Preliminary Exam 2019
Afternoon Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Find z given that

$$\begin{pmatrix} 2 & 2 & 0 \\ 3 & 4 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Problem 2. Find an invertible matrix U such that $U^{-1}AU$ is diagonal, where

$$A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

Problem 3. The linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ is given by

$$T(w, x, y, z) = (w - 3x - y + z, x + 2y + z, w + 5y + 4z).$$

Find a basis for the kernel of T and for the image of T.

Problem 4. Let Π be the plane $2x - y - z = 0$ in $\mathbb{R}^3$. Find vectors $u_1, u_2 \in \Pi$ such that the formula $\gamma(t) = (\cos t)u_1 + (\sin t)u_2$ parametrizes the circle of radius 1 on Π centered at the origin.

Problem 5. Let G be a nontrivial cyclic group generated by an element g satisfying $g^{1028} = 1$ and $g^{550} = 1$. Find the order of G.

Part II.

Solve three of the following six problems.

Problem 6. Let R and S be rings and let $f, g : R \to S$ be ring homomorphisms. Define $f + g : R \to S$ by the formula $(f + g)(r) = f(r) + g(r)$ and $fg : R \to S$ by the formula $(fg)(r) = f(r)g(r)$.
(a) Is $f + g$ a ring homomorphism? Why or why not?
(b) Is fg a ring homomorphism? Why or why not?

Problem 7. Let A and B be $n \times n$ matrices with coefficients in $\mathbb{R}$ satisfying $AB = BA$. Suppose that A is symmetric (in other words, A equals its transpose) and has n distinct eigenvalues. Prove that B is symmetric.

Problem 8. Let $\mathcal{S}_n$ denote the group of permutations of n elements, and given $\sigma \in \mathcal{S}_n$, define an $n \times n$ matrix $A(\sigma)$ by requiring the entry in the ith row and jth column to be 1 if $j = \sigma(i)$ and 0 otherwise. Prove that $\det(A(\sigma)) = \text{sign}(\sigma)$.

Problem 9. Let $v_1 = (1, 2, 1)$, $v_2 = (3, 0, -1)$, and $v_3 = (-2, -4, 1)$, and put $\mathcal{L} = \{n_1v_1 + n_2v_2 + n_3v_3 : n_j \in \mathbb{Z}\}$. Show that the quotient group $\mathbb{Z}^3/\mathcal{L}$ is cyclic, and find its order.

Problem 10. Let F be a field, and consider the ring $R = F[x, y]$. Write (a, b) for the ideal of R generated by $a, b \in R$.
(a) If $F = \mathbb{Q}$ is $R/(x + y, x^2 + y^2)$ finite-dimensional as a vector space over $\mathbb{Q}$? If so, what is its dimension?
(b) If $F = \mathbb{F}_2$ is $R/(x + y, x^2 + y^2)$ finite-dimensional as a vector space over $\mathbb{F}_2$? If so, what is its dimension? (Here $\mathbb{F}_2$ is the field with 2 elements.)

Problem 11. By the *minimal polynomial* of a square matrix A we mean the monic polynomial $f(x)$ of smallest positive degree such that $f(A) = 0$. Also, given a square matrix A with coefficients in $\mathbb{C}$, we say that A is *nilpotent* if $A^n = 0$ for some $n \geqslant 1$, and for A nilpotent we put

$$\exp(A) = \sum_{j \geqslant 0} A^j / j!.$$

If x^m is the minimal polynomial of A then what is the minimal polynomial of $\exp(A)$? Justify your answer.

Part III.

Solve one of the following three problems.

Problem 12. Let d and d' be positive integers such that $[\mathbb{Q}(\sqrt{d}, \sqrt{d'}) : \mathbb{Q}] = 4$. Put $\alpha = \sqrt{d} + \sqrt{d'}$. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{d}, \sqrt{d'})$.

Problem 13. Let G be a group. In each case, say whether the given condition on G implies that G is abelian, justifying your answer either with a proof or a counterexample.

(i) The function $f : G \times G \to G$ given by $f(a, b) = ab$ is a group homomorphism.

(ii) G has a normal subgroup H such that G/H is cyclic.

(iii) G has a normal subgroup H such that G/H is cyclic and $gh = hg$ for all $g \in G$ and $h \in H$.

Problem 14. Let A be a 3×3 matrix with coefficients in a field F, and suppose that $A^3 = I$, the identity matrix. Write $\mathbb{F}_p$ for the field with p elements, where p denotes a prime. For the purposes of this problem, two matrices in Jordan canonical form are considered the same if they differ simply in the order in which the Jordan blocks are listed.

(a) Suppose $F = \mathbb{F}_3$. List the possibilities for the Jordan canonical form of A.

(b) Suppose $F = \mathbb{F}_7$. How many possibilities are there for the Jordan canonical form of A? Justify your answer.

(c) Suppose $F = \mathbb{R}$ and $A \neq I$. Explain why there is no invertible matrix U with coefficients in $\mathbb{R}$ such that UAU^{-1} is in Jordan canonical form.