Preliminary Exam 2019
Morning Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Consider the function f on $\mathbb{R}$ such that $f(x) = x^2 \ln|x|$ if $x \neq 0$ and $f(0) = 0$. Prove that $f'(0)$ exists but $f''(0)$ does not.

Problem 2. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is twice-differentiable and satisfies the differential equation $y'' + y' + y = 0$. If $y(0) = 1$ and $y(\pi/\sqrt{3}) = 0$ then what is $y(2\pi/\sqrt{3})$? Simplify your answer to the extent possible.

Problem 3. Let $f(x) = e^{x^2}$ and $g(x) = \int_0^{\tan x} f(t)\, dt$. Find $g'(x)$, and then find a constant $c \neq 0$ such that $cg'(x) = h(x)e^{h(x)}$ for some function $h(x)$.

Problem 4. Decide whether each series converges, justifying your answer:
(a) $\sum_{n\geqslant 2}(\cos(1/\log n) - 1)n^{-1}$
(b) $\sum_{n\geqslant 2} \sin(1/\log n)n^{-1}$

Problem 5. Let I be an open interval in $\mathbb{R}$ and f a differentiable function on I, and suppose that f' is identically 0. Prove that f is a constant function. You may quote theorems from calculus that are logically prior to the present assertion.

Part II.

Solve three of the following six problems.

Problem 6. If $n \mapsto a_n$ is a surjective or "onto" map from the set of positive integers to the set $\mathbb{Q} \cap [0, 1]$ of rational numbers between 0 and 1, what is the radius of convergence of the power series $\sum_{n\geqslant 1} a_n x^n$? Prove your answer.

Problem 7. Show that

$$\sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)(2n+2)} = \int_0^1 \left(\frac{1}{\sqrt{2-x^2}} - \frac{x}{1+x^2}\right) dx$$

by computing both sides explicitly.

Problem 8. Find the value of the line integral $\int_C x\, dy - y\, dx$, where C is the curve $r = \cos 2\theta$ for $-\pi/4 \leqslant \theta \leqslant \pi/4$, oriented counterclockwise.

Problem 9. Let D be the solid region inside the cone $z = \sqrt{x^2 + y^2}$ and between the two hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$. Given that D has uniform density, find the "center of mass" or "centroid" of D. You may use symmetry considerations to reduce the amount of computation.

Problem 10. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 4$ defined by $z \geqslant 1$, and let $\mathbf{F}(x, y, z) = (-y + z, x + z, z^2)$. Find the value of the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ \ d\sigma,$$

(to use two common notations), where n is the unit outward normal vector, $d\sigma$ is an infinitesimal unit of surface area, and $d\mathbf{S} = \mathbf{n}\ \ d\sigma$.

Problem 11. Show that there are open neighborhoods D and D' of $(0,0) \in \mathbb{R}^2$ such that if $(a,b) \in D'$ then the system of equations

$$\begin{cases} 2e^x - e^{2y} - e^{4x-7y} = a \\ e^{3x} + 4e^y - 5e^{x+y} = b \end{cases}$$

has a unique solution $(x,y) \in D$.

Part III.

Solve one of the following three problems.

Problem 12. Let X be a metric space with metric d, and let $\{x_n\}_{n\geqslant 1}$ and $\{y_n\}_{n\geqslant 1}$ be two Cauchy sequences in X. Show that $\{d(x_n, y_n)\}_{n\geqslant 1}$ is a Cauchy sequences of real numbers. Do *not* use the fact that X can be embedded in a complete metric space.

Problem 13. Let I be an interval in $\mathbb{R}$ and $\{f_n\}_{n\geqslant 1}$ a sequence of continuous real-valued functions on I which is uniformly convergent to a real-valued function f on I. In the following questions, "prove" means "justify by quoting general theorems," and "give a counterexample" includes proving that your counterexample does what you claim.

(a) If $I = [0,1]$ then is f uniformly continuous? Prove or give a counterexample.

(b) If $I = \mathbb{R}$ then is f uniformly continuous? Prove or give a counterexample.

Problem 14. Define $f, g : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$ and $g(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$. Given $c > 0$, consider the surface

$$S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : g(x_1, x_2, \dots, x_n) = c \text{ and } x_i > 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

(a) Show that f attains a minimum value on S even though S is not compact.

(b) Show that the minimum value of f on S occurs at $(c^{1/n}, c^{1/n}, \dots, c^{1/n})$ and at no other point.

(c) Deduce that $(x_1 x_2 \cdots x_n)^{1/n} \leqslant (x_1 + x_2 + \cdots + x_n)/n$ for all $x_1, x_2, \dots, x_n > 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.