# Persistent Homology

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# Overview

Topics:

- Simplicies

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- Filtrations

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- Persistence

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- Simplicies
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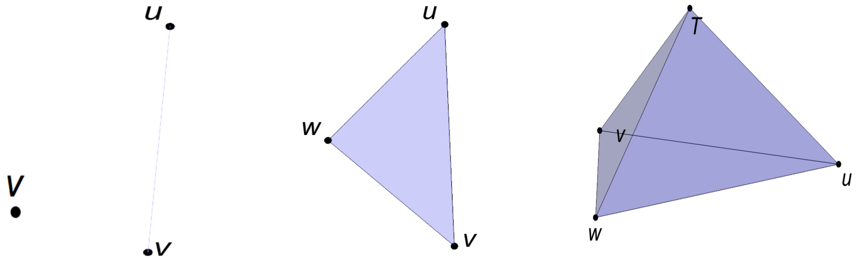
Applications:

- Singh, Gurjeet et al. Topological analysis of population activity in visual cortex *Journal of vision* vol. 8,8 11.1-18. 30 Jun. 2008
- Lockwood, Svetlana and Krishnamoorthy, B. (2015). Topological Features in Cancer Gene Expression Data. Biocomputing 2015. November 2014, 108-119
- Silva, Vin and Ghrist, Robert. (2007). Coverage in sensor networks via persistent homology. Algebraic and Geometric Topology. 7. 10.2140/agt.2007.7.339

# Simplices

## Definition

A *k-simplex* is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$.




A *simplicial complex* is a finite collection of simplices $K$ such that $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$, and $\sigma, \sigma_0 \in K$ implies $\sigma \cap \sigma_0$ is either empty or a face of both.

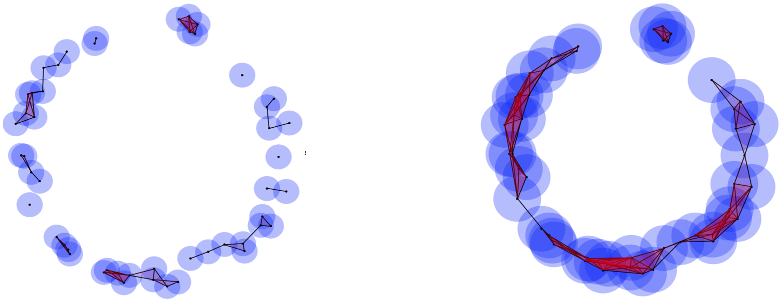
# Vietoris-Rips Complex

## Definition

A *filtration* is a sequence of simplicial complexes $\{K_i\}$ such that $\sigma \in K_i$ only if each face of $\sigma$ is in some $K_j$ for $j \leq i$.

For each point in a finite set $S$ the *Vietoris-Rips Complex* for a fixed $r$ is the simplicial complex

$$\{\sigma \in S \mid \operatorname{diam}(\sigma) \leq 2r\}.$$



# Vietoris-Rips Complex

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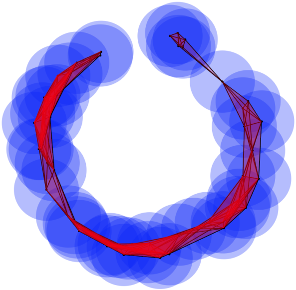
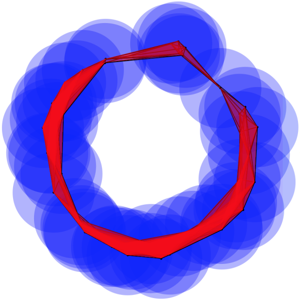




# Simplicial Homology

## Chains

### Definition

Given a simpicial complex $K$, a $p$-chain over $\mathbb{Z}/2\mathbb{Z}$ is a formal sum of $p$-dimensional simplices

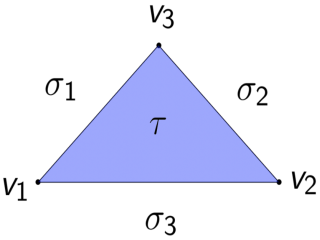
$$\sum_{i=1}^{n} a_i \sigma_i$$

where $a_i \in \{0, 1\}$ and $\sigma_i$ is a $p$-dimensional simplex in $K$.

### Definition

The boundary operator $\partial$ is a linear operator that sends a $p$-dimensional simplex to the formal sum of its $(p-1)$-dimensional faces.

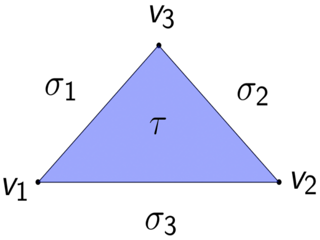
# Simplicial Homology




$$\partial(\tau) = \sigma_1 + \sigma_2 + \sigma_3$$
$$\partial(\sigma_1 + \sigma_2 + \sigma_3) =$$
$$\partial(\sigma_1) + \partial(\sigma_2) + \partial(\sigma_3) =$$
$$(v_1 + v_3) + (v_2 + v_3) + (v_1 + v_2) = 0$$

- A $p$-cycle is a $p$-chain $\sigma$ such that $\partial(\sigma) = 0$. We define $Z_p$ as the group of all $p$-cycles.

# Simplicial Homology

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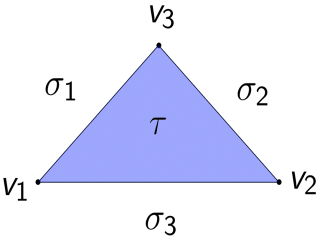
- A $p$-cycle is a $p$-chain $\sigma$ such that $\partial(\sigma) = 0$. We define $\mathrm{Z}_p$ as the group of all $p$-cycles.
- The boundary group $\mathrm{B}_p$ consists of all $p$-chains $\sigma$ such that there exists some $(p+1)$-chain $\tau$ with $\partial(\tau) = \sigma$.

# Simplicial Homology




$$
\begin{gathered}
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- A $p$-cycle is a $p$-chain $\sigma$ such that $\partial(\sigma) = 0$. We define $\mathrm{Z}_p$ as the group of all $p$-cycles.
- The boundary group $\mathrm{B}_p$ consists of all $p$-chains $\sigma$ such that there exists some $(p+1)$-chain $\tau$ with $\partial(\tau) = \sigma$.
- We define the $p$-th homology group as $\mathrm{H}_p(K) = \mathrm{Z}_p(K)/\mathrm{B}_p(K)$.

# Persistence

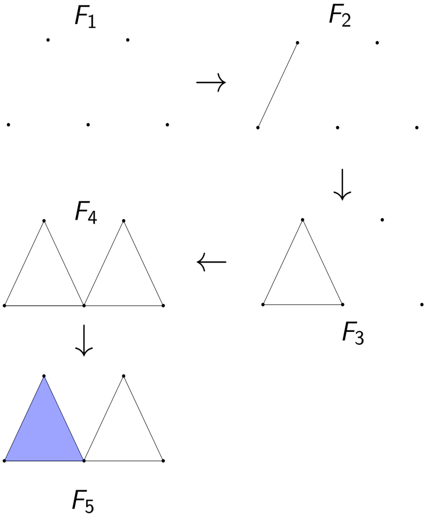
For a filtration $\{F_i\}$, the inclusion maps $f^{i,j} : F_i \to F_j$ are simplicial maps so they induce homomorphisms $f_p^{i,j} : \mathrm{H}_p(F_i) \to \mathrm{H}_p(F_j)$ for each dimension $p$. This produces a sequence

$$\mathrm{H}_p(F_1) \to \mathrm{H}_p(F_2) \to \cdots \to \mathrm{H}_p(F_n) = \mathrm{H}_p(F).$$

- The *p-th persistent homology groups* are the images of the $f_p^{i,j}$ i.e. $\mathrm{H}_p^{i,j} = \mathrm{im} f_p^{i,j}$.
- A class of $p$-cycles $c \in H_p(F_i)$ is *born* at $F_i$ if $c \notin \mathrm{H}_p^{i-1,i}$.
- If a class $c$ is born entering $F_i$ then it *dies entering* $F_j$ if f it merges with an older class as we go from $F_{j-1}$ to $F_j$.
- If $c$ is born at $F_i$ and dies entering $F_j$ then we define the *persistence* of $c$ as $\mathrm{pers}(c) = j - i$.
- We define the *Betti Number* of $\mathrm{H}_p^{i,j}$ as $\beta_p^{i,j} = \mathrm{rank}\ \mathrm{H}_p^{i,j}$.

# Persistence

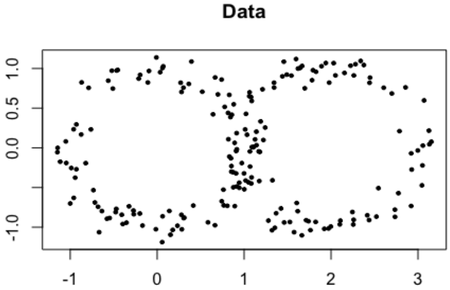
## Example




- A cycle is born at $F_3$ and at $F_4$ and one of them dies entering $F_5$.
- $\mathrm{H}_0(F_1) = (\mathbb{Z}/2\mathbb{Z})^5$
- $\mathrm{H}_1(F_3) = \mathrm{H}_1(F_5) = \mathbb{Z}/2\mathbb{Z}$
- $\mathrm{H}_1(F_4) = (\mathbb{Z}/2\mathbb{Z})^2$
- $\mathrm{H}_1^{3,4} = \mathrm{H}_1^{4,5} = \mathbb{Z}/2\mathbb{Z}$
- $\mathrm{H}_1^{3,5} = \mathrm{H}_1^{2,i} = 0$
- $\mathrm{H}_0^{1,2} = (\mathbb{Z}/2\mathbb{Z})^4$.
- $\beta_1^{3,4} = \beta_1^{4,5} = 1$
- $\beta_1^{3,5} = 0$

# Sampling Points From A Circle

Below are set of data points uniformly sampled points from a wedge sum of circles with uniform noise added. The persistence diagrams are computed using the ‘Statistical Tools for Topological Data Analysis’ R package.



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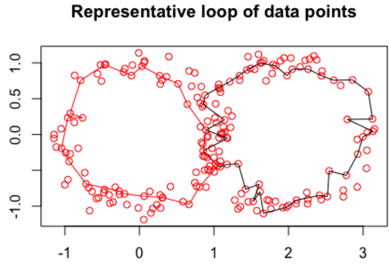
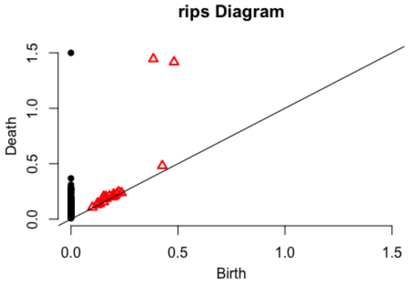
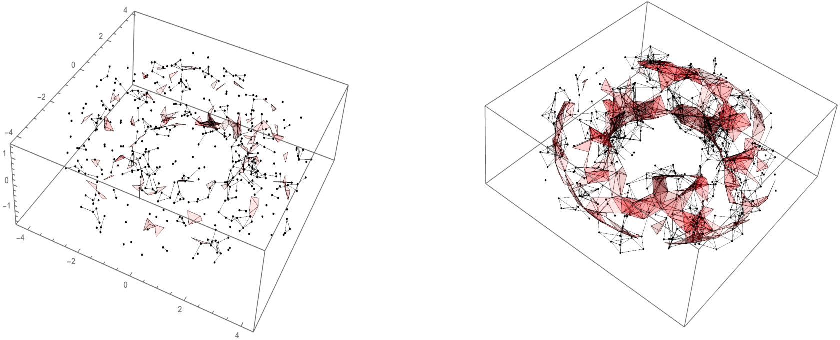


# Sampling From A Torus

## Rips Complex

Rips complex from points sampled from torus with added noise.



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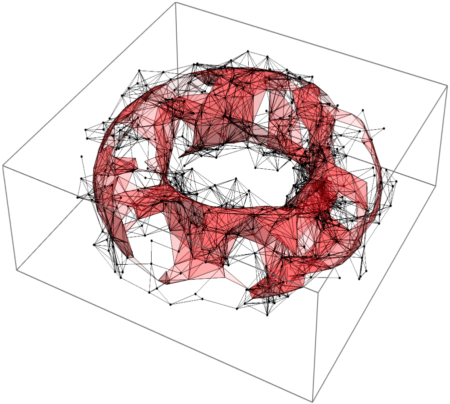
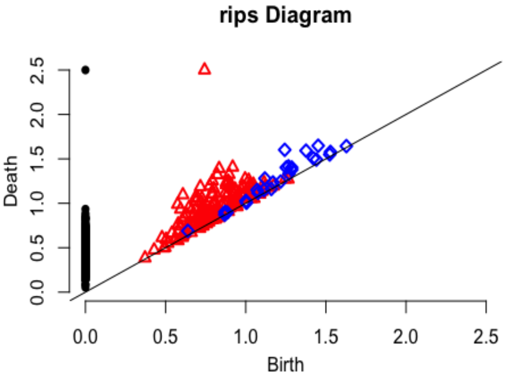


# Sampling From A Torus

## Persistence Diagram



Barcode diagrams display the lifetime of each homology generator. Here 1-cycles are shown in red and 2-cycles are shown in blue.