Combinatorial Game Theory

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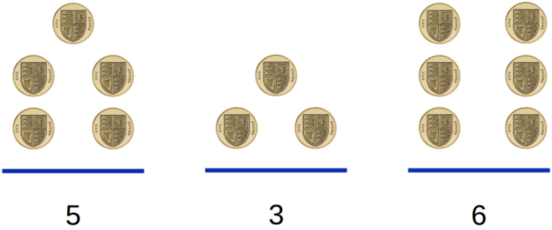
Let's Play a Game



Rules of the Game:

- On your turn, you can take any number of coins provided they all come from the same heap
- The person who takes the last coin from all the heaps, wins. i.e. there are no coins left

Outline

What is a Game?

8 rules for Combinatorial Games:

1. Two players, Left and Right
2. Finite many positions, including a starting position
3. Clearly defined rules for players to move from the current position to its options
4. Left and Right alternate in turns
5. Complete information
6. No chance moves
7. In the normal play convention **the player unable to move loses**
8. There exists an ending condition, i.e. game ends when either player is unable to move

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G is a number when $a, b, c, ... \leq d, e, f, ...$

Winning Strategies

If $G > 0$, there is a winning strategy for Left.

If $G < 0$, there is a winning strategy for Right.

If $G = 0$, there is a winning strategy for the second player.

$G||0$, there is a winning strategy for the first player.

Common Games

$$0 = \{|\}$$

which is the ending condition

$$n + 1 = \{n|\}$$

which is a Left player win

$$-n + 1 = \{| - n\}$$

which is a Right player win

$$* = \{0|0\}$$

which is a first player win

Simplifying Games

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If Left has options {-3, $\frac{1}{2}$, 7}, options -3 and $\frac{1}{2}$ would be dominated by 7.

If Right had options {-6, -2, 0}, options 0 and 2 would be dominated by -6.

Simplicity Rule

Definition

If all the options for both players of some game G are **numbers**, and every Left option is **less than or equal** to every Right option, then G itself is a number, namely the **simplest number greater than** every Left option and **less than** every Right option.

$$G = \frac{2p+1}{2^{n+1}} = \{\frac{2p}{2^{n+1}} | \frac{2p+2}{2^{n+1}}\} = \{\frac{p}{2^n} | \frac{p+1}{2^n}\}$$

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A whole number is simpler than $\frac{a}{2}$.
$\frac{a}{2}$ is simpler than $\frac{a}{4}$ and so on.
The smallest number that satisfies the above is the simplest number.

Examples

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Game of Nim

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E.g. The game we played was:

$$G = \{\star 1, \star 2, \star 3, \star 4, \star 5, \star 6 \mid \star 1, \star 2, \star 3, \star 4, \star 5, \star 6\}$$

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Nimbers are their own negatives.
i.e. $\star 1 + \star 2 + \star 3 = 0$ is the same as $\star 1 + \star 3 = \star 2$

Facts:

- A single non-empty heap is fuzzy.
- Two equal-sized heaps is a zero game.
- Two unequal-sized heaps is fuzzy.

3-Heap Game

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- A single non-empty heap is fuzzy.
- Two equal-sized heaps is a zero game.
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How to lose in a 3-heap game:

- Equalize two heaps
- Empty a heap

Nim Addition Table

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

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Break each nimber into distinct parts whose sizes are powers of 2, beginning with the largest size.

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$\star 10 + \star 6 = \star 8 + \star 2 + \star 4 + \star 2$

Nim Addition

$\star 10 + \star 6 = 12$
Break each nimber into distinct parts whose sizes are powers of 2, beginning with the largest size.
$\star 10 + \star 6 = \star 8 + \star 2 + \star 4 + \star 2$
$\star 10 + \star 6 = \star 8 + \star 4 = 12$

Conclusion

A game is a number if:

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G will be the simplest number greater than every Left option and less than every Right option.

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In the game of Nim:

- Nimbers are their own negatives
- In a 3-heap game, the player who equalizes two heaps or empties a heap loses
- To add nimbers, break each nimber into distinct parts whose sizes are powers of 2 and use arithmetic addition

References

Berlekamp, Elwyn R, John H Conway, and Richard K Guy. *Winning Ways for Your Mathematical Plays*. AK Peters/CRC Press, 2018.

Conway, John H. *On numbers and games*. AK Peters/CRC Press, 2000.

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Questions?