Preliminary Exam 2019
Solutions to Morning Exam

Part I.

Solve four of the following five problems.

Problem 1. Consider the function f on $\mathbb{R}$ such that $f(x) = x^2 \ln|x|$ if $x \neq 0$ and $f(0) = 0$. Prove that $f'(0)$ exists but $f''(0)$ does not.

Solution: The limit of the difference quotient for f at 0 is

$$\lim_{x\to 0} \frac{x^2 \ln|x|}{x} = \lim_{x\to 0} x \ln|x| = 0$$

by L'Hôpital's Rule. So $f'(0)$ exists and equals 0. Now if $x \neq 0$ then $f'(x) = 2x\ln|x| + x$, so the limit of the difference quotient for f' at 0 is

$$\lim_{x\to 0} \frac{2x\ln|x| + x}{x} = 1 + 2\lim_{x\to 0} \ln|x|,$$

which does not exist as a finite quantity (the limit is $-\infty$).

Problem 2. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is twice-differentiable and satisfies the differential equation $y'' + y' + y = 0$. If $y(0) = 1$ and $y(\pi/\sqrt{3}) = 0$ then what is $y(2\pi/\sqrt{3})$? Simplify your answer to the extent possible.

Solution: Since $r^2+r+1 = (r-e^{\lambda})(r-e^{\overline{\lambda}})$ with $\lambda = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, we can write $y = ae^{\lambda t} + be^{\overline{\lambda} t}$ with constants $a, b \in \mathbb{C}$. But $y(0) = 1$ and $y(\pi/\sqrt{3}) = 0$, or in other words.

$$\begin{cases} a+b=1 \\ e^{-\pi/\sqrt{3}}(a-b)i = 0, \end{cases}$$

so $a = b = 1/2$. Therefore

$$y = (e^{(-1/2+i\sqrt{3}/2)t} + e^{(-1/2-i\sqrt{3}/2)t})/2 = e^{-t/2}\cos(\sqrt{3}t/2),$$

so $y(2\pi/\sqrt{3}) = -e^{-\pi/\sqrt{3}}$.

Problem 3. Let $f(x) = e^{x^2}$ and $g(x) = \int_0^{\tan x} f(t)\,dt$. Find $g'(x)$, and then find a constant $c \neq 0$ such that $cg'(x) = h(x)e^{h(x)}$ for some function $h(x)$.

Solution: By the Fundamental Theorem of Calculus and the Chain Rule,

$$g'(x) = f(\tan x)\sec^2 x = e^{\tan^2 x}\sec^2 x.$$

Recalling that $\tan^2 x + 1 = \sec^2 x$, we see that we may take $c = e$ and $h(x) = \sec^2 x$.

Problem 4. Decide whether each series converges, justifying your answer:

(a) $\sum_{n\geqslant 2}(\cos(1/\log n) - 1)n^{-1}$

Solution: From the Taylor series of the cosine function (or simply by Taylor's theorem) we have $|\cos x - 1| \leqslant Mx^2$ for x near 0, so in particular

$$|\cos(1/\log n) - 1|n^{-1} \leqslant Mn^{-1}(\log n)^{-2}.$$

Now $\sum_{n\geqslant 2} n^{-1}(\log n)^{-2}$ converges by the Integral Test, because

$$\int_2^T \frac{dx}{x(\log x)^2} = -(\log x)^{-1}|_2^T = (\log 2)^{-1} - (\log T)^{-1} \to (\log 2)^{-1}$$

as T goes to infinity. Therefore $\sum_{n\geqslant 2}(\cos(1/\log n)-1)n^{-1}$ is absolutely convergent (and hence convergent) by the Comparison Test.

(b) $\sum_{n\geqslant 2}\sin(1/\log n)n^{-1}$

Solution: Since $\lim_{x\to 0}\sin x/x = 1$, we see that for some $M > 0$ we have $\sin x > Mx$ if x is close to 0 and positive. In particular,

$$(M/\log n)n^{-1} < \sin(1/\log n)n^{-1}.$$

But $\sum_{n\geqslant 2} n^{-1}(\log n)^{-1}$ diverges by the Integral Test, because

$$\int_2^T \frac{dx}{x\log x} = \log(\log x)|_2^T = \log\log T - \log\log 2 \to \infty$$

as T goes to infinity. So $\sum_{n\geqslant 2}\sin(1/\log n)n^{-1}$ diverges by the Comparison Test.

Problem 5. Let I be an open interval in $\mathbb{R}$ and f a differentiable function on I, and suppose that f' is identically 0. Prove that f is a constant function. You may quote theorems from calculus that are logically prior to the present assertion.

Solution: Fix $a \in I$. By the Mean Value Theorem, for any $b \in I$ there exists a number c between a and b such that $f(b) - f(a) = f'(c)(b-1)$. Since $f' = 0$, we get $f(b) = f(a)$, and since $b \in I$ was arbitrary we conclude that f is constant.

Part II.

Solve three of the following six problems.

Problem 6. If $n \mapsto a_n$ is a surjective or "onto" map from the set of positive integers to the set $\mathbb{Q}\cap[0,1]$ of rational numbers between 0 and 1, what is the radius of convergence of the power series $\sum_{n\geqslant 1} a_n x^n$? Prove your answer.

Solution: Let ρ be the radius of convergence. Then $1/\rho = \limsup_{n\to\infty} a_n^{1/n}$. Put $L = \limsup_{n\to\infty} a_n^{1/n}$. We claim that $L = 1$, whence $\rho = 1$. Certainly $L \leqslant 1$, because $a_n \leqslant 1$ for all n, whence $a_n^{1/n} \leqslant 1$ for all n also. On the other hand, given $\varepsilon > 0$, there are infinitely many rational numbers between $1-\varepsilon$ and 1, hence infinitely many n such that $1-\varepsilon \leqslant a_n \leqslant 1$. Therefore

$$1-\varepsilon < (1-\varepsilon)^{1/n} \leqslant a_n^{1/n}$$

for infinitely many n. Therefore $L \geqslant 1-\varepsilon$, and since $\varepsilon > 0$ is arbitrary, we conclude that $L \geqslant 1$, whence $L = 1$.

Problem 7. Show that

$$\sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)(2n+2)} = \int_0^1 \left(\frac{1}{\sqrt{2-x^2}} - \frac{x}{1+x^2}\right) dx$$

by computing both sides explicitly.

Solution: The left-hand side can be rewritten as

$$\sum_{n\geqslant 0}(-1)^n\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \sum_{n\geqslant 0}\frac{(-1)^n}{2n+1} - \frac{1}{2}\sum_{n\geqslant 0}\frac{(-1)^n}{n+1}.$$

The first and second series on the right-hand side converge by the Alternating Series Test, and they are the values at $x = 1$ of the Taylor series

$$\tan^{-1} x = \sum_{n\geqslant 0}(-1)^n\frac{x^{2n+1}}{2n+1}$$

and
$$\log(x+1) = \sum_{n\geqslant 0}(-1)^n \frac{x^{n+1}}{n+1}$$
respectively. Thus the left-hand side is $\tan^{-1}(1) - (\log 2)/2 = \pi/4 - \log\sqrt{2}$. (The Taylor series are obtained by integrating the geometric series for $(1+t^2)^{-1}$ and $(1+t)^{-1}$ respectively for $0 \leqslant t \leqslant x$. Strictly speaking, since the geometric series converges only for $|x| < 1$, one should also quote Abel's theorem that a power series that converges on a closed interval is continuous there.)

The right-hand side can be written as
$$\sin^{-1}(x/\sqrt{2})|_0^1 - \frac{\log(1+x^2)}{2}|_0^1 = \pi/4 - (\log 2)/2,$$
which coincides with the left-hand side.

Problem 8. Find the value of the line integral $\int_C x\ dy - y\ dx$, where C is the curve $r = \cos 2\theta$ for $-\pi/4 \leqslant \theta \leqslant \pi/4$, oriented counterclockwise.

Solution: By Green's Theorem, the line integral coincides with
$$\int\int_R (\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y))\ dx\ dy = 2\int\int_R dx\ dy,$$
where R is the region enclosed by C. Thus the line integral coincides with
$$2\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\ dr \quad = \int_{\pi/4}^{\pi/4} \cos^2(2\theta)\ d\theta.$$
Since $\cos^2(2\theta)$ is an even function, the right-hand side coincides with
$$2\int_0^{\pi/4} \cos^2(2\theta)\ d\theta = \int_0^{\pi/4}(1+\cos 4\theta)\ d\theta,$$
which is $\pi/4$.

Problem 9. Let D be the solid region inside the cone $z = \sqrt{x^2+y^2}$ and between the two hemispheres $z = \sqrt{4-x^2-y^2}$ and $z = \sqrt{1-x^2-y^2}$. Given that D has uniform density, find the "center of mass" or "centroid" of D. You may use symmetry considerations to reduce the amount of computation.

Solution: Let $(\overline{x}, \overline{y}, \overline{z})$ be the center of mass. The rotational symmetry of D about the z-axis ensures that $\overline{x} = \overline{y} = 0$. We must compute the quantity $\overline{z} = I/J$, where
$$I = \int\int\int_D z\ dx\ dy\ dz$$
and
$$J = \int\int\int_D dx\ dy\ dz.$$
We compute I and J in spherical coordinates: Thus
$$J = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta = 2\pi(1-\sqrt{2}/2)(7/3)$$
and
$$I = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho^3 \cos\varphi \sin\varphi\ d\rho\ d\varphi\ d\theta = 15\pi/8.$$
So $\overline{z} = I/J = 45/(56(2-\sqrt{2}))$.

Problem 10. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 4$ defined by $z \geqslant 1$, and let $\mathbf{F}(x, y, z) = (-y + z, x + z, z^2)$. Find the value of the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \ d\sigma,$$

(to use two common notations), where n is the unit outward normal vector, $d\sigma$ is an infinitesimal unit of surface area, and $d\mathbf{S} = \mathbf{n}\ \ d\sigma$.

Solution: By Stokes' Theorem, the surface integral equals a line integral:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where ∂S is the boundary of S oriented counterclockwise. Now ∂S is a circle, and we can parametrize it by

$$\mathbf{r}(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 1) \qquad (0 \leqslant t \leqslant 2\pi).$$

Also $\mathbf{F}(\mathbf{r}(t)) = (-\sqrt{3}\sin t + 1, \sqrt{3}\cos t + 1, 1)$ and $\mathbf{r}'(t) = (-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)$, so

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}'(t) = 3 - \sqrt{3}\sin t + \sqrt{3}\cos t.$$

Hence

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (3 - \sqrt{3}\sin t + \sqrt{3}\cos t)\ dt = 6\pi.$$

Problem 11. Show that there are open neighborhoods D and D' of $(0, 0) \in \mathbb{R}^2$ such that if $(a, b) \in D'$ then the system of equations

$$\begin{cases} 2e^x - e^{2y} - e^{4x-7y} = a \\ e^{3x} + 4e^y - 5e^{x+y} = b \end{cases}$$

has a unique solution $(x, y) \in D$.

Solution: Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (2e^x - e^{2y} - e^{4x-7y}, e^{3x} + 4e^y - 5e^{x+y})$. The Jacobian matrix of f is

$$[f'(x, y)] = \begin{pmatrix} 2e^x - 4e^{4x-7y} & -2e^{2y} + 7e^{4x-7y} \\ 3e^{3x} - 5e^{x+y} & 4e^y - 5e^{x+y} \end{pmatrix},$$

so

$$f'(0, 0) = \begin{pmatrix} -2 & 5 \\ -2 & -1 \end{pmatrix},$$

and consequently $\det f'(0, 0) = 12 \neq 0$. Thus $f'(0, 0)$ is invertible Note also that $f(0, 0) = (0, 0)$. Applying the Inverse Function Theorem, we deduce that there are open neighborhoods D and D' of $(0, 0)$ such that f is a C^∞ diffeomorphism of D onto D'. In particular, f is a bijection of D onto D', so given $(a, b) \in D'$ there exists a unique point $(x, y) \in D$ such that $f(x, y) = (a, b)$.

Solution:

Part III.

Solve one of the following three problems.

Problem 12. Let X be a metric space with metric d, and let $\{x_n\}_{n\geqslant 1}$ and $\{y_n\}_{n\geqslant 1}$ be two Cauchy sequences in X. Show that $\{d(x_n, y_n)\}_{n\geqslant 1}$ is a Cauchy sequences of real numbers. Do *not* use the fact that X can be embedded in a complete metric space.

Solution: By the triangle inequality we have

$$d(x_n, y_n) \leqslant d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n),$$

whence

$$d(x_n, y_n) - d(x_m, y_m) \leqslant d(x_n, x_m) + d(y_m, y_n).$$

Similarly

$$d(x_m, y_m) - d(x_n, y_n) \leqslant d(x_n, x_m) + d(y_m, y_n),$$

so we get

$$|d(x_m, y_m) - d(x_n, y_n)| \leqslant d(x_n, x_m) + d(y_m, y_n).$$

Now if $\varepsilon > 0$ is given, there exists N such that if $n, m \geqslant N$ then $d(x_n, x_m) < \varepsilon/2$ and $d(y_m, y_n) < \varepsilon/2$. Hence if $n, m \geqslant N$ then $|d(x_m, y_m) - d(x_n, y_n)| < \varepsilon$, so indeed, $\{d(x_n, y_n)\}_{n\geqslant 1}$ is a Cauchy sequences of real numbers.

Problem 13. Let I be an interval in $\mathbb{R}$ and $\{f_n\}_{n\geqslant 1}$ a sequence of continuous real-valued functions on I which is uniformly convergent to a real-valued function f on I. In the following questions, "prove" means "justify by quoting general theorems," and "give a counterexample" includes proving that your counterexample does what you claim.

(a) If $I = [0, 1]$ then is f uniformly continuous? Prove or give a counterexample.

Solution: Yes, because (i) f, being the limit of a uniformly convergent sequence of continuous functions, is continuous, and (ii) a continuous function on a compact set is uniformly continuous.

(b) If $I = \mathbb{R}$ then is f uniformly continuous? Prove or give a counterexample.

Solution: No, f need not be uniformly continuous. For example, let $f_n(x) = e^x + 1/n$ and $f(x) = e^x$. Then $\{f_n\}$ is uniformly convergent to f, because

$$|f_n(x) - f(x)| = 1/n$$

for all $x \in \mathbb{R}$, whence $|f_n(x) - f(x)| < \varepsilon$ for $n > 1/\varepsilon$. But f is not uniformly continuous. Indeed let $\varepsilon = 1$, and suppose that there exists $\delta > 0$ such that $|e^x - e^y| < 1$ whenever $|x - y| < \delta$. Then $|e^x - e^{x+\delta/2}| < 1$ for all $x \in \mathbb{R}$. So $|1 - e^{\delta/2}| < e^{-x}$ for all $x \in \mathbb{R}$. Taking the limit as $x \to \infty$, we obtain $|1 - e^{\delta/2}| \leqslant 0$, contradicting our assumption that $\delta > 0$.

Problem 14. Define $f, g : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ and $g(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$. Given $c > 0$, consider the surface

$$S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : g(x_1, x_2, \ldots, x_n) = c \text{ and } x_i > 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

(a) Show that f attains a minimum value on S even though S is not compact.

Solution: Choose a number $L > nc^{1/n}$. If $(x_1, x_2, \ldots, x_n) \in S$ and $x_i > L$ for some i then $f(x_1, x_2, \ldots x_n) > L$, whereas $f(c^{1/n}, c^{1/n}, \ldots, c^{1/n}) < L$. Hence writing $x = (x_1, x_2, \ldots, x_n)$, we have

$$\inf\{f(x) : x \in S\} = \inf\{f(x) : x \in S \text{ and } x_i \leqslant L \text{ for } 1 \leqslant i \leqslant n\}.$$

Let X be the set of x on the right-hand side, in other words, the set of $x \in S$ such that $x_i \leqslant L$ for all i. Then X is nonempty because $(c^{1/n}, c^{1/n}, \ldots, c^{1/n}) \in X$. Hence $\{f(x) : x \in X\}$ is nonempty, and since the latter set is bounded below by 0, we see that $\inf\{f(x) : x \in X\}$ does exist. But X is compact (note that the condition $x_i \leqslant L$ amounts to $0 \leqslant x_i \leqslant L$) so $\inf\{f(x) : x \in X\}$ is actually the minimum of f on X and hence on S.

(b) Show that the minimum value of f on S occurs at $(c^{1/n}, c^{1/n}, \ldots, c^{1/n})$ and at no other point.

Solution: If the minimum value of f on S occurs at the point $x \in S$ then by the method of Lagrange multipliers, we have $\nabla f(x) = \lambda \nabla g(x)$. In other words, for $1 \leqslant i \leqslant n$ we have $1 = \lambda c/x_i$. Taking the product of these equations for $1 \leqslant i \leqslant n$, we obtain $1 = \lambda^n c^n/c$, whence $\lambda = c^{-(n-1)/n}$. Substituting this value in $1 = \lambda c/x_i$, we find that $x_i = c^{1/n}$ for all i.

(c) Deduce that $(x_1 x_2 \cdots x_n)^{1/n} \leqslant (x_1 + x_2 + \cdots + x_n)/n$ for all $x_1, x_2, \ldots, x_n > 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Solution: Let $c = x_1 x_2 \cdots x_n$. Then $x \in S$, so we know from (b) that $f(x) \geqslant nc^{1/n}$, the minimum value of f on S. Furthermore, this minimum is attained if and only if $x = (c^{1/n}, c^{1/n}, \ldots, c^{1/n})$. Thus $f(x)/n \geqslant c^{1/n}$, with equality if and only if all the x_i are equal.