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The main premise in *Mathematical Finance* is that markets tolerate only one special type of functional dependence between derivative and underlying assets. Plainly, the price of the underlying asset dictates the value of the derivative asset more or less in the way Newton's law $F = m\,a$ determines the trajectory of an object immersed in a force field. It is important to point out that the term *asset* is not necessarily a reference to a share (or any number of shares) of some company's common stock and *derivative asset* is not necessarily a contract to buy or sell such shares at some pre-negotiated price. For example, the “asset” may be a factory making a new drug and the “derivative asset” may be the right to buy such a factory under certain terms and conditions. There are many variations: the underlying asset may be freely traded crude oil and the derivative asset may be the right to extract oil from a particular location; or, the underlying asset may be a home in a desirable neighborhood and the derivative asset may be a building lot on which one can build such a home for a fixed construction cost. And so on. What all these examples have in common is that the value of the underlying asset is subject to random fluctuations and, at the same time, the value of the derivative asset is completely determined by the present value of the underlying. The main question, then, is, given the present value of the underlying, what is the value of the derivative and when is the best time to exercise the rights incorporated in the derivative instrument? In other words, given the present value of a particular home in town X what is the fair price of a building lot on which one can build – now, or at any time in the future – a home of that same type? Once the lot is purchased, when would be optimal to build the home? *Financial Calculus* is a symbolic language in which the actions of the market forces are expressed more or less in the way in which the actions of physical forces and the trajectory followed by a particle immersed in a force-field are expressed in terms of derivatives and integrals in the language invented by Leibniz and Newton, which is what we call *Calculus*.

Let's suppose that the value of the underlying asset changes at rate

$$\frac{\Delta V_t}{V_t} = \frac{V_{t+\Delta t} - V_t}{V_t} \approx \rho\,\Delta t + \sigma\,\Delta w_t \ .$$

In this expression Δw_t represents *volatility*, that is to say *risk*, associated with investing in the underlying between time t and time $t + \Delta t$. Since Δt is assumed to be infinitesimally small, so is also the quantity Δw_t, which is treated as a random variable distributed with normal (bell-shaped) probability density centered at the point 0 and having standard deviation $\sqrt{\Delta t}$. In more technical parlance, Δw_t is the time-step $(w_{t+\Delta t} - w_t)$ of some standard Brownian motion process that drives the fluctuations in the value of the underlying. Thus, the standard deviation of the return $\Delta V_t / V_t$ is $\sigma\sqrt{\Delta t}$. If someone invests V units of money in the underlying at time t, during the next Δt days this investment will change on average by $\rho\,V\,\Delta t$

units of money. The actual change however will be random and will fluctuate around this amount with standard deviation $\approx \sigma V \sqrt{\Delta t}$ ($\approx$ means that quantities of order $(\Delta t)^p$ for $p > 1$ are neglected) measured, of course, in the same units of money. If V units of money are invested in a risk-free security – such as a bank deposit, say – for Δt days this investment will bring fixed – and guaranteed – gain of $V_{t+\Delta t} - V_t \approx r V \Delta t$ units of money, where r denotes the (fixed) short-term interest rate. Of course, unless $\rho V \Delta t > r V \Delta t$ no rational investor will assume the risk associated with investing in the underlying. The difference $(\rho - r) V \Delta t$ is the *risk-premium* which investors collect – note well: *on average* – in exchange for assuming $\sigma V \sqrt{\Delta t}$ standard deviations of risk incorporated in the fluctuation $\sigma V \Delta w_t$. Consequently, $\Delta t (\rho - r)/\sigma$ is the price of the risk associated with the fluctuation Δw_t alone — notice that Δw_t is 0 on average. The same argument shows that if, in addition, the underlying pays dividend at rate δ then the price of the risk incorporated in the random quantity Δw_t would be $\Delta t (\rho + \delta - r)/\sigma$, for, in that case, investing V units of money in the underlying generates $\delta V \Delta t$ units of money in addition to the expected $\rho V \Delta t$ units of money.

Now consider a derivative asset whose value follows in a completely predictable way the value, V, of the underlying asset, so that the value of the derivative asset can be expressed as $F(V)$ for some pricing function $F(\cdot)$. We will show next that $F(.)$ cannot be any function and must satisfy a particular equation. First, notice that from the Itô formula (roughly, because of Taylor's formula and the fact that $(\Delta w_t)^2$ behaves as Δt) one has

$$\Delta F(V_t) \equiv F(V_{t+\Delta t}) - F(V_t) \approx F'(V_t) \times \Delta V_t + \frac{1}{2} F''(V_t) \times (\Delta V_t)^2 ,$$

which, after discarding terms containing $(\Delta t)^2$, yields

$$\Delta F(V_t) \approx \rho V F'(V_t) \times \Delta t + \frac{1}{2} \sigma^2 V^2 F''(V_t) \times \Delta t + \sigma V F'(V_t) \times \Delta w_t$$

Thus, keeping $V_t \equiv V$ fixed, $\Delta F(V_t)$ fluctuates around its average $\rho V F'(V) \Delta t + \frac{1}{2} \sigma^2 V^2 F''(V) \Delta t$ by the amount $\sigma V F'(V) \Delta w_t$ — the key here is that Δw_t is independent from the value $V_t \equiv V$. However, the risk that the fluctuation $\sigma V F'(V_t) \Delta w_t$ carries is valued at

$$\sigma V F'(V_t) \times \frac{\rho + \delta - r}{\sigma} \times \Delta t = V F'(V_t) \times (\rho + \delta - r) \times \Delta t ,$$

which is $\sigma V F'(V_t)$ times the value of the risk incorporated in the fluctuation Δw_t. This is the amount by which the average gain from keeping $F(V)$ units of money invested in the derivative asset for Δt days must exceed the gain from keeping $F(V)$ units of money invested in a bank deposit for Δt days, which is $\approx rF(V) \Delta t$ units of money. Thus, one must have

$$\rho V F'(V) \Delta t + \frac{1}{2} \sigma^2 V^2 F''(V) \Delta t - rF(V) \Delta t = V F'(V) \times (\rho + \delta - r) \times \Delta t , \qquad (1)$$

which, after an obvious cancellation and division by Δt, gives

$$\frac{1}{2} \sigma^2 V^2 F''(V) + V F'(V) \times (r - \delta) - rF(V) = 0 . \qquad (2)$$

The most remarkable feature of this equation is that it does not contain the expected rate of capital gain ρ – all terms containing this parameter were canceled at the previous step. Consequently, the value of the derivative instrument depends on the interest rate r, the dividend δ and the volatility σ in the value of the underlying, *but not on the expected gain from investing in the underlying*. Note (this is important!) that equality in (1), and consequently in (2), holds only under the assumption that the derivative is not to be exercised between time t and time $t + \Delta t$. Exercising during this time interval reduces the risk and therefore also the reward, so that, in general, "=" in (1) and (2) must be replaced by "⩽" whenever exercising the option is allowed at any time.

Since in the interval of values V for which "not exercising" is preferable, i.e., in the interval of values V for which the equality in (2) holds, the relation (2) turns into a homogeneous second order equation, its general solution must depend on two free constants. Consequently, in order to determine the pricing function $F(\cdot)$, one must specify two boundary conditions. Of course, the boundary conditions depend on the nature of the problem. For example, one such condition might be $F(0) = 0$ and the second condition might be the requirement $F(V_E) = V_E - K$, which simply says that if the derivative instrument — treated in this case as an option to buy the underlying asset at price K — is exercised at the moment when the price of the underlying has reached the level V_E, then at that moment the derivative is worth $V_E - K$ units of money (translation: if the home is built for a fixed cost K when the price of homes is V_E, then the gain is $V_E - K$). However, the threshold V_E is not known and, consequently, in order to determine the function $F(\cdot)$ one needs a third condition. It is not hard to justify in purely financial terms the requirement $F'(V_E) = 1$. This determines completely not only the pricing function $F(\cdot)$ but also the price-threshold V_E at which exercising the option is optimal.

Of course, all of the above is just the tip of the iceberg. There are many modifications and extensions of equation (2) developed in connection with particular problems in finance. For instance, if the derivative can be exercised only at some fixed future moment T, then the valuation function $F(\cdot)$ must be time-dependent, i.e., the value of the derivative can be expressed as $F(t, V_t)$. In this case the left side of (2) will contain also the term $\frac{d}{dt} F(t, V)$. In this later case, the second boundary condition becomes $F(T, V) = V - K$ and the respective solution, which prescribes the price of a standard European call option as a function of the current price of the underlying and the time left to maturity, is given by the renowned *Black-Scholes option pricing formula*. There is a very rich and highly nontrivial mathematical theory behind equation (2), which is intimately related to the general theory of stochastic processes and the theory of partial differential equations with free boundaries, bridged by the famous *Feynman-Kac formula* most physicists are familiar with. This theory has attracted a lot of attention, because the conclusions that it leads to can not be derived with simpler or more intuitive financial arguments. ◆

Suggested Reading

[1] A. K. Dixit and R. S. Pindyck, *Investment Under Uncertainty* (Princeton University Press, 1994)

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[3] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance* (Springer-Verlag, 1998)

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