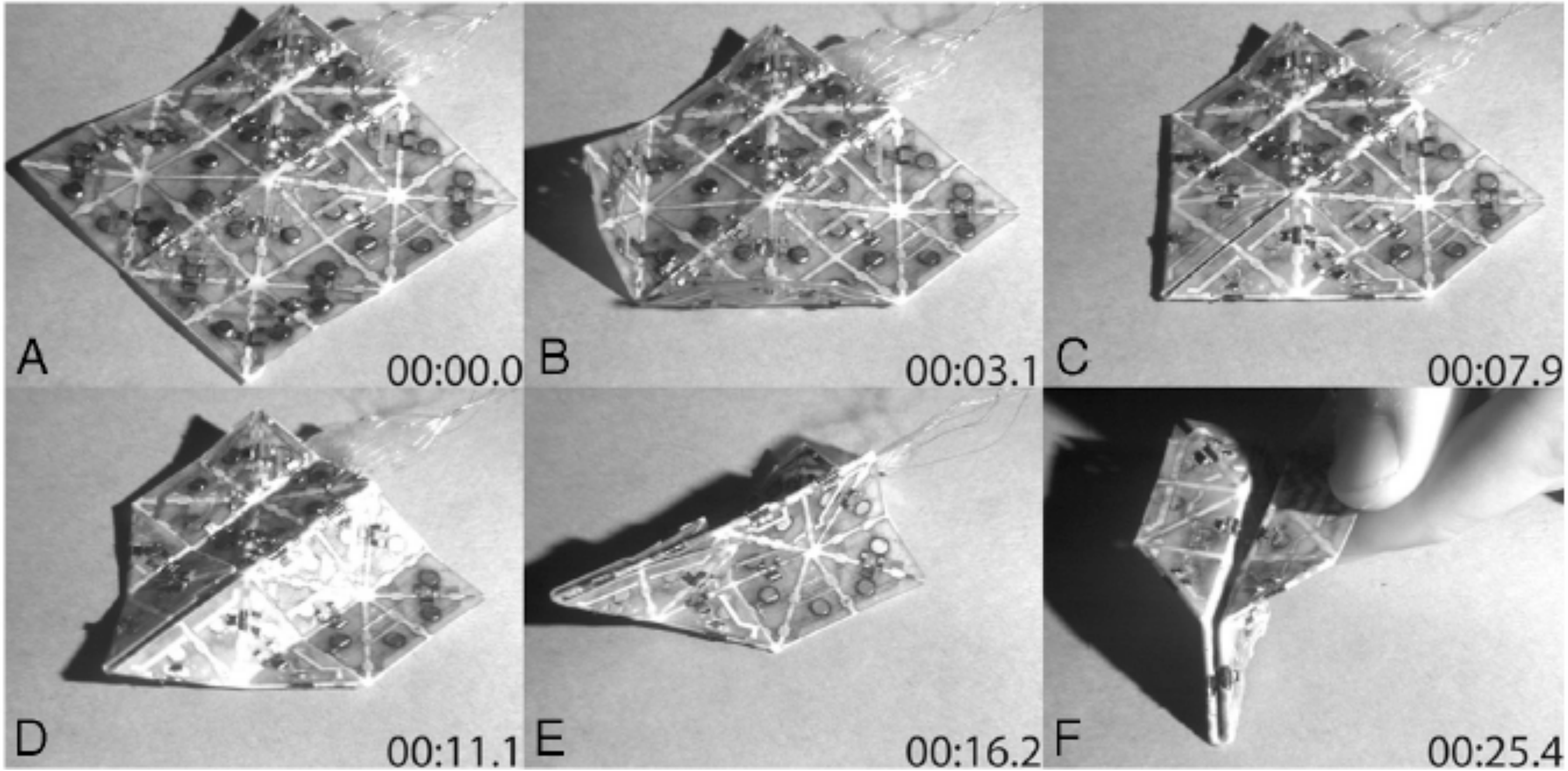




Hawkes et al. "Programmable matter by folding," PNAS, 107(28), 12441-12445, (2010).

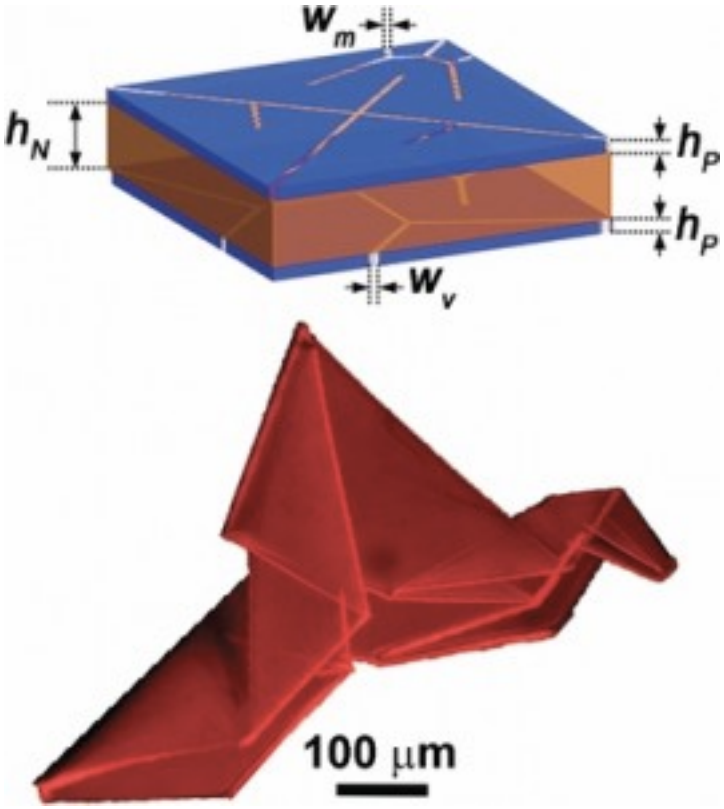
Folding

Nitinol Shape Memory Alloy

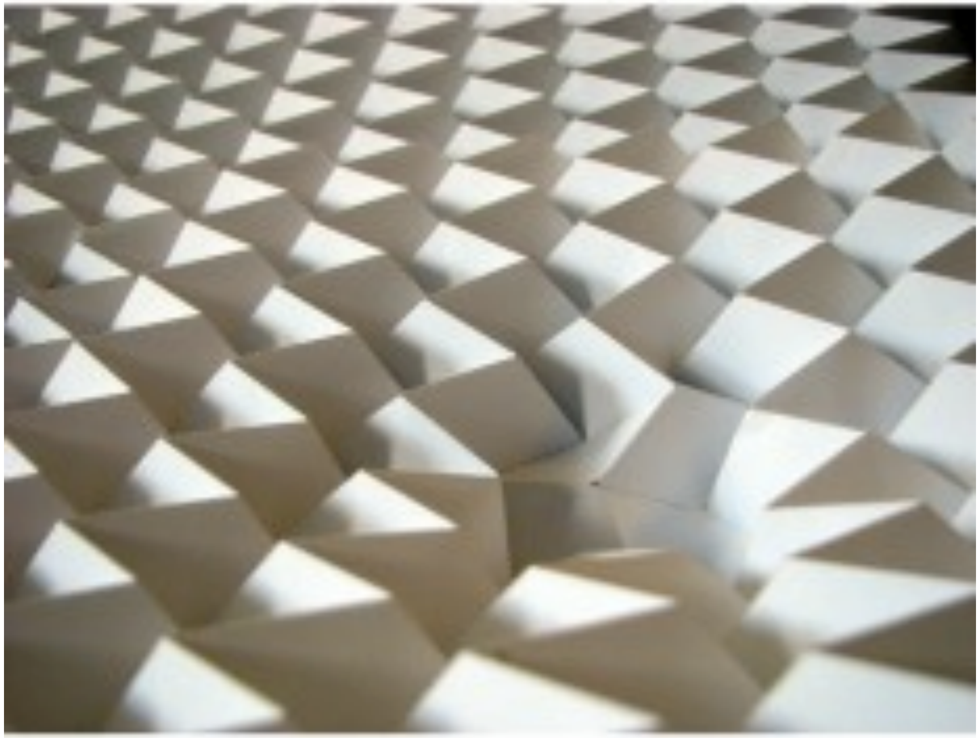


Hawkes et al. “Programmable matter by folding,” PNAS, 107(28), 12441-12445, (2010).

Self-folding Origami Bird

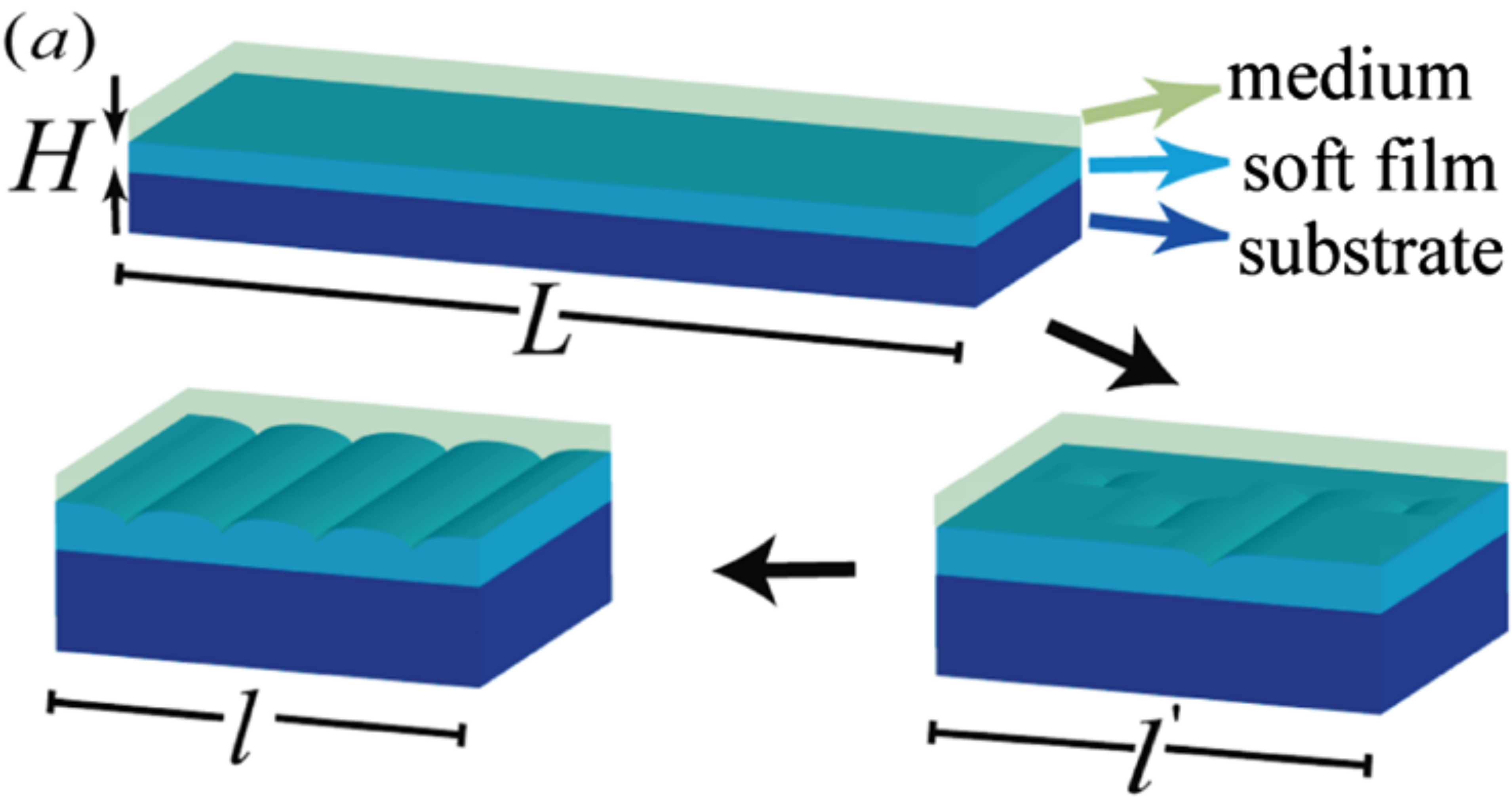


Origami & Mechanical Metamaterials



Na, Jun-Hee, et al. "Programming Reversibly Self-Folding Origami with Micropatterned Photo-Crosslinkable Polymer Trilayers." Advanced Materials 27.1 (2015): 79-85.
Silverberg, Jesse L., et al. "Using origami design principles to fold reprogrammable mechanical metamaterials." science 345.6197 (2014): 647-650.

Creasing

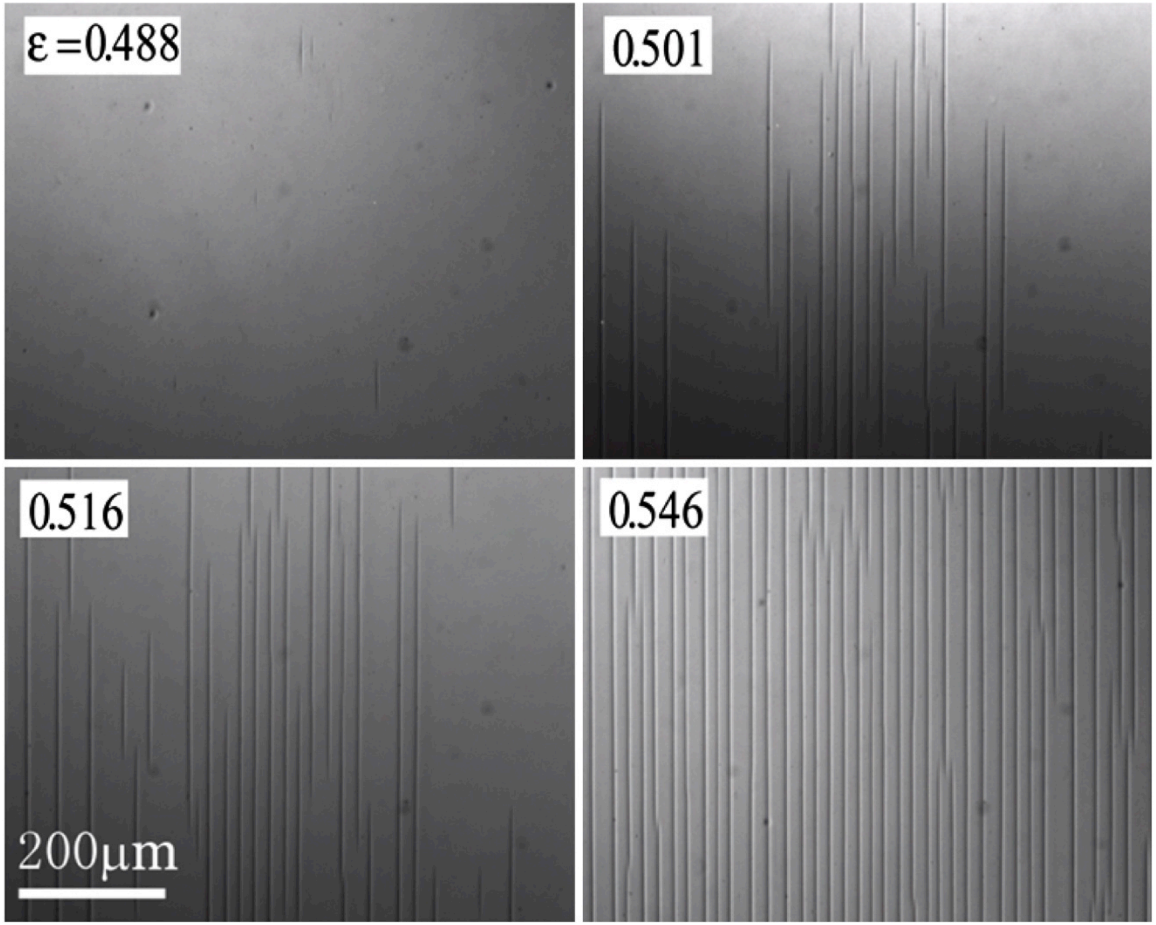


D. Chen et al. "Surface Energy as a Barrier to Creasing of Elastomer Films: An Elastic Analog to Classical Nucleation," PRL **109**, 038001, (2012).

Creasing

Crease vs. Wrinkle

- Fundamentally different type of instability.
- Both are **bifurcations** from a state of homogenous compression.
- Wrinkles bifurcate by a field of **strain small in amplitude**, and **nonlocal in space**.
- Creases bifurcate by a field of **strain large in amplitude** and **localized in space.**
- Wrinkles are predicted theoretically, they are **preceded by creases**.



Wrinkles form by a **linear perturbation**, creases form by **nucleation and growth.**

D. Chen et al. "Surface Energy as a Barrier to Creasing of Elastomer Films: An Elastic Analog to Classical Nucleation," PRL **109**, 038001, (2012).

Creasing

Crease vs. Wrinkle

- Nucleation and growth of creases are analogous to classical nucleation theory for a **thermodynamic phase transition**.
- Forming a crease reduces the elastic energy, yet increases the surface area – barrier to nucleation.
- Absence of surface energy, crease forms at fixed strain:

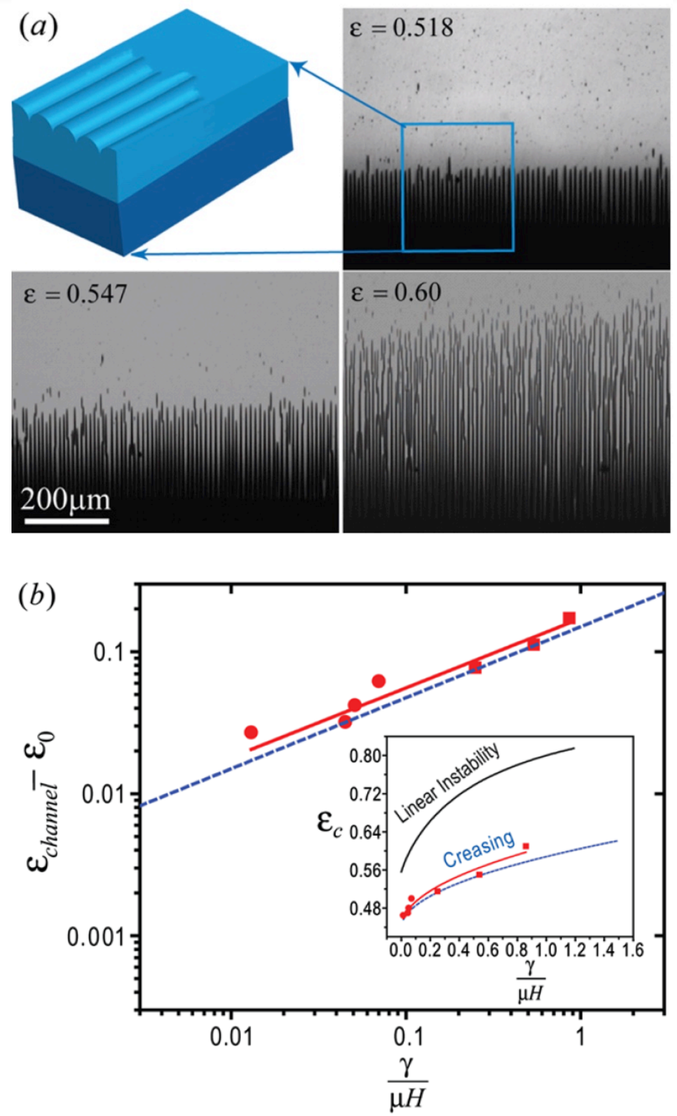
$$\varepsilon_0 = 0.438$$

- Nucleation size:

$$a_{\text{nuc}} \approx \frac{1}{2}\frac{A}{B}\frac{\gamma}{\mu(\varepsilon - \varepsilon_0)}$$

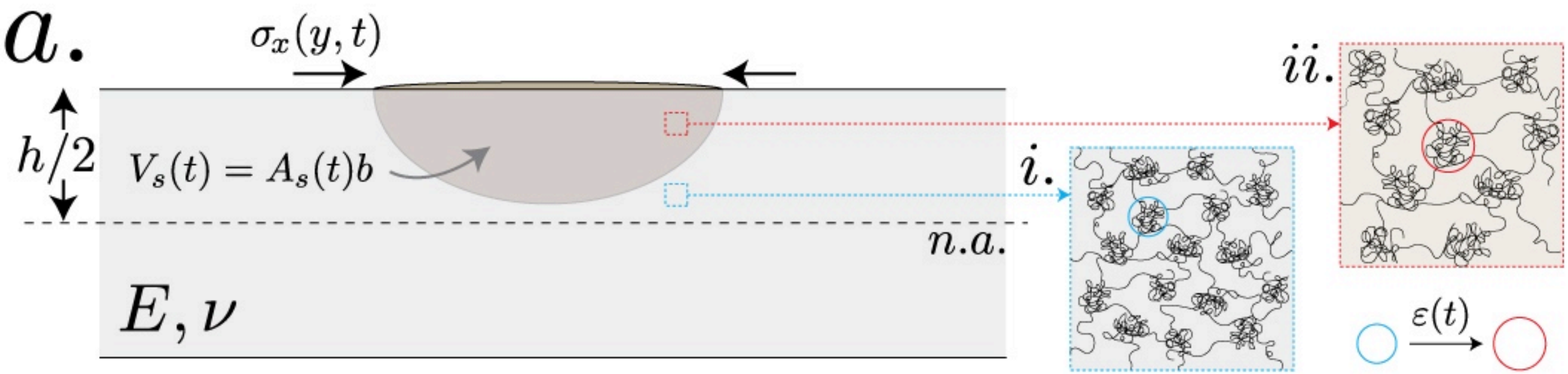
- Overstrain:

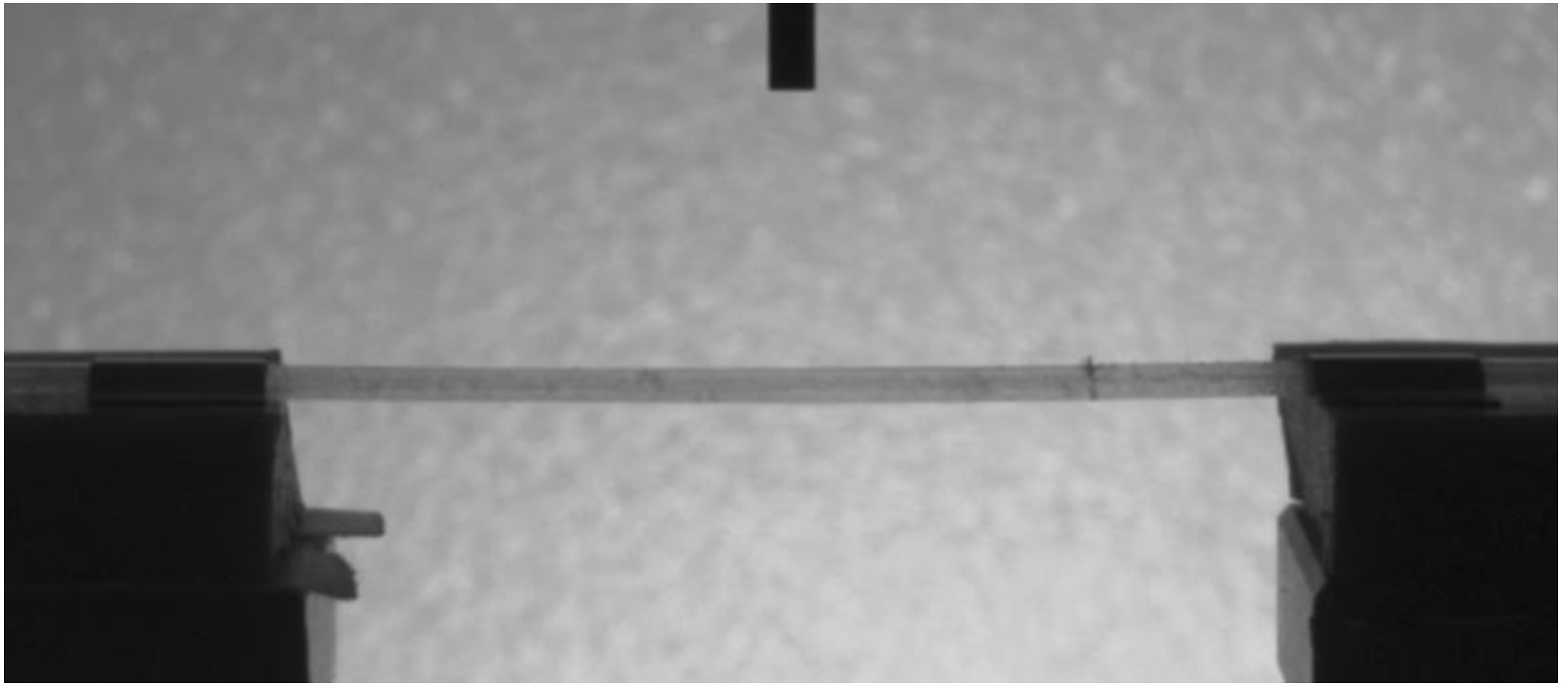
$$\varepsilon - \varepsilon_0 \sim \underbrace{\left(\frac{\gamma}{\mu H}\right)^{1/2}}_{\text{elastocapillary}}$$



D. Chen et al. "Surface Energy as a Barrier to Creasing of Elastomer Films: An Elastic Analog to Classical Nucleation," PRL **109**, 038001, (2012).

Creasing

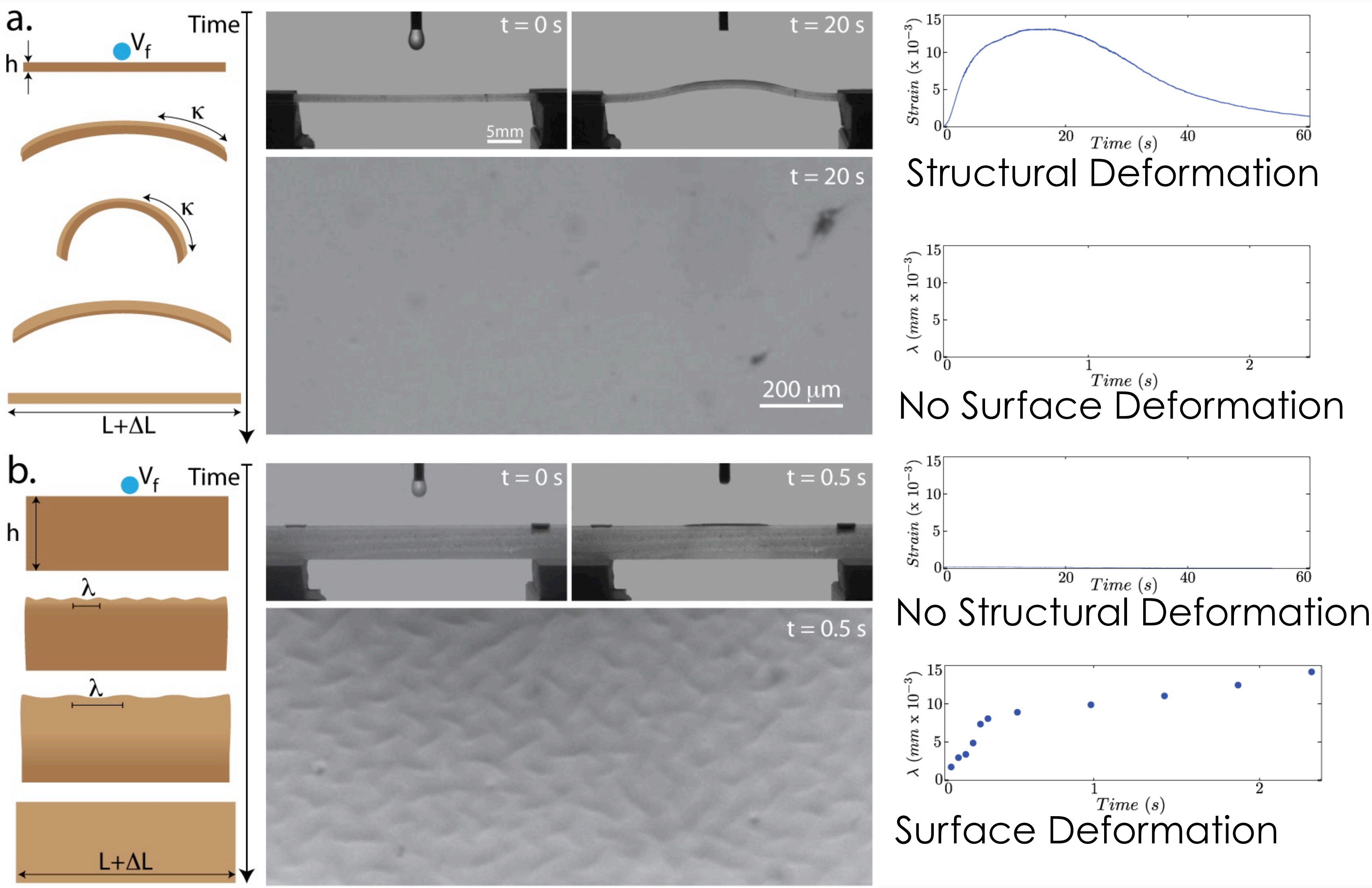




A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

BOSTON UNIVERSITY

Creasing



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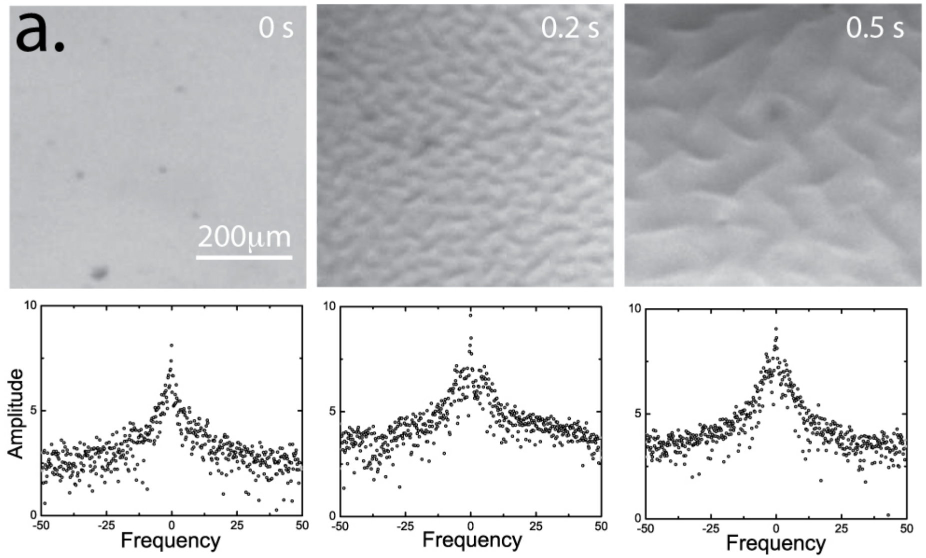
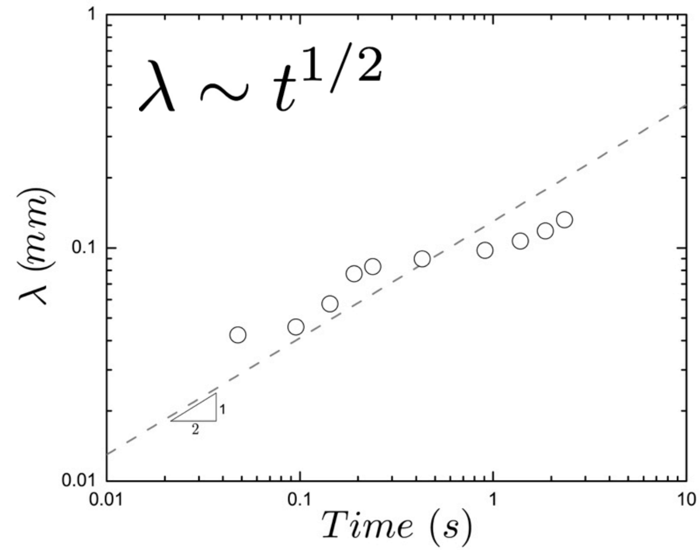
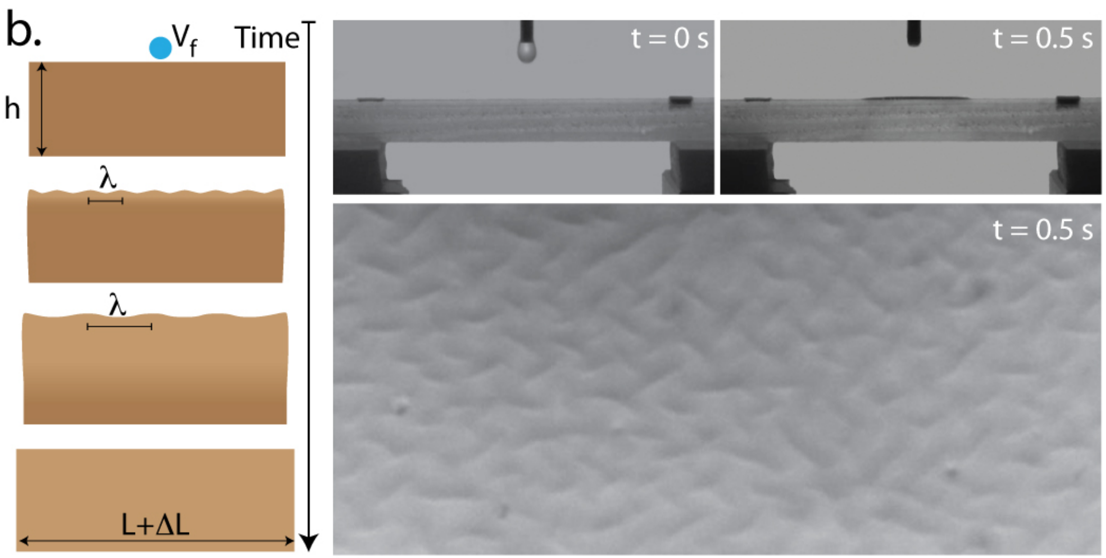
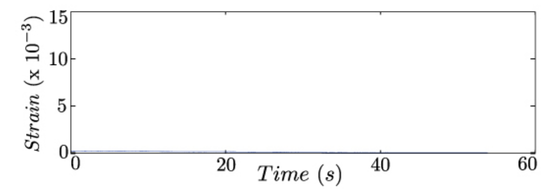
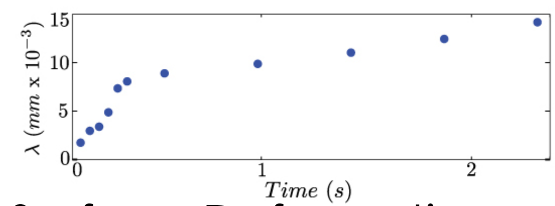








No Structural Deformation



Surface Deformation

A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

H. Tanaka, H. Tomita, A. Takasu, T. Hayashi, and T. Nishi. *Physical Review Letters*, **68**, 18, 1992.

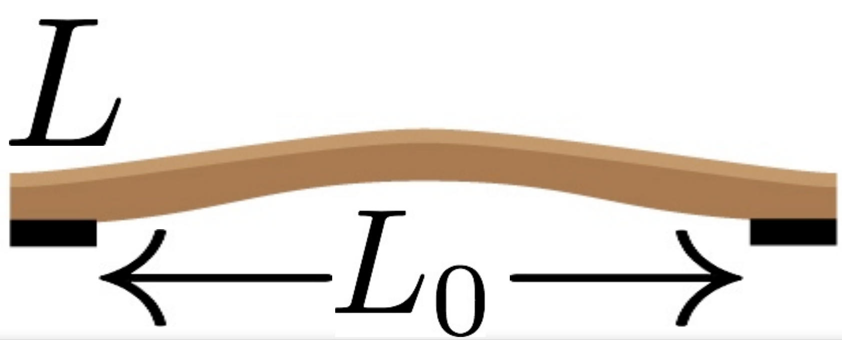
Creasing

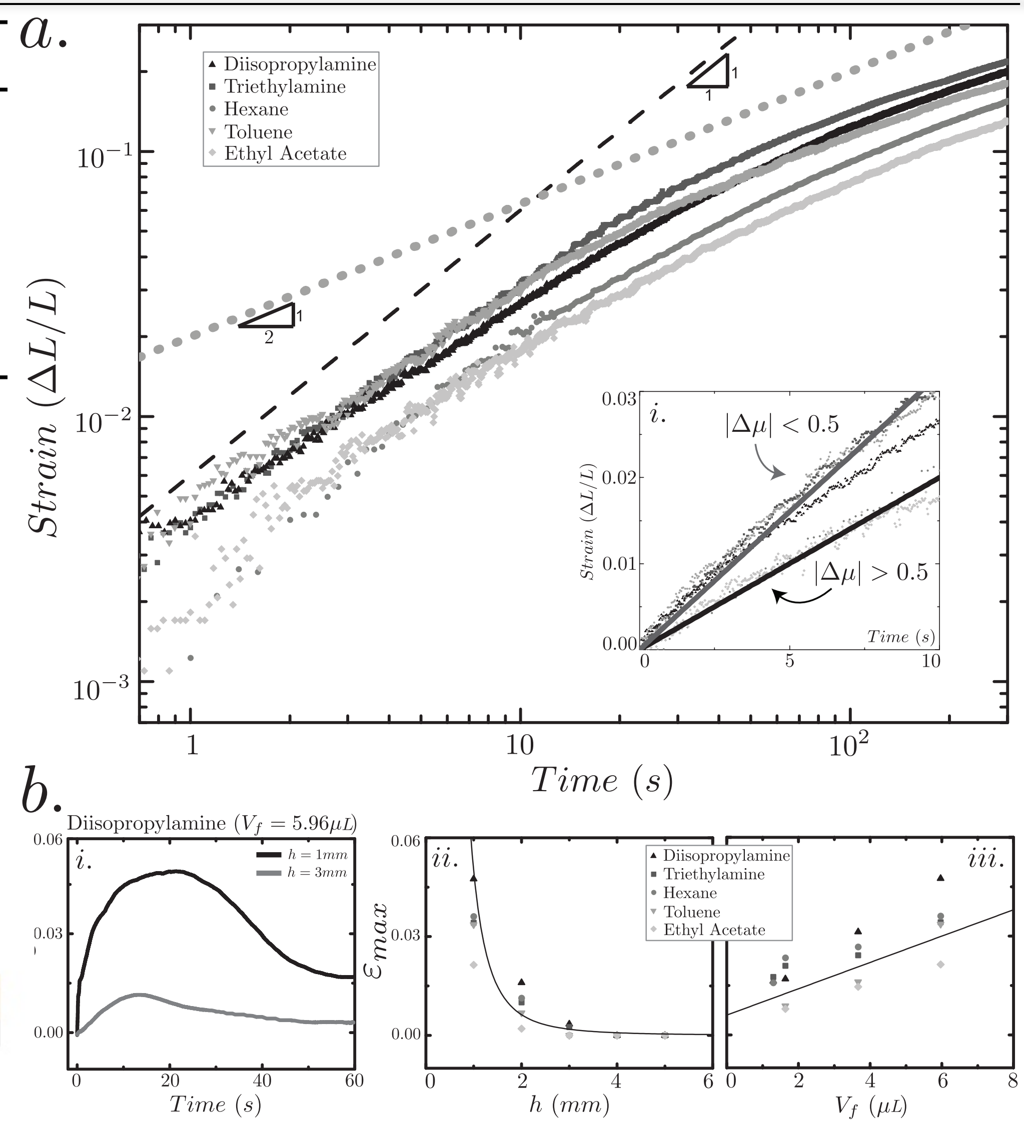
Material	δ_s ($cal^{1/2}cm^{-3/2}$)	μ (D)	ε_{eq}
PDMS	7.3	0.6-0.9	–
Diisopropylamine	7.3	1.2	**1.13**
Triethylamine	7.5	0.7	**0.58**
Hexanes	7.3	0.0	**0.35**
Toluene	8.9	0.4	**0.31**
Ethyl acetate	9.0	1.8	**0.18**

$$\delta_s = \text{Solubility parameter}$$

$$\mu = \text{Solvent polarity}$$

$$\varepsilon_{eq} = \text{Strain at equilibrium}$$





A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Creasing

Can the fluid bend the structure?

Bending

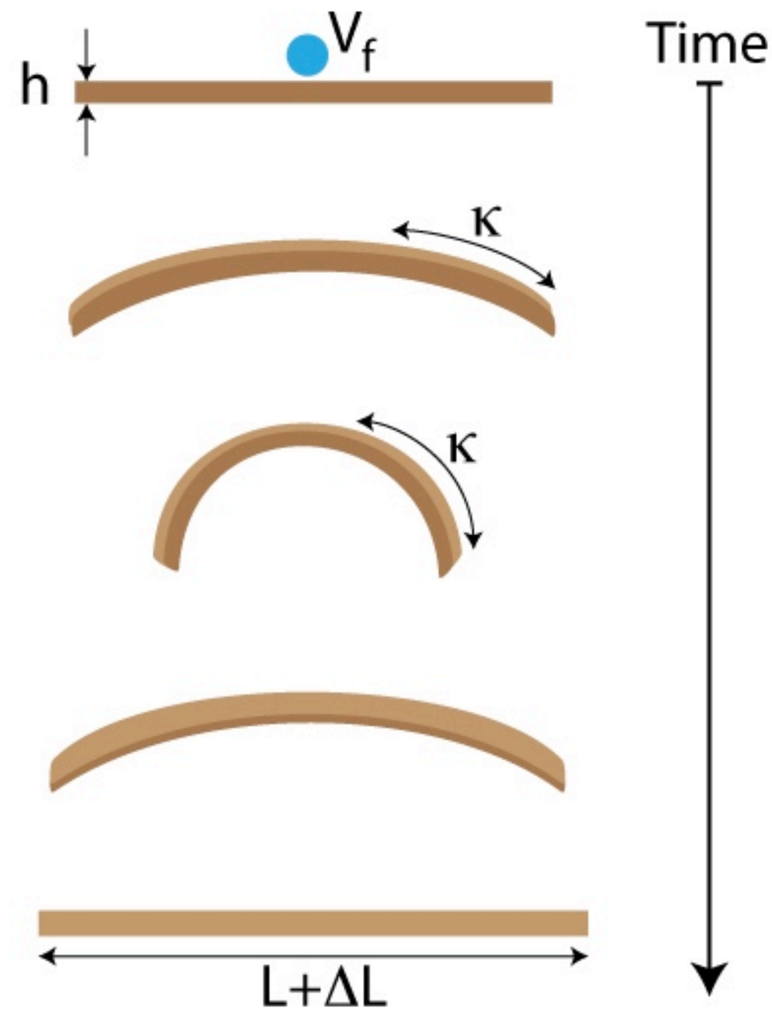
$$\mathcal{U}_b = \frac{B}{2}\int_L \theta'(s)^2 \mathrm{d}s \sim \overline{E}h^3$$

Swelling

$$\mathcal{U}_s = \int_{V_f} \sigma\varepsilon_{eq}\, \mathrm{d}V_f \sim E\varepsilon_{eq}^2 V_f$$

Length scale:

$$\ell_{es} \sim \left(\varepsilon_{eq}^2 V_f\right)^{1/3}$$



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Creasing

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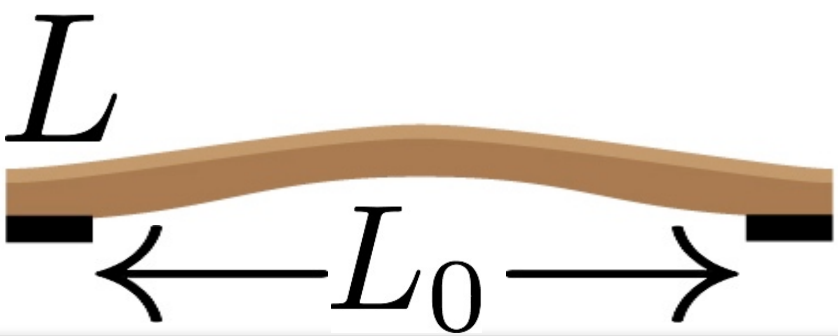
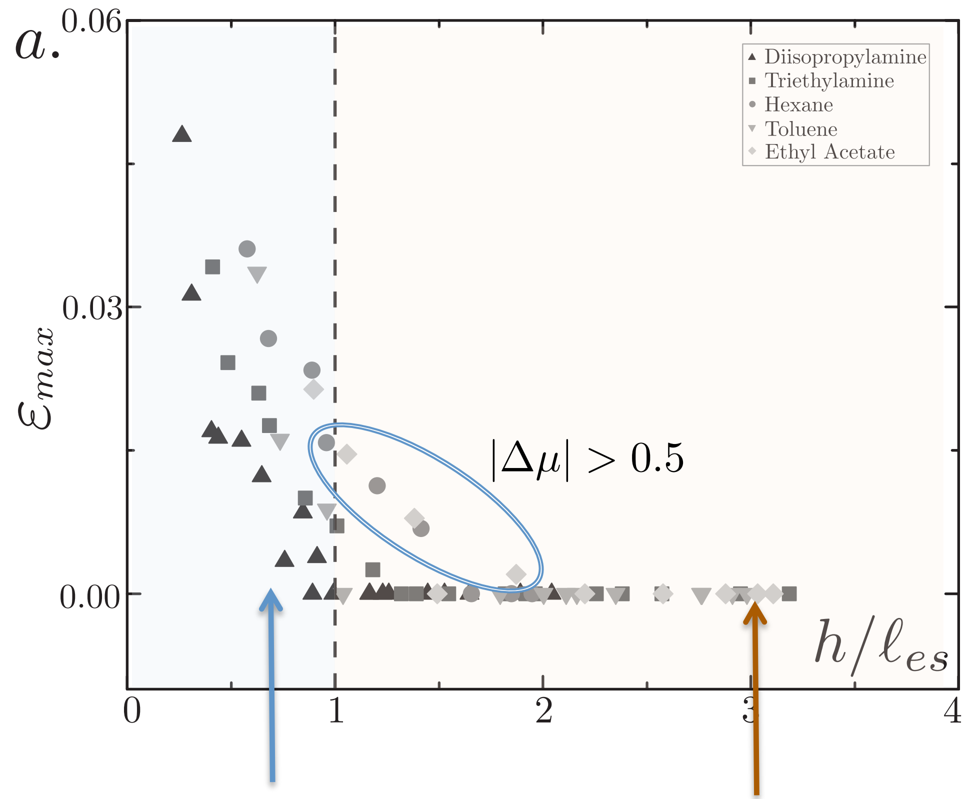
$$\ell_{es} \sim \left(\varepsilon_{eq}^2 V_f\right)^{1/3}$$



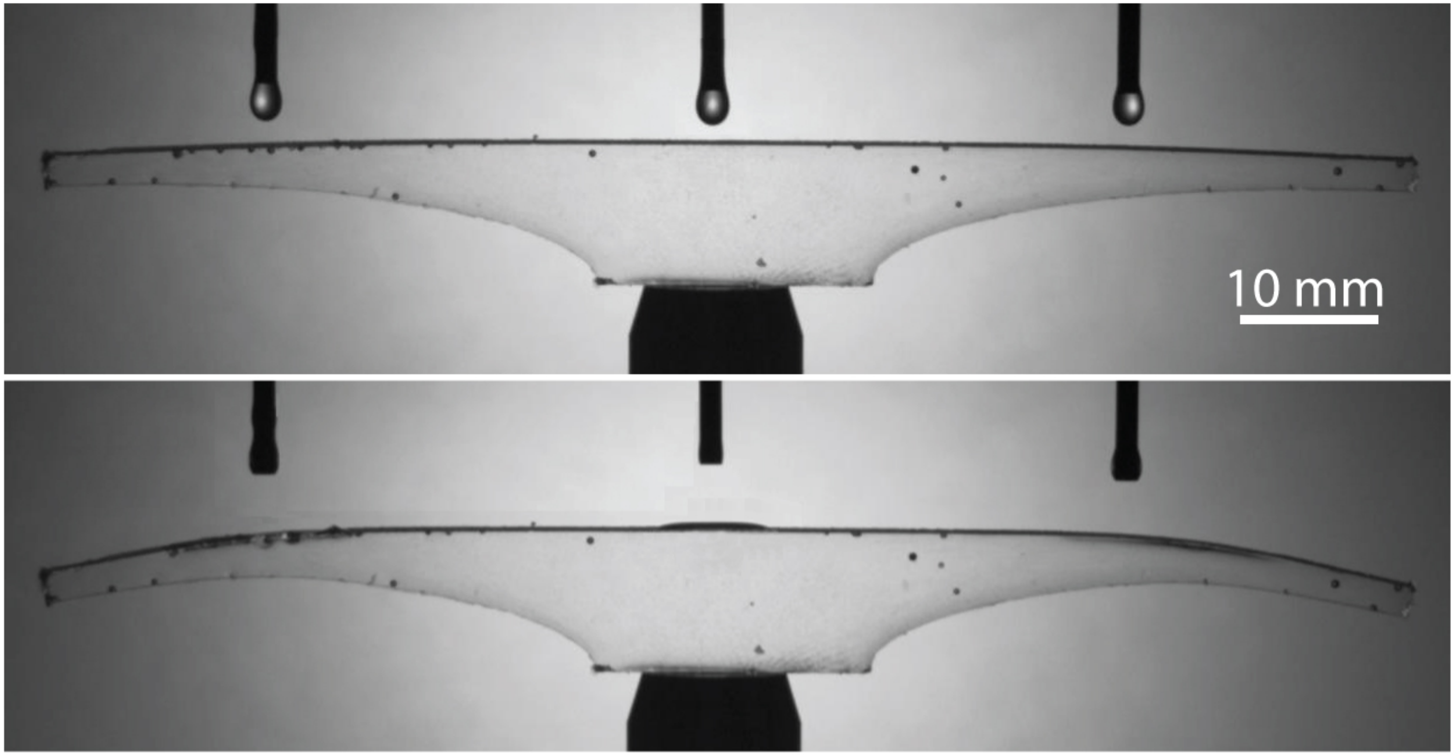


Structural Deformation (Bending/Buckling)

Surface Deformation (Creasing)

A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Creasing



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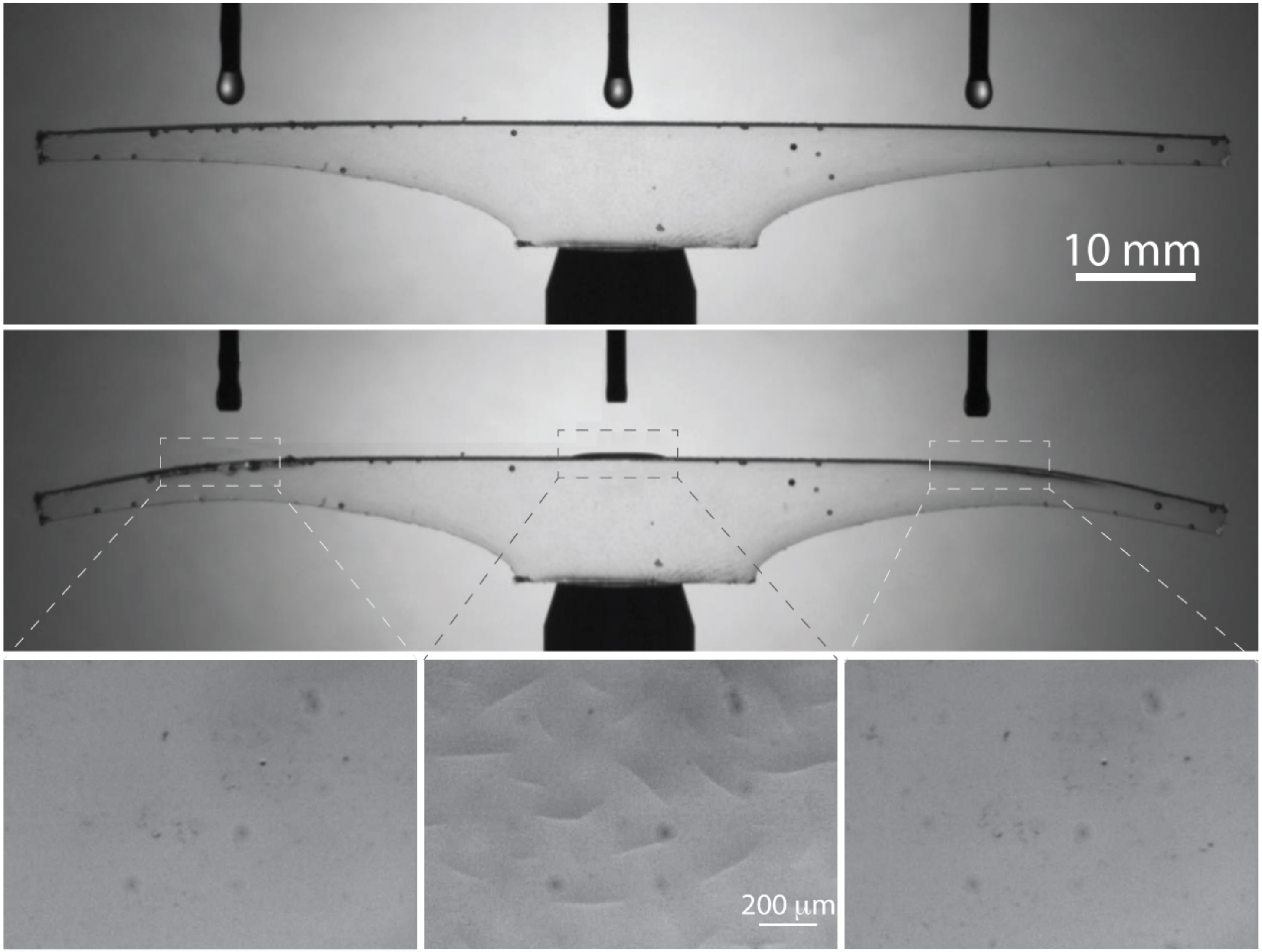
Creasing



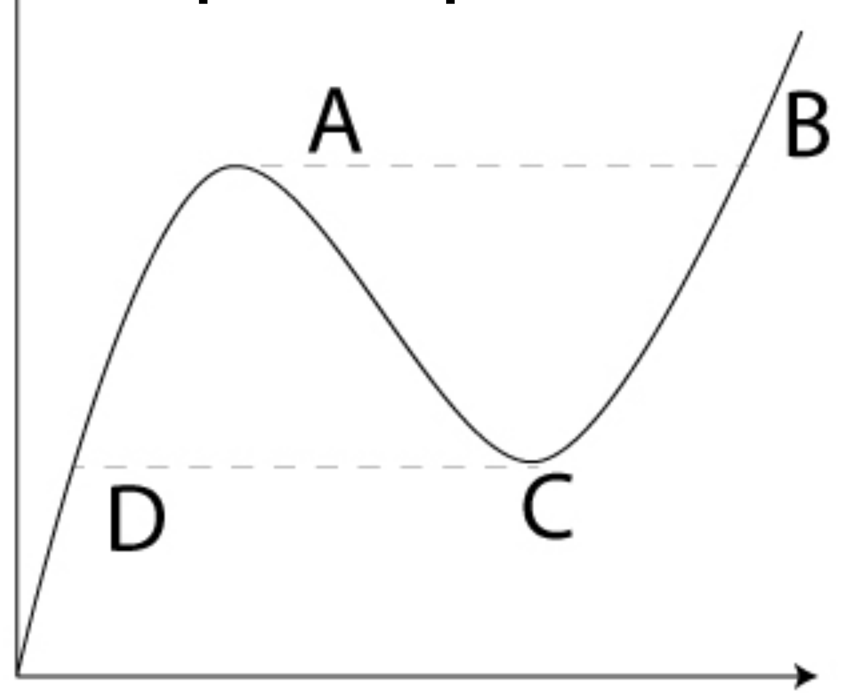
A. Pandey and D.P. Holmes. "Swelling-Induced Deformations: A Materials-Defined Transition from Structural Instability to Surface Instability," Soft Matter, **9**, 5524, (2013).

Snapping

Snappy Functionality

Limit Point Instability

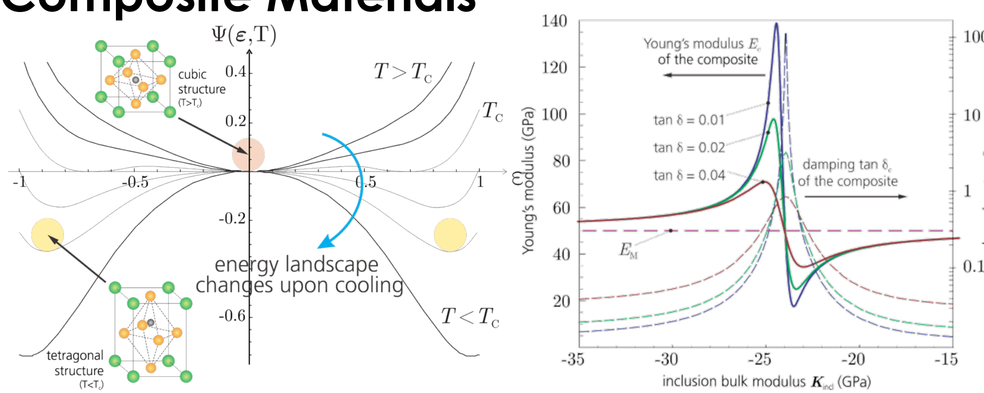
Rapid Jump From A to B

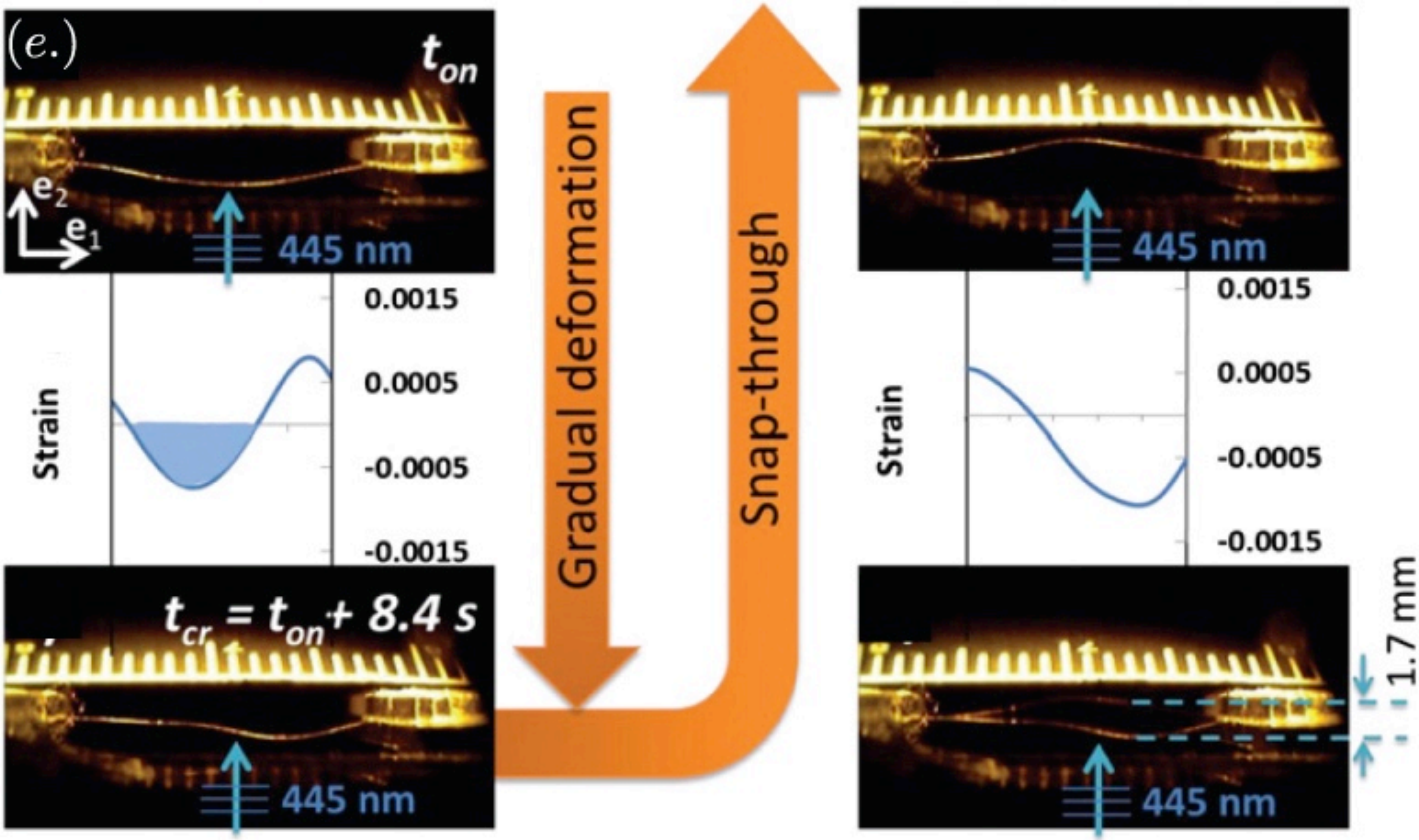


Venus Flytrap

Dielectric Elastomer

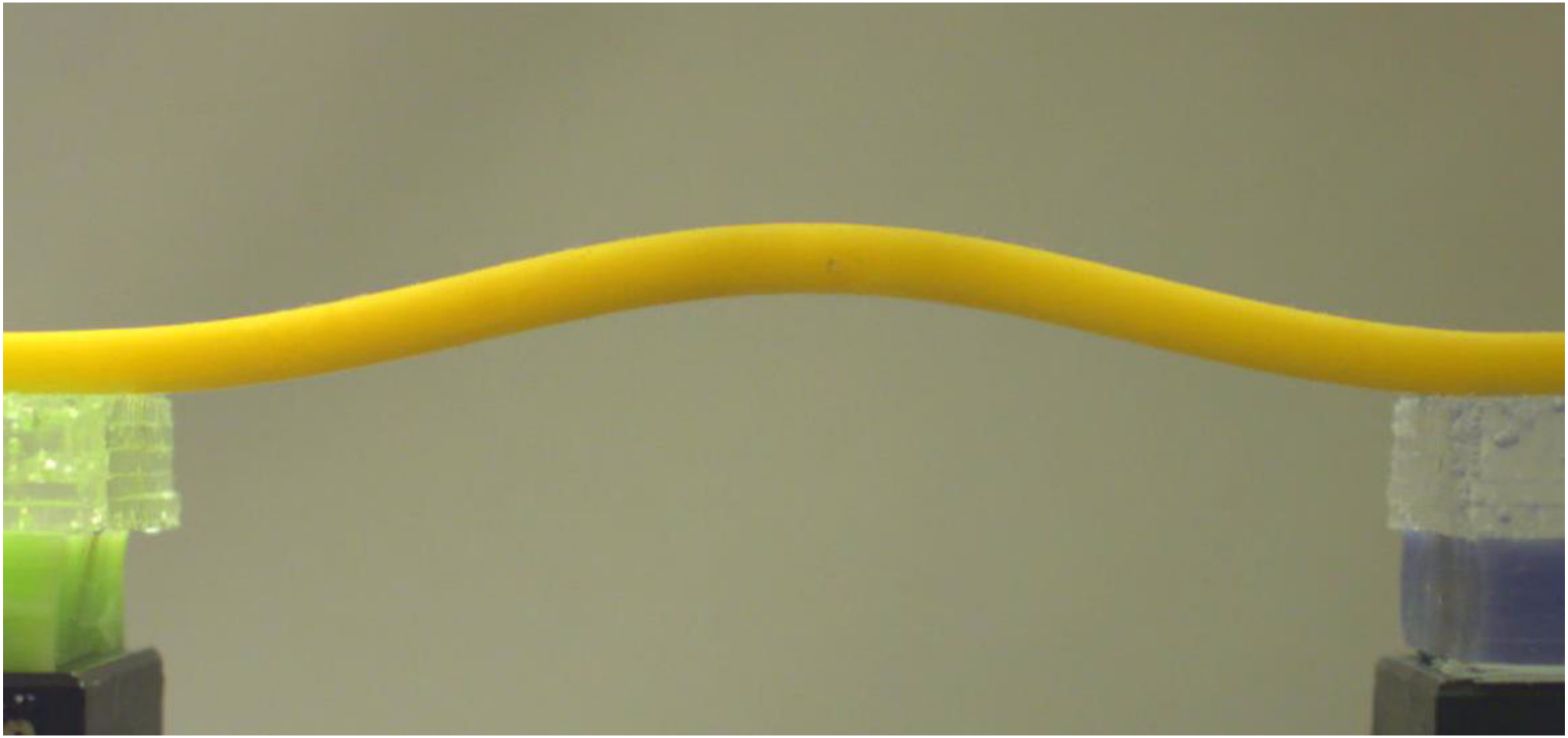
Composite Materials



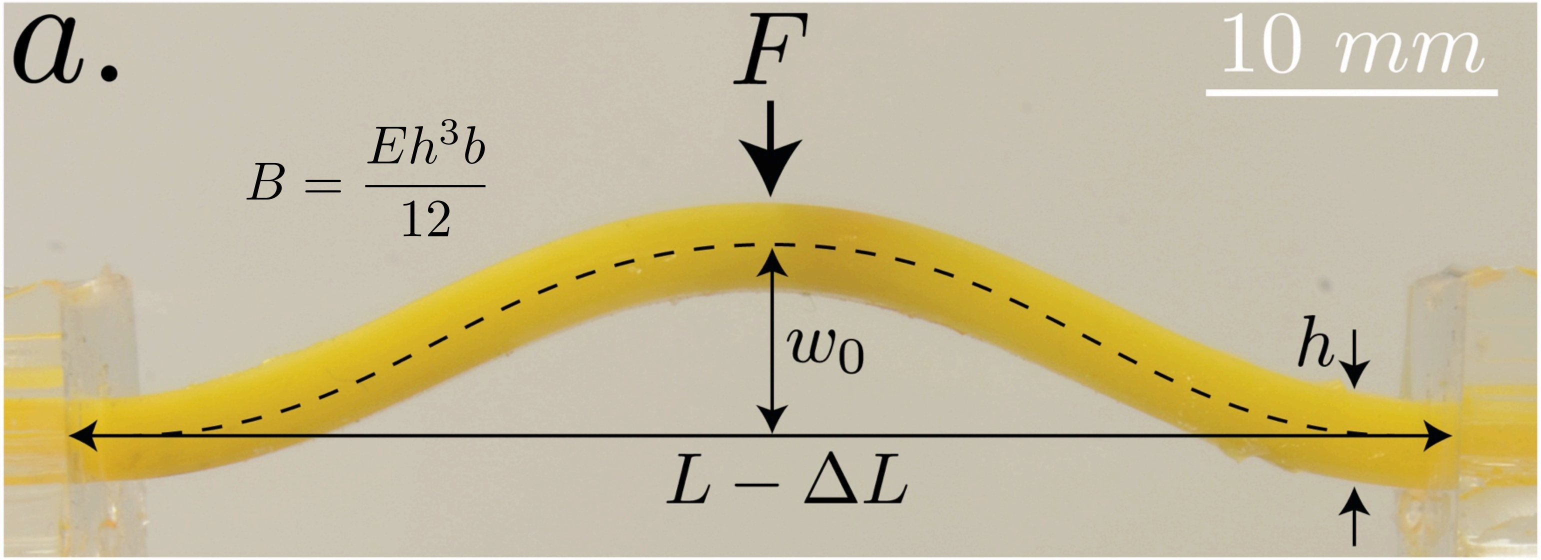


(e.)
t_{on}
e_2
e_1
445 nm
0.0015
0.0005
-0.0005
-0.0015
Strain
Gradual deformation
Snap-through
$t_{cr} = t_{on} + 8.4$ s
445 nm
445 nm
0.0015
0.0005
-0.0005
-0.0015
Strain
1.7 mm
445 nm

Snapping



Statics



Two Geometric Quantities:

End Shortening:

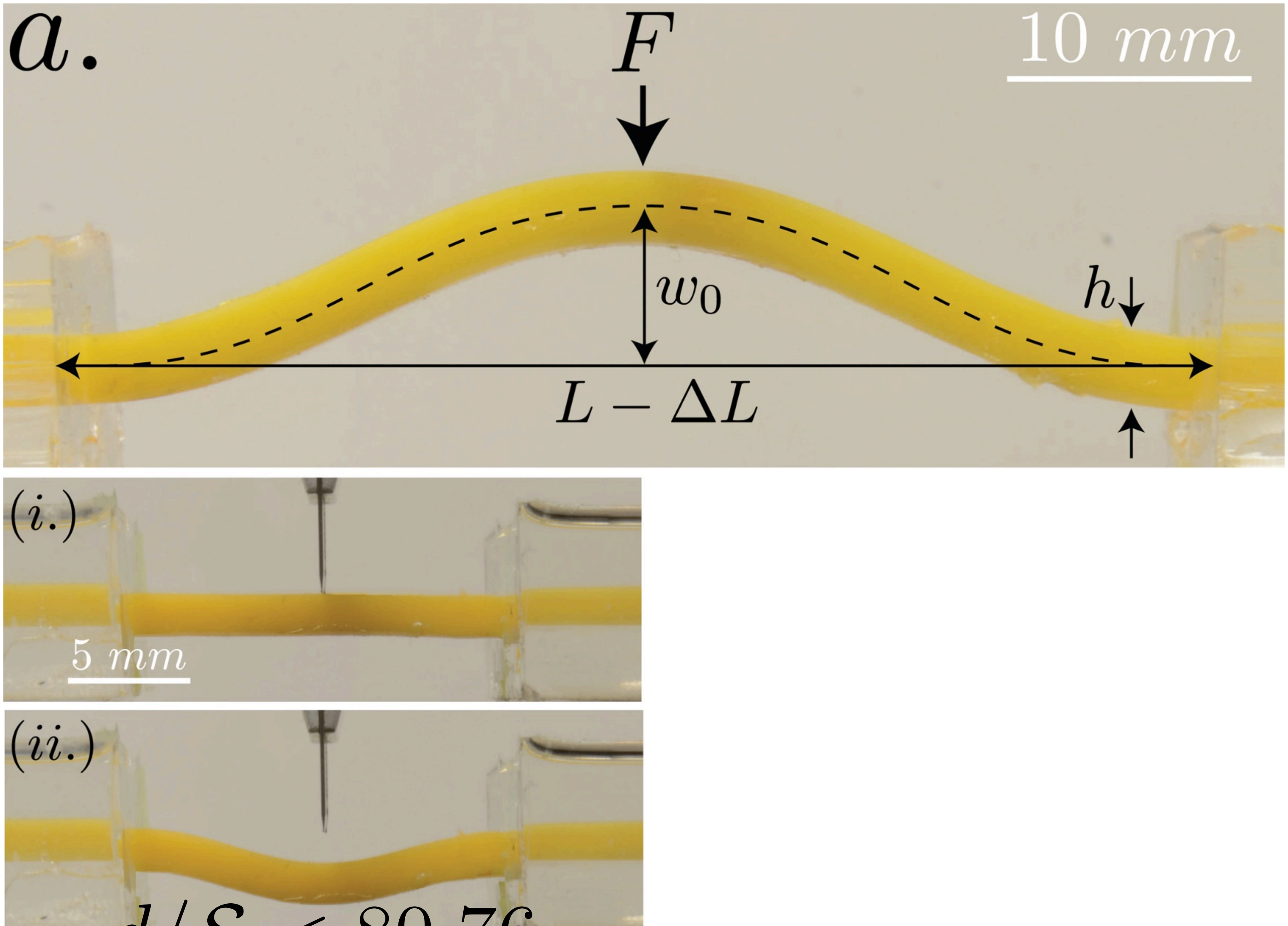
$$d = \frac{\Delta L}{L}$$

Stretchability:

$$\mathcal{S} = \frac{B}{EhL^2b} = \frac{h^2}{12L^2}$$

$$\boxed{d/\mathcal{S}}$$

A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).



$$d/S < 80.76$$

A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**. *arXiv:1310.3703*, 2013.

Statics

$$\underbrace{\rho_s h \frac{\partial^2 w}{\partial t^2}}_{dynamics} + \underbrace{B \frac{\partial^4 w}{\partial x^4} + T \frac{\partial^2 w}{\partial x^2}}_{potential\ energy} = \underbrace{F\delta(x)}_{indentation}$$

von Kármán strains:

$$\varepsilon_{xx} = \partial_x u + \frac{1}{2}\left(\partial_x w\right)^2 = -\mathcal{S}\tau^2$$

Dimensionless tension:

$$\tau^2 = \frac{TL^2}{B}$$

Dimensionless indentation:

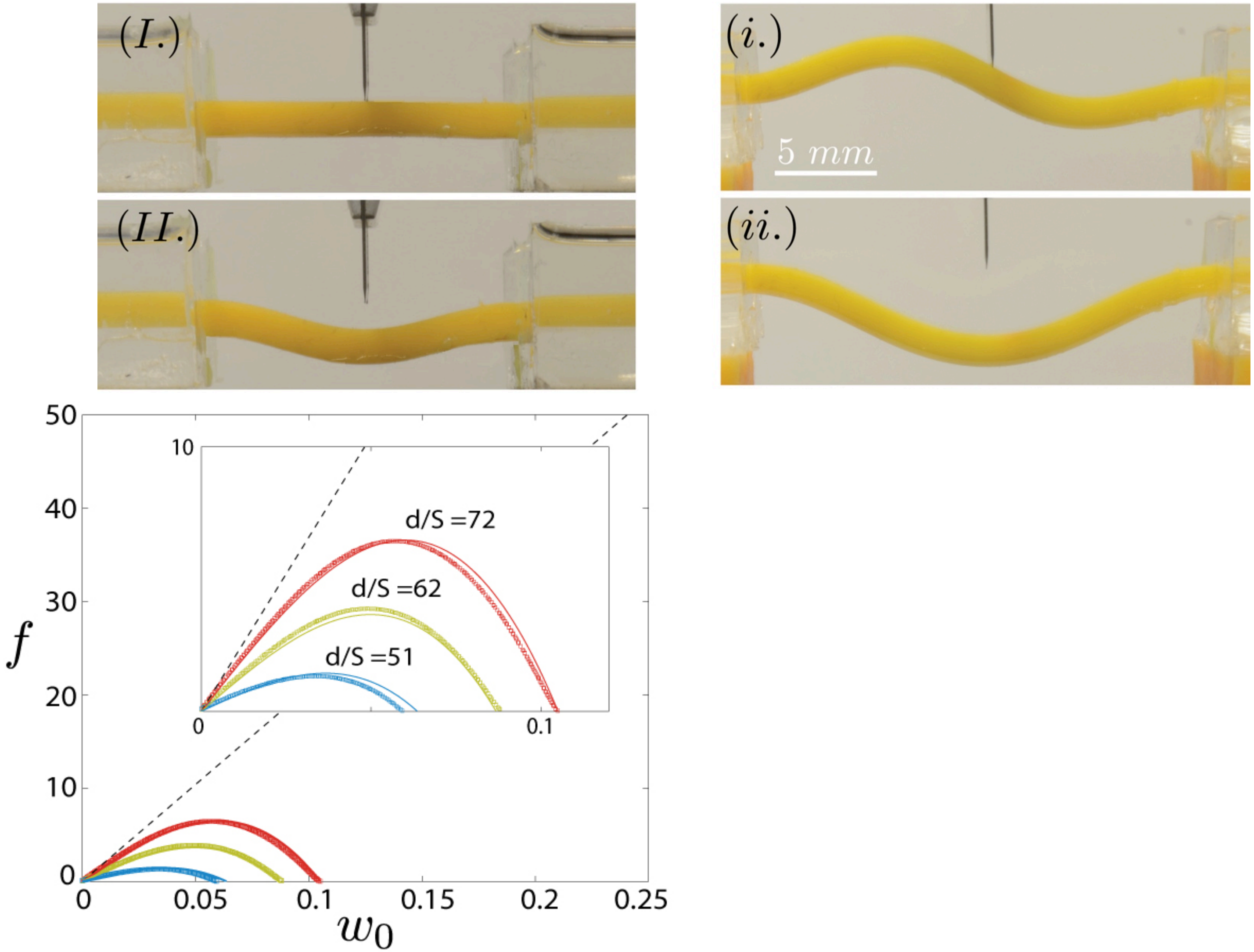
$$f = \frac{FL^2}{B}$$

Transition from Symmetric to Asymmetric shape: **Minimize Bending Energy**

$$U_B = \frac{1}{2} w_0 f + \tau^2\left(d - \mathcal{S}\tau^2\right)$$

A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**. *arXiv:1310.3703*, 2013.

Force vs. Displacement



A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
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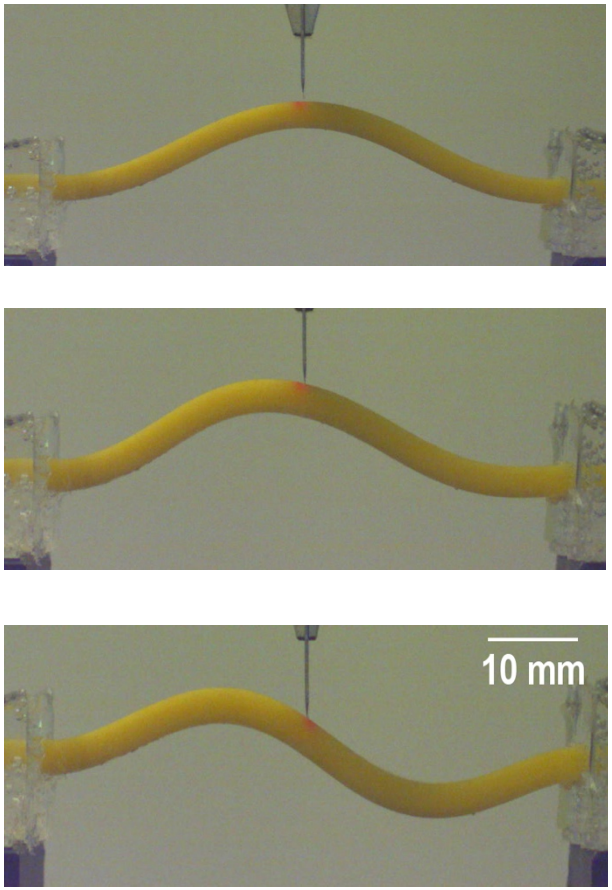

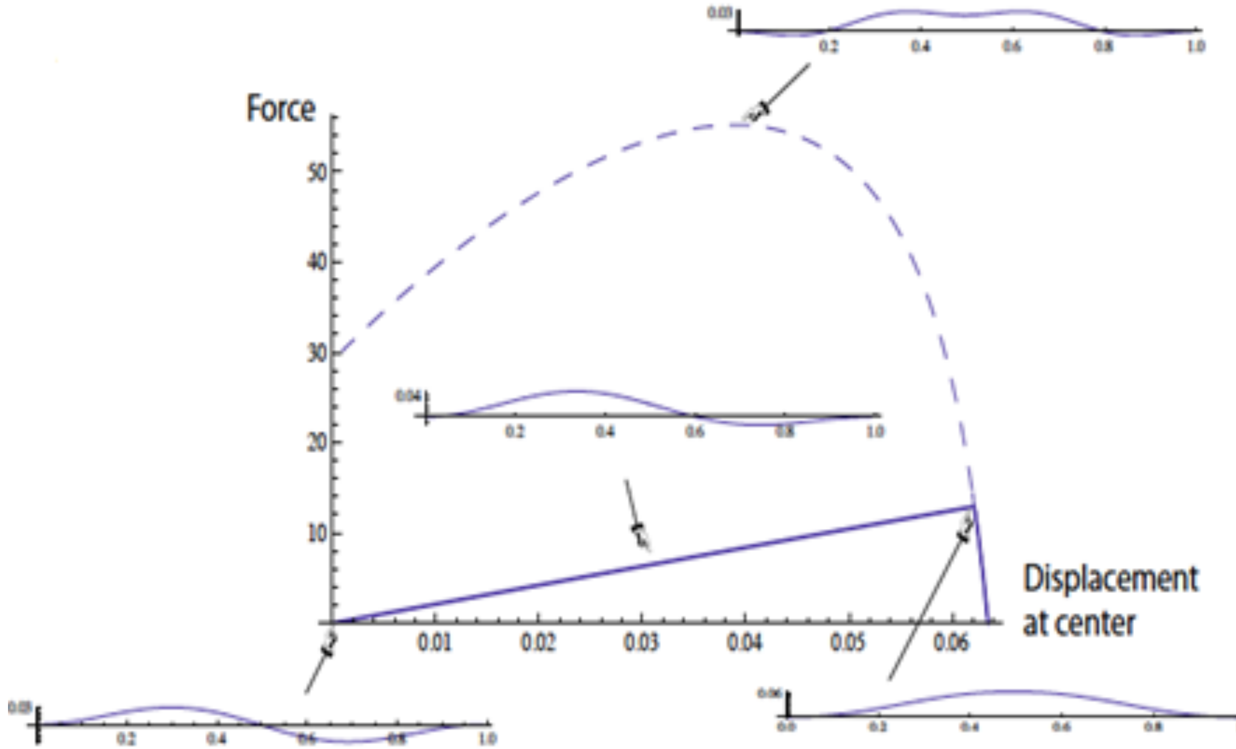

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Statics

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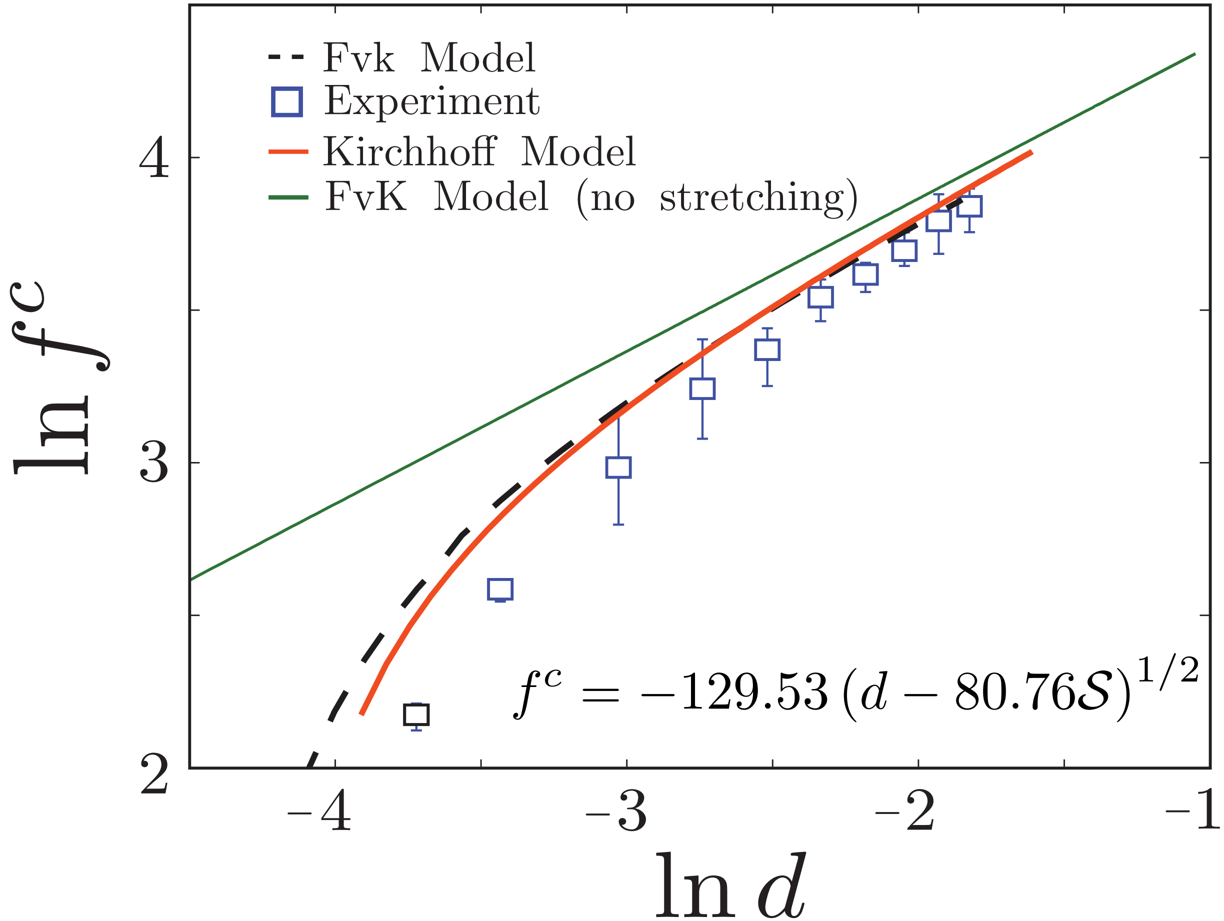

A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
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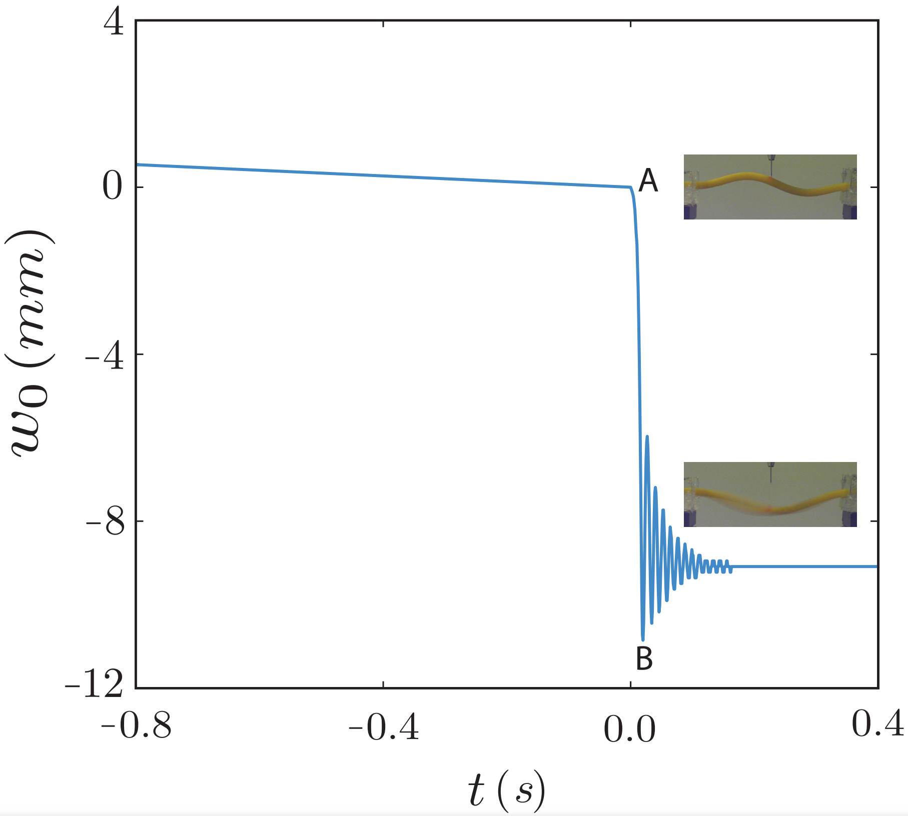


$$\underbrace{\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + \tau^2\frac{\partial^2 w}{\partial x^2}}_{non-dimensional} = 0$$

Solutions:

$$w(x,t) = \underbrace{w_\alpha(x)}_{shape} + \underbrace{\epsilon w_p(x)e^{\sigma t}}_{perturbation}$$

$$\tau = \underbrace{\tau_\alpha}_{comp.} + \underbrace{\epsilon\tau_p e^{\sigma t}}_{perturbation}$$

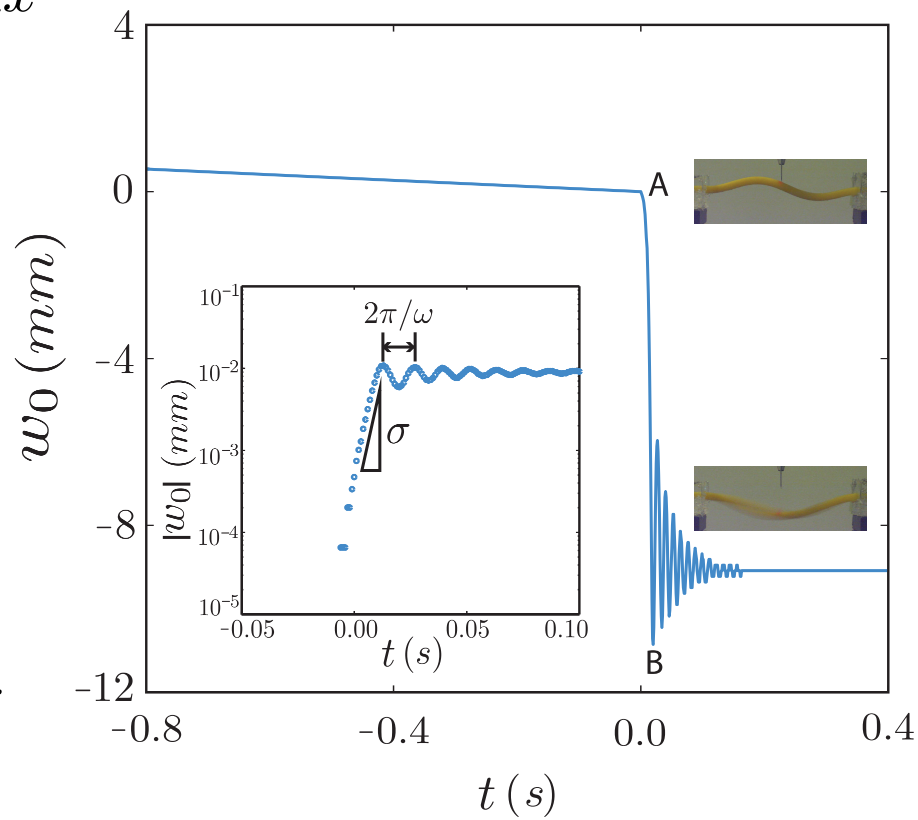
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**. *arXiv:1310.3703*, 2013.

Dynamics

At leading order in ϵ, eigenvalue problem for σ, with eigenfuction $w_p(x)$ satisfying

$$\frac{\mathrm{d}^4 w_p}{\mathrm{d}x^4} + \tau_p^2 \frac{\mathrm{d}^2 w_p}{\mathrm{d}x^2} + \sigma^2 w_p = -2\tau_\alpha \tau_p \frac{\mathrm{d}^2 w_\alpha}{\mathrm{d}x^2}$$

$$\int_{-1/2}^{1/2} \frac{\mathrm{d}w_\alpha}{\mathrm{d}x} \frac{\mathrm{d}w_p}{\mathrm{d}x} \mathrm{d}x = -2\mathcal{S}\tau_\alpha \tau_p$$

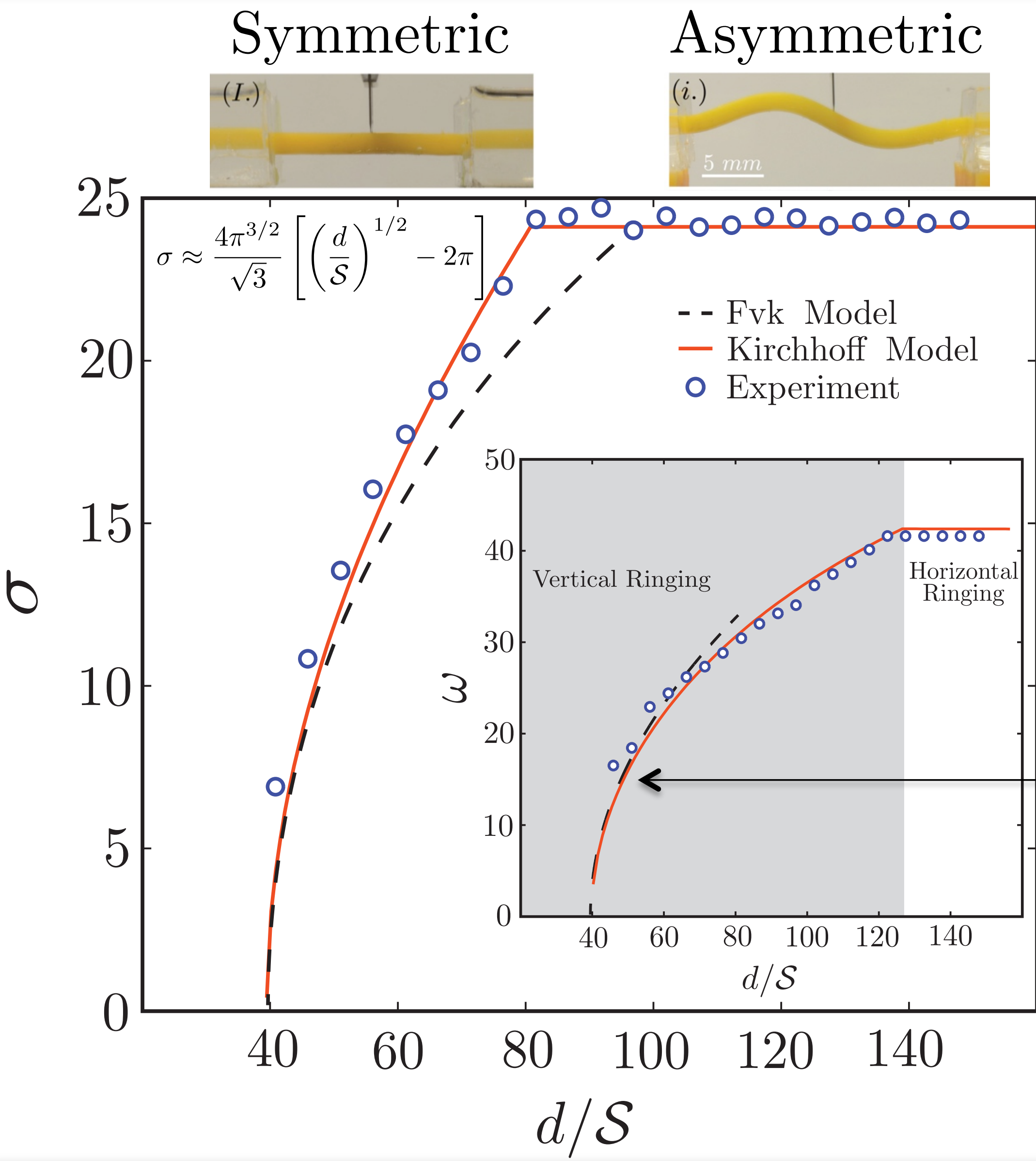


Boundary Conditions

$$w_p(\pm 1/2) = w_p'(\pm 1/2) = 0$$

Transcendental equation for σ

Dynamics

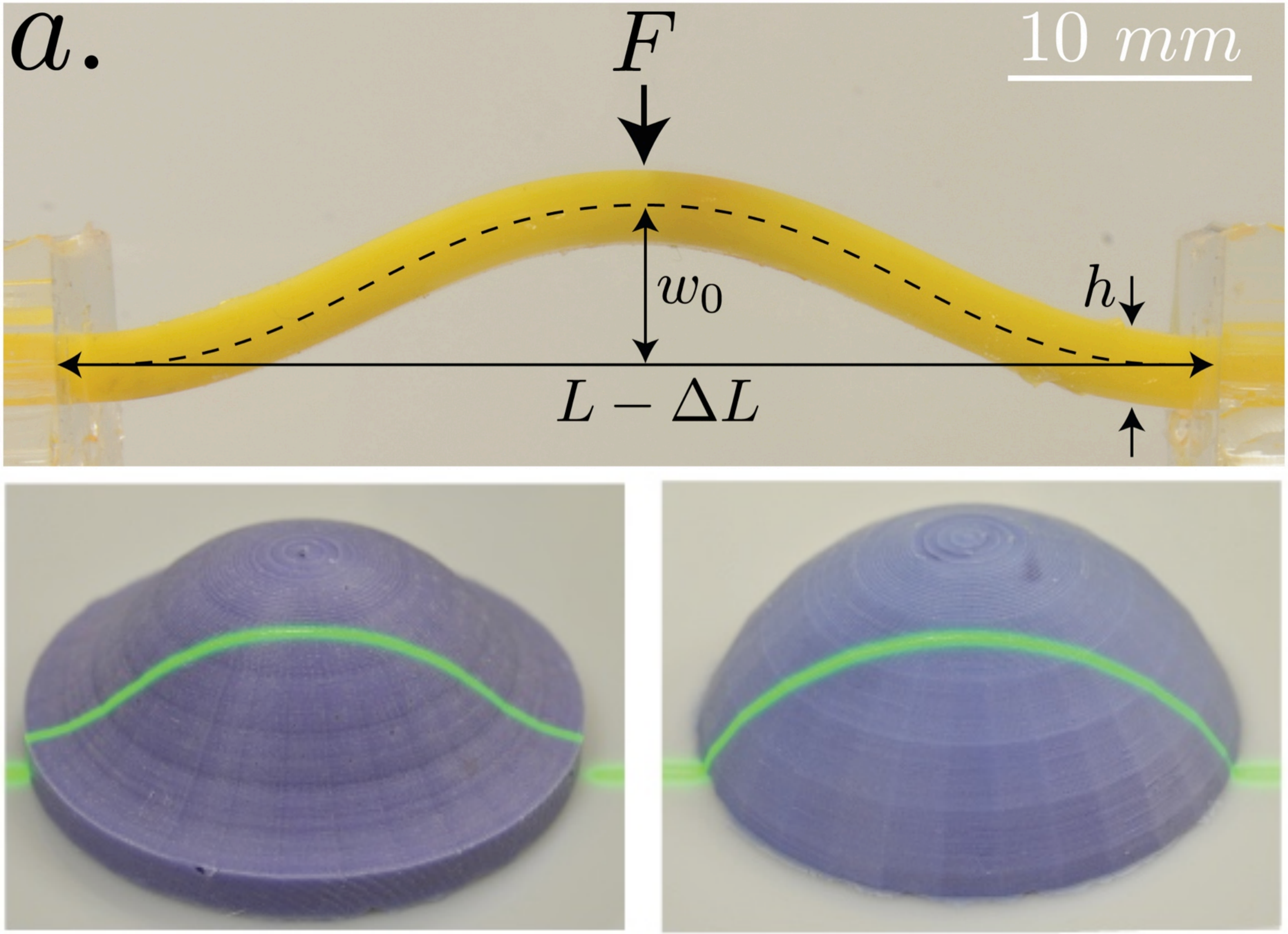


$\sigma \approx 24.113$

$$t_r^{(b)} = \frac{2\pi}{\omega} t_* \approx 0.513 \frac{L^2}{ch}$$

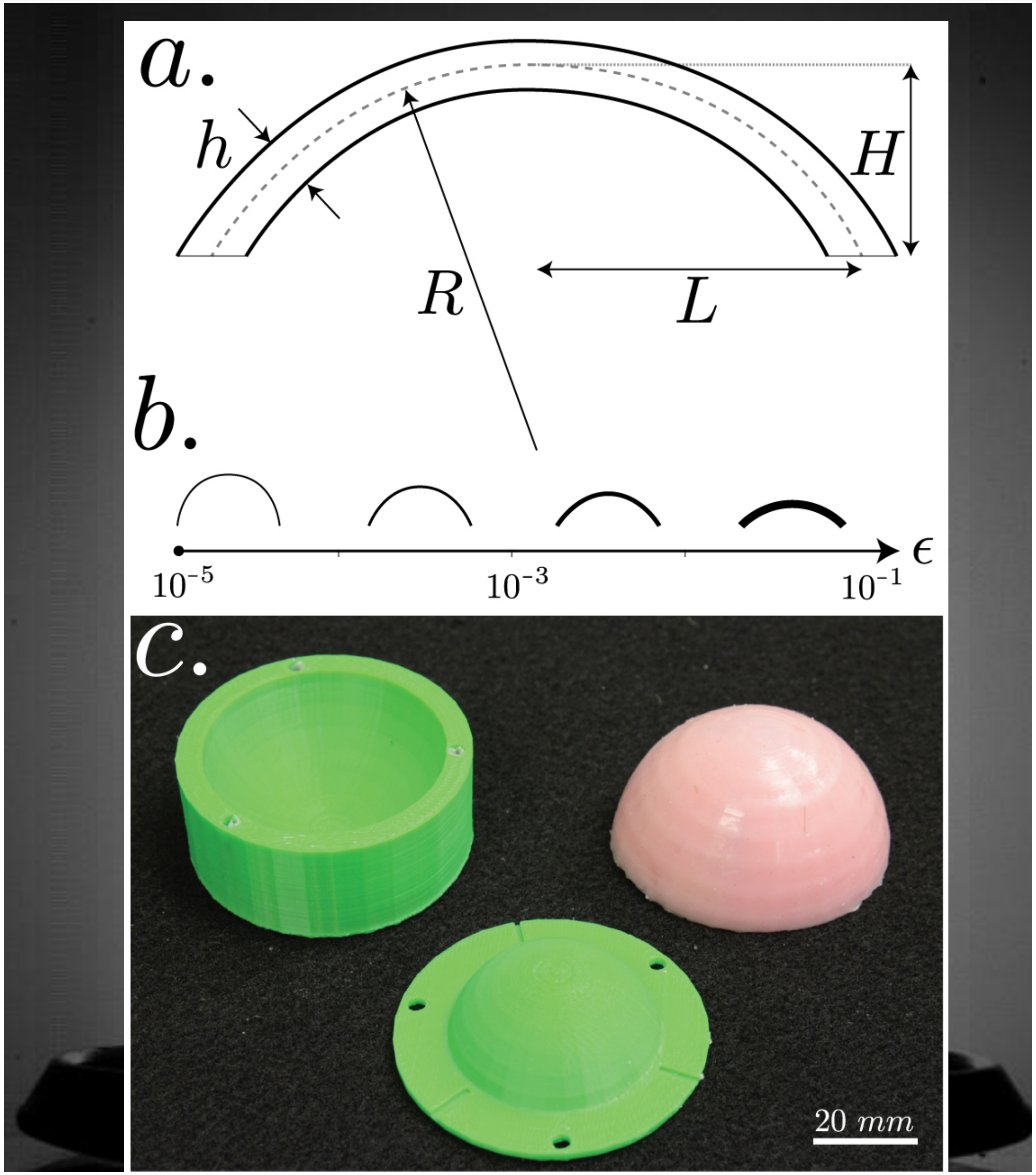
$$\omega \approx \frac{2^{3/2}\pi}{3^{1/2}} \left(\frac{d}{\mathcal{S}} - 4\pi^2\right)^{1/2}$$

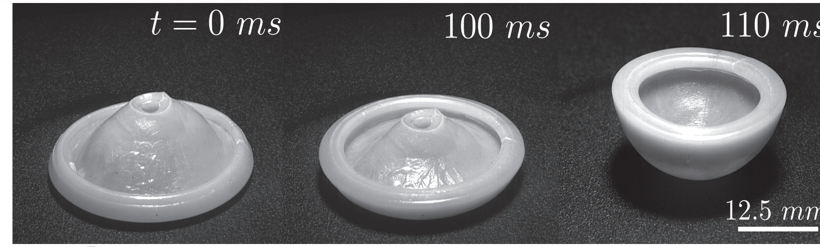
From Beams to Shells

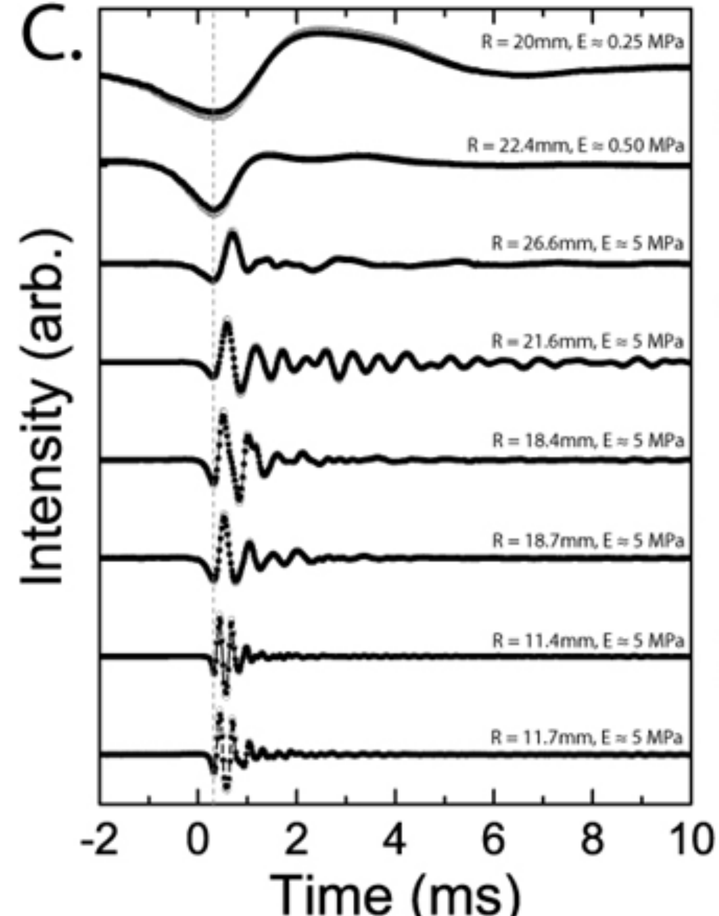


A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**. *arXiv:1310.3703*, 2013.

Snapping Dynamics







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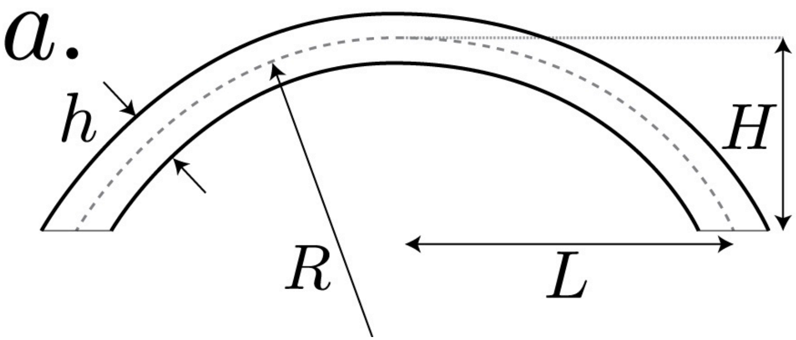


Donnell-Mushtari-Vlasov

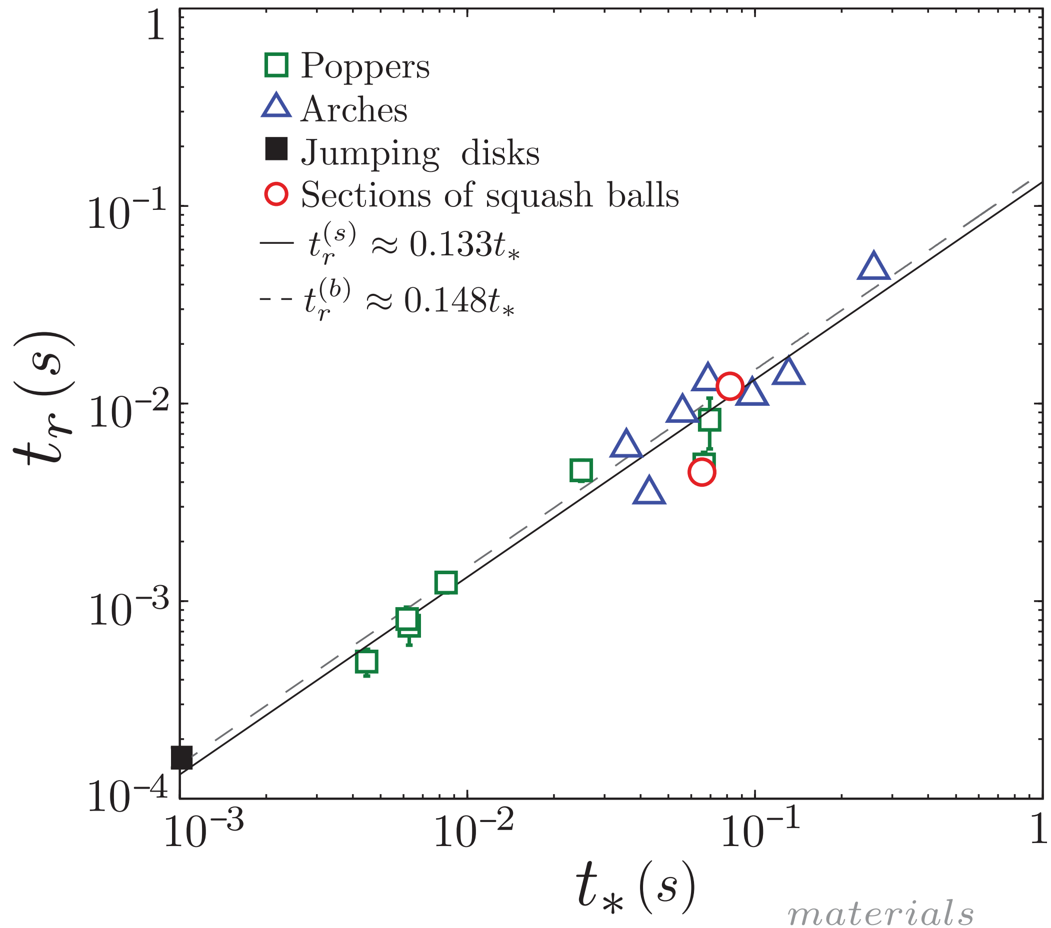
$$\rho_s h \frac{\partial^2 w}{\partial t^2} + B\nabla^4 w + \underbrace{\frac{Eh}{R^2} w}_{curvature} = 0$$

Boundary Conditions

$$\underbrace{\nabla^2 w|_{r=L/2}}_{moment-free} = \underbrace{\frac{\mathrm{d}}{\mathrm{d}r}\left(\nabla^2 w\right)|_{r=L/2}}_{shear-free} = 0$$

Ringing Timescale $(\lambda \approx 3.196)$

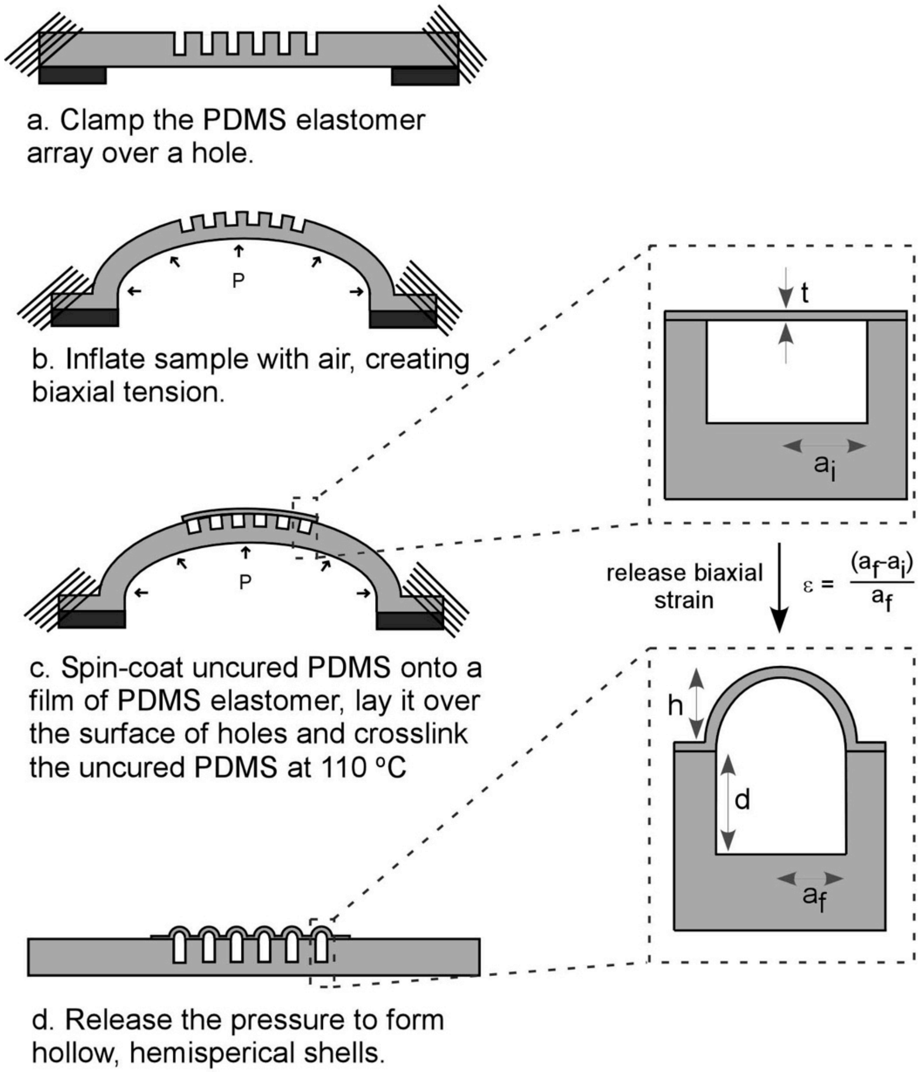
$$t_r^{(s)} = \frac{\pi}{2}\frac{\sqrt{1-\nu^2}}{\lambda^2} t_* \left(1 + \frac{3}{4}\frac{1-\nu^2}{\lambda^4}\frac{L^4}{h^2 R^2}\right)^{-1/2}$$

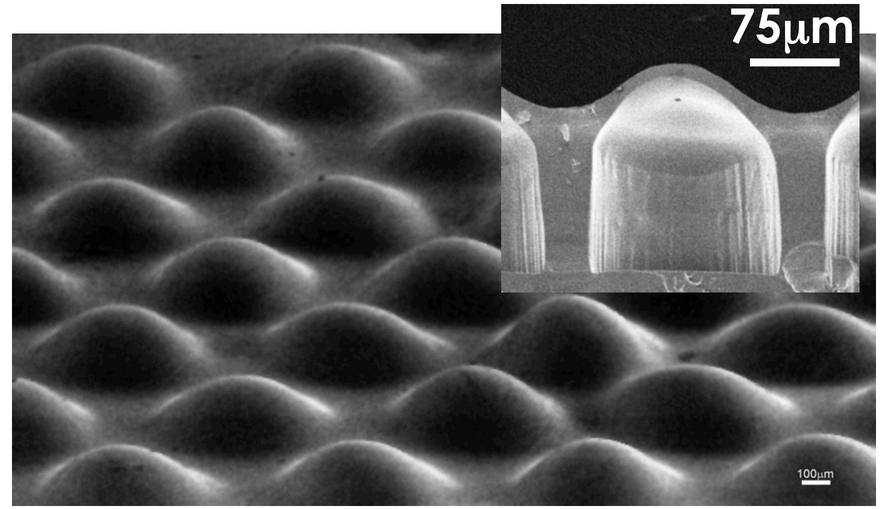


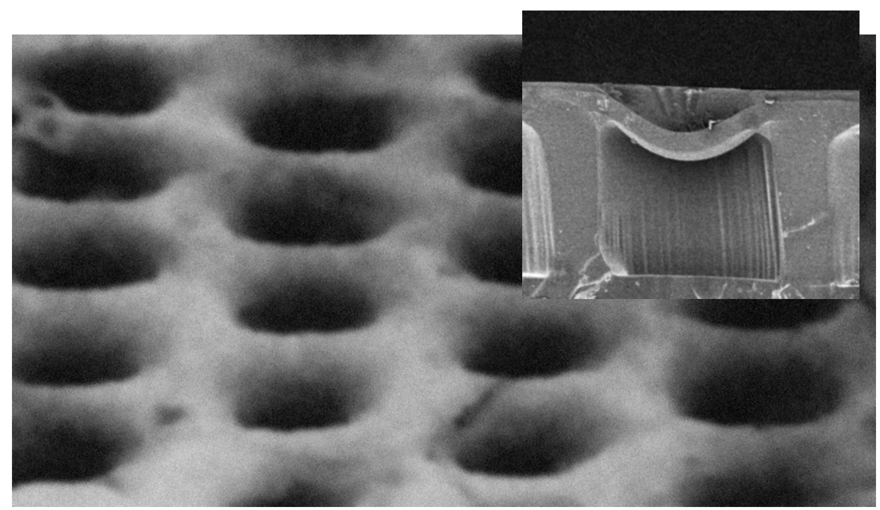
Timescale: $t_r \sim \underbrace{\frac{L^2}{h}}_{geometry} \overbrace{\sqrt{\frac{\rho_s}{E}}}^{materials}$

A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**. *arXiv:1310.3703*, 2013.

Snapping Surfaces

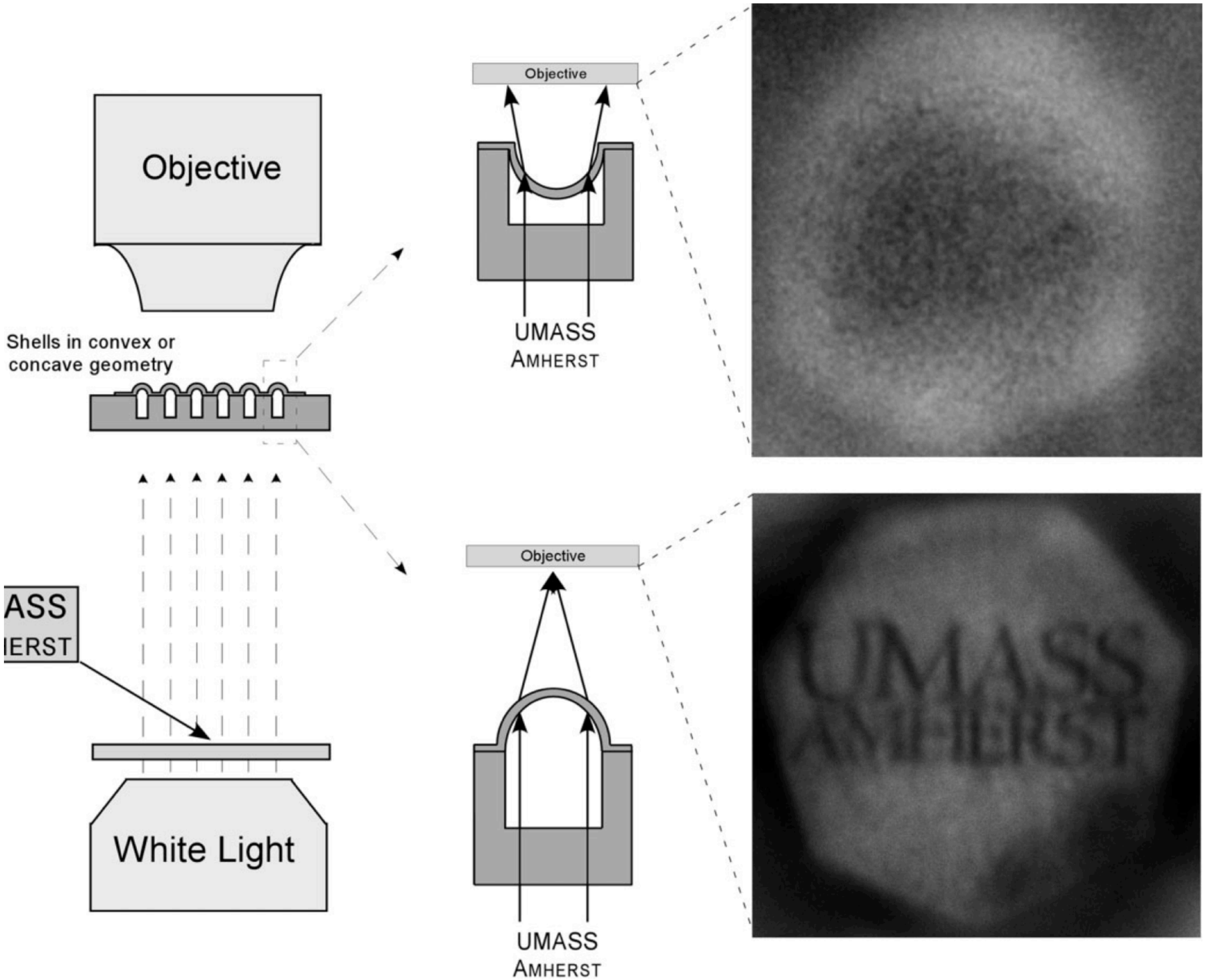






D.P. Holmes and A.J. Crosby, "Snapping Surfaces," *Advanced Materials*, **19**, 21, 3589-3593, 2007.

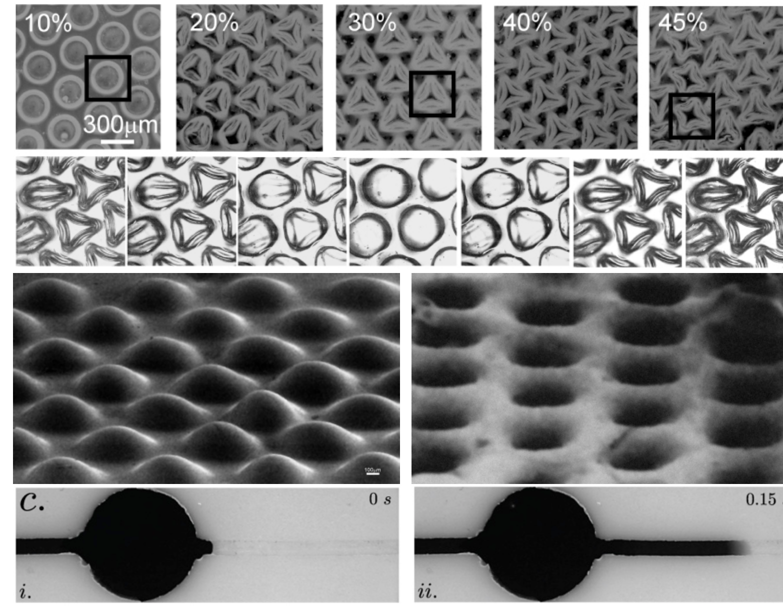
Rapid Optical Switch



D.P. Holmes and A.J. Crosby, "Snapping Surfaces," *Advanced Materials*, **19**, 21, 3589-3593, 2007.

Summary

- **Understanding** and **controlling** the shapes of **slender** structures.
- Harness **instabilities** for advanced **functionality**.
- **Large**, **fast**, **reversible** deformations for **adaptable materials.**



Relevant Publications by PI

- **D.P. Holmes** and A.J. Crosby, "Snapping Surfaces," *Advanced Materials*, **19**, 21, 3589-3593, 2007.
- **D.P. Holmes**, and A.J Crosby, "Crumpled Surface Structures", *Soft Matter*, 4, 82, (2008).
- A. Pandey, D. Moulton, D. Vella, and **D.P. Holmes**, "Dynamics of Snapping Beams and Jumping Poppers", *EPL (Europhysics Letters)*, **105**(2), 24001, (2014).
- **D.P. Holmes**, B. Tavakol, G. Froehlicher, and H.A. Stone, "Control and Manipulation of Microfluidic Fluid Flow via Elastic Deformations ", *Soft Matter*, **9**(29), 7049, (2013).
- B. Tavakol, M. Bozlar, G. Froehlicher, H.A. Stone, I. Aksay, and **D.P. Holmes**. "Buckling Instability of Dielectric Elastomeric Plates for Flexible Microfludic Pumps." *Under Review*, 2014.
- B. Tavakol, **D.P. Holmes**, and H.A. Stone, "Extended Lubrication Theory", *Under Review: Physics of Fluids*, (2015).

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Dominic Vella (Oxford)
Derek Moulton (Oxford)

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