Preliminary Exams 2021  
Algebra Exam (3.5 hours)

Part I.

Solve four of the following five problems.

**Problem 1.** Find the general solution in $\mathbb{R}^4$ to the system of equations

\[
\begin{align*}
    w + 2x - y + 3z &= 4 \\
    2w + 4x - y + 4z &= 0.
\end{align*}
\]

Your answer should have the form $(w, x, y, z) = v + rv' + sv''$, where $v$, $v'$, and $v''$ are vectors and $r$ and $s$ denote arbitrary scalars.

**Problem 2.** Let $V$ be the vector space over $\mathbb{R}$ with basis $\cos^2 x$, $\sin^2 x$, and $\cos x \sin x$, and let $T : V \to V$ be the linear map $T(f) = 3f - f'$, where $f' = df/dx$.

Find the determinant of $T$.

**Problem 3.** Consider the matrix $A = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$.

Find an invertible matrix $C$ and a diagonal matrix $D$ such that $A^{100} = CDC^{-1}$.

**Problem 4.** Let $V$ be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (1, 1, 1, 1)$ and $v_2 = (1, -1, -1, 1)$.

(a) Find an orthonormal basis $u_1$, $u_2$ for $V$ relative to the dot product.

(b) Put $w = (1, 2, 2, 5)$. Find scalars $a_1$, $a_2$ such that $(w - a_1u_1 - a_2u_2) \cdot v = 0$ for all $v \in V$.

**Problem 5.** Let $M$ be the subgroup of $\mathbb{Z}$ generated by 101010 and 111111.

Find the order of $\mathbb{Z}/M$.

Part II.

Solve three of the following six problems.

**Problem 6.** Show that the ring $A = \mathbb{R}[x]/(x^2)$ has exactly 3 ideals.

**Problem 7.** Give 3 different examples of $6 \times 6$ matrices $J$ in Jordan canonical form with characteristic polynomial $X^6$ and minimal polynomial $X^3$. (Here two examples $J$ and $J'$ are understood to be “different” if there does not exist an invertible $6 \times 6$ matrix $U$ such that $J' = UJU^{-1}$.)

**Problem 8.** Let $SL_2(\mathbb{R})$ be the group of $2 \times 2$ matrices with real coefficients and determinant 1, and put

$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Write $t g$ for the transpose of an element $g \in SL_2(\mathbb{R})$.

(a) Show that $\sigma g \sigma^{-1} = t g^{-1}$ for $g \in SL_2(\mathbb{R})$.

(b) Explain why there is no element $\tau \in SL_2(\mathbb{R})$ such that $\tau g \tau^{-1} = t g$ for all $g \in SL_2(\mathbb{R})$. Is there an element $\tau$ such that $\tau g \tau^{-1} = g^{-1}$? Why or why not?

**Problem 9.** Let $S_n$ be the group of permutations of the set $\{1, 2, 3, \ldots, n\}$. In each case, give an example of an element of $S_n$ of order $d$, or explain why none exists: (i) $n = 10$, $d = 30$, (ii) $n = 11$, $d = 33$. 


Problem 10. Let $A$ be an $n \times n$ symmetric matrix with real coefficients and $n$ distinct eigenvalues. Suppose that $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are eigenvectors of $A$ satisfying $x_j > 0$ and $y_j > 0$ for $1 \leq j \leq n$. Prove that $x$ is a scalar multiple of $y$.

Problem 11. Let $L$ be the subgroup of $\mathbb{Z}^3$ generated by $(3, 2, 9)$, $(2, 2, 2)$, and $(3, 2, 3)$. Find a direct sum of cyclic groups isomorphic to $\mathbb{Z}^3/L$.

Part III.

Solve one of the following three problems.

Problem 12. Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{R}$ with minimal polynomial $(x - 1)^2x$ and trace 2. Prove that $A = UBU^{-1}$ for some invertible $n \times n$ matrix $U$.


(a) Prove that the set $HK = \{hk : h \in H, k \in K\}$ is a subgroup of $G$.

(b) Give an example to show that if $H$ is not normal then the set $HK$ need not be a subgroup of $G$.

Problem 14. In each case determine the degree $[F(\alpha) : F]$, where $\alpha^4 = -1$ and $F$ is the field (i) $\mathbb{R}$, (ii) $\mathbb{Q}$, (iii) $\mathbb{F}_3$ (the field with 3 elements), and (iv) $\mathbb{F}_{17}$ (the field with 17 elements).