Persistent Homology

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Overview

Topics:

- Simplicies
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- Filtrations
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- Homology
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- Simplicies
- Filtrations
- Homology
- Persistence

Applications:
Overview

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- Persistence

Applications:
**Simplices**

**Definition**

A \( k \)-simplex is the convex hull of \( k + 1 \) affinely independent points in \( \mathbb{R}^d \).

A simplicial complex is a finite collection of simplices \( K \) such that \( \sigma \in K \) and \( \tau \leq \sigma \) implies \( \tau \in K \), and \( \sigma, \sigma_0 \in K \) implies \( \sigma \cap \sigma_0 \) is either empty or a face of both.
Definition

A \textit{filtration} is a sequence of simplicial complexes \( \{K_i\} \) such that \( \sigma \in K_i \) only if each face of \( \sigma \) is in some \( K_j \) for \( j \leq i \).

For each point in a finite set \( S \) the \textit{Vietoris-Rips Complex} for a fixed \( r \) is the simplicial complex

\[
\{ \sigma \in S \mid \text{diam}(\sigma) \leq 2r \}.
\]
**Vietoris-Rips Complex**

**Definition**

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For each point in a finite set \( S \) the *Vietoris-Rips Complex* for a fixed \( r \) is the simplicial complex

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Simplicial Homology

Chains

Definition
Given a simplicial complex $K$, a $p$-chain over $\mathbb{Z}/2\mathbb{Z}$ is a formal sum of $p$-dimensional simplices

$$
\sum_{i=1}^{n} a_i \sigma_i
$$

where $a_i \in \{0, 1\}$ and $\sigma_i$ is a $p$-dimensional simplex in $K$.

Definition
The boundary operator $\partial$ is a linear operator that sends a $p$-dimensional simplex to the formal sum of its $(p-1)$-dimensional faces.
Simplicial Homology

A $p$-cycle is a $p$-chain $\sigma$ such that $\partial(\sigma) = 0$. We define $Z_p$ as the group of all $p$-cycles.

\[
\partial(\tau) = \sigma_1 + \sigma_2 + \sigma_3 \\
\partial(\sigma_1 + \sigma_2 + \sigma_3) = \\
\partial(\sigma_1) + \partial(\sigma_2) + \partial(\sigma_3) = \\
(\nu_1 + \nu_3) + (\nu_2 + \nu_3) + (\nu_1 + \nu_2) = 0
\]
A $p$-cycle is a $p$-chain $\sigma$ such that $\partial(\sigma) = 0$. We define $Z_p$ as the group of all $p$-cycles.

The boundary group $B_p$ consists of all $p$-chains $\sigma$ such that there exists some $(p+1)$-chain $\tau$ with $\partial(\tau) = \sigma$. 

$$
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$$

$$
\partial(\sigma_1 + \sigma_2 + \sigma_3) =
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$$

$$
(v_1 + v_3) + (v_2 + v_3) + (v_1 + v_2) = 0
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The boundary group $B_p$ consists of all $p$-chains $\sigma$ such that there exists some $(p+1)$-chain $\tau$ with $\partial(\tau) = \sigma$.

We define the $p$-th homology group as $H_p(K) = Z_p(K)/B_p(K)$. 

\[ \partial(\tau) = \sigma_1 + \sigma_2 + \sigma_3 \]
\[ \partial(\sigma_1 + \sigma_2 + \sigma_3) = \]
\[ \partial(\sigma_1) + \partial(\sigma_2) + \partial(\sigma_3) = \]
\[ (v_1 + v_3) + (v_2 + v_3) + (v_1 + v_2) = 0 \]
Persistence

For a filtration \( \{F_i\} \), the inclusion maps \( f^{i,j} : F_i \rightarrow F_j \) are simplicial maps so they induce homomorphisms \( f_{p}^{i,j} : H_p(F_i) \rightarrow H_p(F_j) \) for each dimension \( p \). This produces a sequence

\[
H_p(F_1) \rightarrow H_p(F_2) \rightarrow \cdots \rightarrow H_p(F_n) = H_p(F).
\]

- The \( p \)-th persistent homology groups are the images of the \( f_{p}^{i,j} \) i.e. \( H_{p}^{i,j} = \text{im} f_{p}^{i,j} \).
- A class of \( p \)-cycles \( c \in H_p(F_i) \) is born at \( F_i \) if \( c \notin H_{p}^{i-1,i} \).
- If a class \( c \) is born entering \( F_i \) then it dies entering \( F_j \) if it merges with an older class as we go from \( F_{j-1} \) to \( F_j \).
- If \( c \) is born at \( F_i \) and dies entering \( F_j \) then we define the persistence of \( c \) as \( \text{pers}(c) = j - i \).
- We define the Betti Number of \( H_{p}^{i,j} \) as \( \beta_{p}^{i,j} = \text{rank} H_{p}^{i,j} \).
A cycle is born at $F_3$ and at $F_4$ and one of them dies entering $F_5$.

- $H_0(F_1) = (\mathbb{Z}/2\mathbb{Z})^5$
- $H_1(F_3) = H_1(F_5) = \mathbb{Z}/2\mathbb{Z}$
- $H_1(F_4) = (\mathbb{Z}/2\mathbb{Z})^2$
- $H_1^{3,4} = H_1^{4,5} = \mathbb{Z}/2\mathbb{Z}$
- $H_1^{3,5} = H_1^{2,i} = 0$
- $H_0^{1,2} = (\mathbb{Z}/2\mathbb{Z})^4$
- $\beta_1^{3,4} = \beta_1^{4,5} = 1$
- $\beta_1^{3,5} = 0$
Sampling Points From A Circle

Below are set of data points uniformly sampled points from a wedge sum of circles with uniform noise added. The persistence diagrams are computed using the ‘Statistical Tools for Topological Data Analysis’ R package.
Sampling Points From A Circle

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**Representative loop of data points**

**rips Diagram**
Rips complex from points sampled from torus with added noise.
Sampling From A Torus
Rips Complex

Rips complex from points sampled from torus with added noise.
Sampling From A Torus

Persistence Diagram

[Image of a persistence diagram with axes labeled Birth and Death, showing red and blue data points with a diagonal line indicating persistence intervals.]

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Barcode diagrams display the lifetime of each homology generator. Here 1-cycles are shown in red and 2-cycles are shown in blue.