Preliminary Exam 2017
Solutions to Morning Exam

Part I.

Solve four of the following five problems.

Problem 1. Verify that

$$\int_0^{2\pi} \cos^2 x\, dx = 6 \sum_{n \geqslant 0} (-1)^n \frac{3^{-(2n+1)/2}}{2n+1}$$

by computing both sides.

Solution: The left-hand side can be computed using either integration by parts $(u = \cos x, v = \sin x)$, or the identity $\cos^2 x = (1 + \cos(2x))/2$, the antiderivatives obtained being respectively $(x + \cos x \sin x)/2$ and $x/2 + \sin(2x)/4$ up to an additive constant. Thus the left-hand side is π. The right-hand side is $6f(1/\sqrt{3})$, where

$$f(x) = \sum_{n \geqslant 0} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Differentiating, we find

$$f'(x) = \sum_{n \geqslant 0} (-1)^n x^{2n} = \frac{1}{1+x^2},$$

so $f(x) = \int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$, and since $f(0) = 0$ we get $C = 0$. Thus the right-hand side is $6\tan^{-1}(1/\sqrt{3}) = \pi$, which is the left-hand side.

Problem 2. Suppose that $y = y(t)$ is a differentiable function on $\mathbb{R}$ satisfying $y'(t) - \sin(2t)y(t) = e^{\sin^2 t}$. If $y(0) = 0$ what is $y(\pi)$?

Solution: Since $-\int \sin(2t)\, dt = \cos^2 t + C$, we multiply both sides of the differential equation by $e^{\cos^2 t}$, obtaining

$$\frac{d}{dt}(y(t)e^{\cos^2 t}) = e.$$

Therefore $y(t)e^{\cos^2 t} = et + C$ and $y(t) = (et + C)e^{-\cos^2 t}$. Setting $t = 0$ and using $y(0) = 0$, we obtain $C = 0$, so $y(t) = ete^{-\cos^2 t}$ or in other words $y(t) = te^{\sin^2 t}$. Putting $t = \pi$ gives $y(\pi) = \pi$.

Problem 3. Let D be the upper half of the standard unit ball in $\mathbb{R}^3$, defined by the inequalities $x^2 + y^2 + z^2 \leqslant 1$ and $z \geqslant 0$. Assuming that D is of constant density, find the "centroid" or "center of mass" of D. You may use symmetry considerations and a standard volume formula to reduce the amount of calculation.

Solution: Write the centroid as $(\overline{x}, \overline{y}, \overline{z})$. Symmetry considerations (i. e. the invariance of D under rotation about the z-axis) give $\overline{x} = \overline{y} = 0$, and since the volume of D is $(4\pi/3)/2 = 2\pi/3$, we have

$$\overline{z} = \frac{3}{2\pi} \int\int\int_D z\, dx\, dy\, dz.$$

Switching to spherical coordinates, we find that $\overline{z}$ is

$$\frac{3}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho\cos\phi)(\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta = 3/8.$$

Problem 4. Let f be a continuous function on $\mathbb{R}$, define $F(x) = \int_0^x f(t)\ dt$, and suppose that a and b are real numbers with $a < b$. Apply the Mean Value Theorem to F on $[a, b]$, simplifying your answer and expressing the result entirely in terms of f. Then interpret the result geometrically.

Solution: The Mean Value Theorem asserts that $F(b) - F(a) = F'(c)(b - a)$ for some $c \in (a, b)$. By the Fundamental Theorem of Calculus, $F' = f$, so we get

$$\int_a^b f(x)\ dx = f(c)(b - a).$$

So the area under the graph of f from a to b is equal to the area under the horizontal line $y = f(c)$ for some c between a and b. We can also write

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c),$$

and then we are saying that the average value of f from a to b is an actual value of f between a and b (the Intermediate Value Theorem for Integrals).

Problem 5. Let $\varepsilon(n)$ be the nth digit in the decimal expansion of π, so that $\varepsilon(1) = 3$, $\varepsilon(2) = 1$, $\varepsilon(3) = 4$, $\varepsilon(4) = 1$, $\varepsilon(5) = 5$, and so on. Does the infinite series $\sum_{n\geqslant 1}(-1)^{\varepsilon(n)}(\ln(1 + 1/n) - 1/n)$ converge? Why or why not?

Solution: Since a series converges if it converges absolutely, it suffices to see that $\sum_{n\geqslant 1} |\ln(1 + 1/n) - 1/n|$ converges. Now $\ln(1 + x) = x - x^2/2 + x^3/3 - \ldots$ for $|x| < 1$, and consequently $|\ln(1+x) - x| \leqslant Cx^2$ for some constant C and all x near 0. Thus for large n we have

$$|\ln(1 + 1/n) - 1/n| \leqslant C/n^2.$$

Since the series $\sum_{n\geqslant 1} 1/n^2$ converges (p-series with $p = 2 > 1$) we conclude that $\sum_{n\geqslant 1} |\ln(1 + 1/n) - 1/n|$ converges and hence that the given series converges.

Part II.

Solve three of the following six problems.

Problem 6. Find the value of the line integral $\int_C (y + e^x)dx + (x^2 - x + e^y)dy$, where C is the ellipse $x^2/4 + y^2/9 = 1$ in the xy-plane, oriented counterclockwise.

Solution: Let R be the region $x^2/4 + y^2/9 \leqslant 1$. By Green's theorem, the given line integral is

$$\int\int_R \left(\frac{\partial(x^2 - x + e^y)}{\partial x} - \frac{\partial(y + e^x)}{\partial y}\right)\ dx\ dy = \int\int_R (2x - 2)\ dx\ dy.$$

The integral of $2x$ over R is 0 because $2x$ is odd and R is symmetric about the y-axis. So the integral is -2 times the area of R. The area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab, so we conclude that the value of the given integral is $-2(6\pi) = -12\pi$.

Problem 7. Let f and g be real-valued functions on $\mathbb{R}$. Assume $|f(x)| \leqslant M$ for some constant $M > 0$ and $\lim_{x\to 0} g(x) = 0$.

(a) Using the formal definition of "limit," prove that $\lim_{x\to 0} f(x)g(x) = 0$.

Solution: Let $\varepsilon > 0$ be given. Since $\lim_{x\to 0} g(x) = 0$, there exists $\delta > 0$ such that if $0 < |x| < \delta$ then $|g(x)| < \varepsilon/M$. Hence if $0 < |x| < \delta$ then $|f(x)g(x)| < M(\varepsilon/M) = \varepsilon$, and we conclude that $\lim_{x\to 0} f(x)g(x) = 0$.

(b) Use (a) to show that the function

$$r(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at 0.

Solution: Let $f(x) = \sin(1/x)$ for $x \neq 0$, and put $f(0) = 0$. Also put $M = 1$. Then $|f(x)| \leqslant M$. Let $g(x) = x$, so that $\lim_{x\to 0} g(x) = 0$. We have

$$\lim_{h\to 0} \frac{r(0+h) - r(0)}{h} = \lim_{h\to 0} f(h)g(h) = 0$$

by (a), so r is differentiable at 0 and $r'(0) = 0$.

Problem 8. Find the maximum and minimum values of $f(x) = xz + yz$ on the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\}$.

Solution: Let $g(x, y, z) = x^2 + y^2 + z^2$. According to the method of Lagrange multipliers, the points where an extreme value occurs are among the points $P \in S$ where $\nabla f(P) = \lambda \nabla g(P)$ for some $\lambda \in \mathbb{R}$. Thus we seek solutions to the system

$$\begin{cases} z = \lambda 2x \\ z = \lambda 2y \\ x + y = \lambda 2z \\ x^2 + y^2 + z^2 = 4. \end{cases}$$

The first and third equations show that if $\lambda = 0$ then $z = 0$ and $x = -y$, whence the fourth equation gives $P = \pm(\sqrt{2}, -\sqrt{2}, 0)$, so that $f(P) = 0$. On the other hand, if $\lambda \neq 0$, then the first and second equations give $x = z/(2\lambda)$ and $y = z/(2\lambda)$, whence the third equation gives $z/(2\lambda^2) = z$. Since $z \neq 0$ (else $x = y = 0$ also, contradicting the fourth equation) we get $\lambda = \pm 1/\sqrt{2}$. The first two equations then give $x = y$ and $z = \pm\sqrt{2}x$, so the fourth equations gives $4x^2 = 4$. Thus $x = \pm 1$ and $P = \pm(1, 1, \varepsilon\sqrt{2})$ with $\varepsilon \in \{\pm 1\}$. Taking account of all possible signs, we find $f(P) = \pm 2\sqrt{2}$. So $2\sqrt{2}$ is the maximum value of f and $-2\sqrt{2}$ is the minimum value.

Problem 9. Let $\{x_n\}$ be the sequence of positive real numbers defined by $x_1 = 1$ and, for $n \geqslant 1$,

$$x_{n+1} = \frac{1}{x_n + x_n^{-1}}.$$

Show that $\{x_n\}$ converges. To what number does it converge?

Solution: Since $x_n > 0$ for all n, we have

$$x_{n+1} = \frac{x_n}{x_n^2 + 1} < x_n.$$

Thus $\{x_n\}$ is a decreasing sequence bounded below (by 0), and so it converges. If $x = \lim_{n\to\infty} x_n$ then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n}{x_n^2 + 1} = \frac{x}{x^2 + 1}.$$

The only solution in $\mathbb{R}$ to $x = x/(x^2+1)$ is $x = 0$, so $\{x_n\}$ converges to 0.

Problem 10. Define functions $f_n : [0,1] \to \mathbb{R}$ for $n \geqslant 1$ by

$$f_n(x) = \begin{cases} x^n \ln x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Is f_n is continuous at 0? Justify your answer.

Solution: Yes. By L'Hôpital's Rule,

$$\lim_{x\to 0^+} f_n(x) = \lim_{x \to 0^+} \frac{\ln x}{x^{-n}} = \lim_{x\to 0} \frac{x^n}{-n} = 0,$$

which is $f_n(0)$.

b) Is $\{f_n\}$ a uniformly convergent sequence of functions? Justify your answer.

Solution: Yes. Since $\ln x < 0$ for $x \in (0,1)$, we see that $f_n(x) \leqslant 0$. Also, $f_n'(x) = nx^{n-1}\ln x + x^{n-1}$ and therefore $f_n'(x) = 0$ if and only if $x = e^{-1/n}$. We see in fact that $f_n'(x) < 0$ for $x < e^{-1/n}$ and $f_n'(x) > 0$ for $x > e^{-1/n}$, so the minimum value of the continuous function f_n on $[0,1]$ is $f(e^{-1/n}) = -1/(ne)$. Thus $|f_n(x)| \leqslant 1/(ne)$ for $x \in [0,1]$. Since the upper bound $1/(ne)$ is independent of x and goes to 0 as n goes to infinity, we see that $\{f_n\}$ is uniformly convergent to 0 on $[0,1]$.

Problem 11. Find the value of the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{dS}$, where the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (e^y + xz, e^x - yz, z)$, the surface S is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and the normal vector points outward.

Solution: By the Divergence Theorem, the given integral equals

$$\int\int\int_D \nabla \cdot \mathbf{F}\, dV = \int\int\int_D 1\, dx\, dy\, dz,$$

where D is the interior of S and dV is the volume element. The right-hand side is

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} 1\, dx\, dy\, dz = \int_0^1 \frac{(1-z)^2}{2}\, dz,$$

which is $1/6$.

Part III.

Solve one of the following three problems.

Problem 12. Let S be the set of finite sums of the form $\sum_{n=a}^{b} 1/n$, where $1 \leqslant a \leqslant b$. Prove that S is dense in the set of nonnegative real numbers.

Solution: Given $x \in [0,\infty)$ and $\varepsilon > 0$, we must show that there exists $s \in S$ such that $|x - s| < \varepsilon$. If $x = 0$ we choose n such that $1/n < \varepsilon$ and we take $s = 1/n$ (i. e. $a = b = n$). Now suppose $x > 0$, and choose a so that $1/a < \min(x, \varepsilon)$. The set

$$B = \{c \geqslant a : \sum_{n=a}^{c} 1/n < x\}$$

is nonempty because $a \in B$ and is finite because $\sum_{n=a}^{\infty} 1/n = \infty$, i. e. the harmonic series diverges. Put $b = \max(B)$ and $s = \sum_{n=a}^{b} 1/n$. Then $0 < s < x$. On the other hand, $\sum_{n=a}^{b+1} 1/n > x$ by the definition of b. But $\sum_{n=a}^{b+1} 1/n = s + 1/(b+1)$, so we conclude that $s < x < s + 1/(b+1)$, and thus

$$0 < x - s < \frac{1}{b+1} < \frac{1}{b} \leqslant \frac{1}{a} < \varepsilon.$$

Hence $|x - s| < \varepsilon$.

Problem 13. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) = (x^3 + e^y, y^5 - e^x)$. Prove that f is an open mapping. In other words, show that if U is an open subset of $\mathbb{R}^2$ then so is $f(U)$.

Solution: We must show that for every point $P = (x, y) \in U$ there is an open neighborhood N of $f(P)$ such that $N \subset f(U)$. Now the Jacobian determinant of f is

$$\det \begin{pmatrix} 3x^2 & e^y \\ -e^x & 5y^4 \end{pmatrix} = 15x^2y^4 + e^{x+y},$$

and the right-hand side is > 0, and in particular $\neq 0$, for all (x, y). So by the Inverse Function Theorem, there are open neighborhoods V of P and and W of $f(P)$ such that $f|V$ is a C^∞-diffeomorphism, and thus in particular a homeomorphism, of V onto W. Thus $N = f(U \cap V)$ is an open neighborhood of $f(P)$ contained in $f(U)$.

Problem 14. Let X be a complete metric space with metric d satisfying the following condition: For every $\varepsilon > 0$ there is a collection of finitely many open balls of radius ε which covers X. Prove that X is compact.

Solution: Given a sequence $\{x_n\}$ in X we will choose a subsequence $\{y_n\}$ such that for every $N \geqslant 1$, if $m, n \geqslant N$ then $d(y_n, y_m) < 2/N$. Since X is complete it will follow that the Cauchy subsequence $\{y_n\}$ converges, whence X is compact.

To construct the subsequence $\{y_n\}$, we proceed inductively. First, choose finitely many open balls of radius 1 which cover X. Then one of the open balls contains infinitely many terms of the sequence $\{x_n\}$, and so we can choose a subsequence $\{y_n^{(1)}\}$ satisfying

$$d(y_n^{(1)}, y_m^{(1)}) < 2$$

for all $n, m \geqslant 1$.

Now suppose that we have chosen sequences $\{y_n^{(i)}\}$ for $1 \leqslant i \leqslant N$ such that $\{y_n^{(i)}\}$ is a subsequence of $\{y_n^{(i-1)}\}$ for $1 \leqslant i \leqslant N$ (with $\{y_n^{(0)}\}$ understood to be $\{x_n\}$) and

$$d(y_n^{(i)}, y_m^{(i)}) < 2/i$$

for all $n, m \geqslant 1$. Choose finitely many open balls of radius $1/(N+1)$ which cover X. Then one of the open balls contains infinitely many terms of the sequence $\{y_n^{(N)}\}$, and so we can choose a subsequence $\{y_n^{(N+1)}\}$ satisfying

$$d(y_n^{(N+1)}, y_m^{(N+1)}) < 2/(N+1)$$

for all $n, m \geqslant 1$.

Finally, put $y_\nu = y_\nu^{(\nu)}$. Then for every $N \geqslant 1$ and all $n, m \geqslant N$, the terms y_n and y_m are terms of the sequence $\{y_\nu^{(N)}\}$, and consequently they satisfy $d(y_n, y_m) < 2/N$, as desired.