Preliminary Exam 2019
Solutions to Afternoon Exam

Part I.

Solve four of the following five problems.

Problem 1. Find z given that

$$\begin{pmatrix} 2 & 2 & 0 \\ 3 & 4 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Solution: By Cramer's Rule,

$$z = \frac{\det \begin{pmatrix} 2 & 2 & 1 \\ 3 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}}{\det \begin{pmatrix} 2 & 2 & 0 \\ 3 & 4 & 3 \\ 0 & 1 & 2 \end{pmatrix}} = -9/2.$$

Of course the answer can also be found by row reduction.

Problem 2. Find an invertible matrix U such that $U^{-1}AU$ is diagonal, where

$$A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is $x^2 - 1 = (x-1)(x+1)$, so the eigenvalues are ± 1. A row reduction of $A - I$ shows that the vector $(3, -1)$ spans the 1-eigenspace. Similarly, a row reduction of $A + I$ shows that the vector $(1, -1)$ spans the (-1)-eigenspace. Therefore the matrix

$$U = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$$

is the desired change-of-basis matrix.

Problem 3. The linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ is given by

$$T(w, x, y, z) = (w - 3x - y + z, x + 2y + z, w + 5y + 4z).$$

Find a basis for the kernel of T and for the image of T.

Solution: The row reduced upper echelon form of the matrix of T (relative to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^2$) is

$$A = \begin{pmatrix} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the kernel of T consists of vectors (w, x, y, z) satisfying

$$\begin{cases} w + 5y + 4z = 0 \\ x + 2y + z = 0 \end{cases}.$$

Solving for the pivotal variables in terms of the nonpivotal variables, we get $w = -5y - 4z$ and $x = -2y - z$. So the kernel consists of vectors of the form

$$(-5y - 4z, -2y - z, y, z) = y(-5, -2, 1, 0) + z(-4, -1, 0, 1),$$

and we conclude that $\{(-5, -2, 1, 0), (-4, -1, 0, 1)\}$ is a basis for the kernel of T. The transpose of the columns in the original matrix A corresponding to the pivots are $(1, 0, 1)$ and $(-3, 1, 0)$, so $\{(1, 0, 1), (-3, 1, 0)\}$ are a basis for the image of T.

Problem 4. Let Π be the plane $2x - y - z = 0$ in $\mathbb{R}^3$. Find vectors $u_1, u_2 \in \Pi$ such that the formula $\gamma(t) = (\cos t)u_1 + (\sin t)u_2$ parametrizes the circle of radius 1 on Π centered at the origin.

Solution: We take $w_1 = (1, 2, 0)$ and $w_2 = (1, 0, 2)$ as a basis for Π and apply the Gram-Schmidt process orthogonalize this basis. Thus $v_1 = w_1$ and

$$v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1}\, v_1 = (1, 0, 2) - (1/5)(1, 2, 0),$$

so $v_2 = (4/5, -2/5, 2)$. Finally, we normalize the basis v_1, v_2, putting

$$u_1 = \frac{1}{||v_1||}\, v_1 = (1/\sqrt{5}, 2/\sqrt{5}, 0)$$

and

$$u_2 = \frac{1}{||v_2||}\, v_2 = (2/\sqrt{30}, -1/\sqrt{30}, 5/\sqrt{30}).$$

Problem 5. Let G be a nontrivial cyclic group generated by an element g satisfying $g^{1028} = 1$ and $g^{550} = 1$. Find the order of G.

Solution: Since $g^{1028} = 1$ and $g^{550} = 1$ we have $g^{1028-550} = 1$ also, i. e. $g^{478} = 1$. Then $g^{550-478} = 1$, i. e. $g^{72} = 1$. Now $478 - 6 \cdot 72 = 46$, so we have both $g^{78} = 1$ and $g^{46} = 1$, so $g^{78-46} = 1$, i. e. $g^{32} = 1$. Next, since $g^{46} = 1$ and $g^{32} = 1$, we have $g^{46-32} = 1$, i. e. $g^{14} = 1$. Also $32 - 2 \cdot 14 = 4$, so $g^4 = 1$. Finally, since $14 - 3 \cdot 4 = 2$, we get $g^2 = 1$, and since G is nontrivial we conclude that the order of G is 2.

Part II.

Solve three of the following six problems.

Problem 6. Let R and S be rings and let $f, g : R \to S$ be ring homomorphisms. Define $f + g : R \to S$ by the formula $(f + g)(r) = f(r) + g(r)$ and $fg : R \to S$ by the formula $(fg)(r) = f(r)g(r)$.

(a) Is $f + g$ a ring homomorphism? Why or why not?

Solution: No, $f+g$ is not a ring homomorphism. Indeed $(f+g)(1) = 1+1$, which is not equal to 1 unless $S = \{0\}$. Of course one can also observe that $(f + g)(ab) = f(a)f(b) + g(a)g(b)$, which is usually not equal to $(f(a) + g(a))(f(b) + g(b))$ (for example, if $2 \neq 0$ in S, take $a = b = 1$).

(b) Is fg a ring homomorphism? Why or why not?

Solution: Again no. Indeed $(fg)(a + b) = (f(a) + f(b))(g(a) + g(b))$, which is not usually the same thing as $f(a)g(a) + f(b)g(b)$. (Again, if $2 \neq 0$ in S, take $a = b = 1$.)

Problem 7. Let A and B be $n \times n$ matrices with coefficients in $\mathbb{R}$ satisfying $AB = BA$. Suppose that A is symmetric (in other words, A equals its transpose) and has n distinct eigenvalues. Prove that B is symmetric.

Solution: The condition $AB = BA$ implies that if v is in the λ-eigenspace of A then so is Bv. Indeed

$$A(Bv) = B(Av) = B(\lambda v) = \lambda(Bv).$$

But A has n distinct eigenvalues, so the eigenspaces of A are one-dimensional, and consequently the above calculation implies that Bv is a multiple of v. In other words, the eigenspaces of A are also eigenspaces of B. Since nonzero eigenvectors for distinct eigenvalues of A are linearly independent, they form a basis for $\mathbb{R}^n$, and in fact an orthogonal basis, since A is symmetric. We may choose the eigenvectors to have length 1, and then they form an orthonormal basis for $\mathbb{R}^n$. Let U be the $n \times n$ matrix having the elements of this orthonormal basis as its columns. Then $U^{-1}AU$ is diagonal, and $U^{-1}BU$ is diagonal also, because the columns of U are also eigenvectors of B. Write $U^{-1}BU = D$, with D diagonal. Then $B = UDU^{-1}$, and taking transposes of both sides, we have

$$B^{\mathrm{t}} = (U^{-1})^{\mathrm{t}} D U^{\mathrm{t}} = UDU^{-1} = B$$

because $U^{\mathrm{t}} = U^{-1}$ (U is orthogonal).

Problem 8. Let $\mathcal{S}_n$ denote the group of permutations of n elements, and given $\sigma \in \mathcal{S}_n$, define an $n \times n$ matrix $A(\sigma)$ by requiring the entry in the ith row and jth column to be 1 if $j = \sigma(i)$ and 0 otherwise. Prove that $\det(A(\sigma)) = \mathrm{sign}(\sigma)$.

Solution: Write a_{ij} for the entry in the ith row and jth column of $A(\sigma)$. By definition,

$$\det(A(\sigma)) = \sum_{\rho \in \mathcal{S}_n} \mathrm{sign}(\rho) a_{1\rho(1)} a_{2\rho(2)} \cdots a_{n\rho(n)}.$$

Since $a_{ij} = 0$ unless $j = \sigma(i)$, there is only one nonzero term in the above sum, namely the term where $\rho = \sigma$, and this term is $\mathrm{sign}(\sigma)$.

Problem 9. Let $v_1 = (1, 2, 1)$, $v_2 = (3, 0, -1)$, and $v_3 = (-2, -4, 1)$, and put $\mathcal{L} = \{n_1 v_1 + n_2 v_2 + n_3 v_3 : n_j \in \mathbb{Z}\}$. Show that the quotient group $\mathbb{Z}^3/\mathcal{L}$ is cyclic, and find its order.

Solution: We consider the 3×3 matrix having v_1, v_2, and v_3 as columns. Using row and column operations over the integers, we can put this matrix in diagonal form, where the diagonal entries are 1, 1, and 18. Therefore

$$\mathbb{Z}^3/\mathcal{L} \cong \mathbb{Z}^3/(\mathbb{Z} \oplus \mathbb{Z} \oplus 18\mathbb{Z}) \cong \mathbb{Z}/18\mathbb{Z}.$$

So $\mathbb{Z}^3/\mathcal{L}$ is cyclic of order 18.

Problem 10. Let F be a field, and consider the ring $R = F[x, y]$. Write (a, b) for the ideal of R generated by $a, b \in R$.

(a) If $F = \mathbb{Q}$ is $R/(x + y, x^2 + y^2)$ finite-dimensional as a vector space over $\mathbb{Q}$? If so, what is its dimension?

Solution: Consider the ring homomorphism

$$\mathbb{Q}[x] \to \mathbb{Q}[x, y]/(x + y, x^2 + y^2)$$

sending every element to its coset modulo $(x + y, x^2 + y^2)$. This map is surjective because the coset of y is the coset of $-x$ and consequently the coset of a polynomial $f(x, y)$ is the coset of $f(x, -x)$. Furthermore the coset of $x^2 + y^2$ in $\mathbb{Q}[x, y]/(x + y)$ is the coset of $2x^2$, so x^2 is 0 in $\mathbb{Q}[x, y]/(x + y, x^2 + y^2)$. So (x^2) is contained in the kernel of the above map, and in fact (x^2) is the precise kernel, because on elements of the form $a + bx$ $(a, b \in \mathbb{Q})$ the above map is injective. We conclude that

$$\mathbb{Q}[x]/(x^2) \cong \mathbb{Q}[x, y]/(x + y, x^2 + y^2),$$

whence the dimension of the right-hand side is 2.

(b) If $F = \mathbb{F}_2$ is $R/(x + y, x^2 + y^2)$ finite-dimensional as a vector space over $\mathbb{F}_2$? If so, what is its dimension? (Here $\mathbb{F}_2$ is the field with 2 elements.)

Solution: The argument in (a) carries over without change until we assert that the coset of $x^2 + y^2$ is $2x^2$: At that point we are saying that the coset of $x^2 + y^2$ is 0. Thus $\mathbb{F}_2[x, y]/(x + y, x^2 + y^2) \cong \mathbb{F}_2[x, y]/(y) \cong \mathbb{F}_2[x]$, so the dimension of $\mathbb{F}_2[x, y]/(x + y, x^2 + y^2)$ is infinite.

Problem 11. By the *minimal polynomial* of a square matrix A we mean the monic polynomial $f(x)$ of smallest positive degree such that $f(A) = 0$. Also, given a square matrix A with coefficients in $\mathbb{C}$, we say that A is *nilpotent* if $A^n = 0$ for some $n \geqslant 1$, and for A nilpotent we put

$$\exp(A) = \sum_{j \geqslant 0} A^j/j!.$$

If x^m is the minimal polynomial of A then what is the minimal polynomial of $\exp(A)$? Justify your answer.

Solution: The minimal polynomial of A and of $\exp(A)$ depends only on the Jordan normal form of A, so we may assume that A is in Jordan normal form. Then A is a diagonal array of nilpotent Jordan blocks N with 1's on the superdiagonal and 0's elsewhere (i. e. the entry in the ith row and j column is 1 if $j = i + 1$ and 0 otherwise). Furthermore, the minimal polynomial of A is x^m, where m is the size of the largest Jordan block in A. One readily sees that $\exp(A)$ is a diagonal array of the blocks $\exp(N)$ where N runs over the Jordan blocks in A, so we are reduced to showing that if N is a nilpotent Jordan block of size m, then the minimal polynomial of $\exp(N)$ is $(x - 1)^m$. Now

$$\exp(N) = I + N + \frac{1}{2!}N^2 + \cdots + \frac{1}{m!}N^m = I + N + B,$$

where the entry in the ith row and jth column of B is 0 unless $j \geqslant i + 2$. Thus the binomial theorem gives $(\exp(N) - I)^k = N^k + C$, where the entry in the ith row and jth column of C is 0 unless $j \geqslant i + k + 1$. Since the entry in the ith row and jth column of N^k is 0 unless $j = i + k$, we deduce that $(\exp(N) - I)^k = 0$ if and only if $k \geqslant m$, as claimed.

Part III.

Solve one of the following three problems.

Problem 12. Let d and d' be positive integers such that $[\mathbb{Q}(\sqrt{d}, \sqrt{d'}) : \mathbb{Q}] = 4$. Put $\alpha = \sqrt{d} + \sqrt{d'}$. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{d}, \sqrt{d'})$.

Solution: By the binomial theorem, we have

$$\alpha^3 = d\sqrt{d} + 3d\sqrt{d'} + 3d'\sqrt{d} + d'\sqrt{d'},$$

and consequently,

$$\begin{cases} \alpha = \sqrt{d} + \sqrt{d'} \\ \alpha^3 = (d + 3d')\sqrt{d} + (3d + d')\sqrt{d'}. \end{cases}$$

Using Cramer's Rule, we deduce that

$$\sqrt{d} = \frac{\det\begin{pmatrix}\alpha & 1\\ \alpha^3 & (3d+d')\end{pmatrix}}{\det\begin{pmatrix}1 & 1\\ (d+3d') & (3d+d')\end{pmatrix}} = \frac{\alpha(3d+d')-\alpha^3}{2(d-d')}$$

and

$$\sqrt{d'} = \frac{\det\begin{pmatrix}1 & \alpha\\ (d+3d') & \alpha^3\end{pmatrix}}{\det\begin{pmatrix}1 & 1\\ (d+3d') & (3d+d')\end{pmatrix}} = \frac{\alpha^3-\alpha(d+3d')}{2(d-d')}.$$

So $\mathbb{Q}(\alpha)$ contains, and hence equals, $\mathbb{Q}(\sqrt{d},\sqrt{d'})$.

Problem 13. Let G be a group. In each case, say whether the given condition on G implies that G is abelian, justifying your answer either with a proof or a counterexample.

(i) The function $f : G\times G \to G$ given by $f(a,b) = ab$ is a group homomorphism.

Solution: Yes, this does imply that G is abelian. Since f is a homomorphism, we have

$$f(a,b)f(a^{-1},b^{-1}) = f(aa^{-1},bb^{-1}) = f(1,1) = 1$$

for all $a,b\in G$. This means that $(ab)(a^{-1}b^{-1}) = 1$. Multiplying by ba on the right, we obtain $ab = ba$.

(ii) G has a normal subgroup H such that G/H is cyclic.

Solution: No, this does not imply that G is abelian. Take $G = \mathcal{S}_3$, the group of permutations of a set of three elements. Let H be the unique subgroup of order 3, which is therefore normal. Then G/H has order 2 and is therefore cyclic, but G is not abelian.

(iii) G has a normal subgroup H such that G/H is cyclic and $gh = hg$ for all $g\in G$ and $h\in H$.

Solution: Yes, this does imply that G is abelian. Since G/H is cyclic choose $c\in G$ so that cH is a generator of G/H. Then every element of G has the form c^jh for some $j\in\mathbb{Z}$ and some $h\in H$. Given two elements of G, say $a = c^jh$ and $b = c^{j'}h'$, we then have

$$ab = (c^jh)(c^{j'}h') = c^{j+j'}(hh') = c^{j'+j}(h'h) = (c^{j'}h')(c^jh) = ba,$$

so G is abelian.

Problem 14. Let A be a 3×3 matrix with coefficients in a field F, and suppose that $A^3 = I$, the identity matrix. Write $\mathbb{F}_p$ for the field with p elements, where p denotes a prime. For the purposes of this problem, two matrices in Jordan canonical form are considered the same if they differ simply in the order in which the Jordan blocks are listed.

(a) Suppose $F = \mathbb{F}_3$. List the possibilities for the Jordan canonical form of A.

Solution: Since $A^3 - I = (A-I)^3$ the Jordan canonical form of A is either I or

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Suppose $F = \mathbb{F}_7$. How many possibilities are there for the Jordan canonical form of A? Justify your answer.

Solution: Since $x^3 - 1$ has three distinct roots, say $1, \omega, \omega^2 \in \mathbb{F}_7$, the Jordan form of A is diagonal. Let a, b, and c represent the number of times that 1, ω, and ω^2 appear on the diagonal. Then $a, b, c \geqslant 0$ and $a + b + c = 3$. Thus the number of possible Jordan canonical forms is

$$\binom{5}{3} = 10.$$

(c) Suppose $F = \mathbb{R}$ and $A \neq I$. Explain why there is no invertible matrix U with coefficients in $\mathbb{R}$ such that UAU^{-1} is in Jordan canonical form.

Solution: As in (b), the fact that $A^3 - I$ has no repeated roots tells us that the Jordan form of A is diagonal, and since $A \neq I$ at least one of the diagonal entries is $e^{\pm 2\pi i/3}$. Hence there cannot be a matrix U as above such that UAU^{-1} is diagonal, because at least one of the diagonal entries would have to be $e^{\pm 2\pi i/3}$, whereas the coefficients of UAU^{-1} are real.