**Preliminary Exam 2016**
**Solutions to Morning Exam**

**Part I.**

Solve four of the following five problems.

**Problem 1.** Find the volume of the "ice cream cone" defined by the inequalities $x^2 + y^2 + z^2 \leqslant 1$ and $x^2 + y^2 \leqslant z^2/3$ for $z \geqslant 0$.

*Solution*: Since $\tan^{-1}\sqrt{3} = \pi/3 = \pi/2 - \pi/6$, the volume is

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = (2\pi)(1/3)(1 - \sqrt{3}/2),$$

which is $(2 - \sqrt{3})\pi/3$.

One can also do this problem in cylindrical coordinates: Since the intersection of the surfaces $r^2 = z^2/3$ and $r^2 + z^2 = 1$ projects to the circle $r^2 = 1/4$, we get

$$\int_0^{2\pi}\int_0^{1/2}\int_{\sqrt{3}r}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta = (2\pi)\int_0^{1/2}(r\sqrt{1-r^2} - \sqrt{3}r^2)\, dr,$$

which is

$$(2\pi)(-\frac{1}{3}(1-r^2)^{3/2} - \frac{r^3}{\sqrt{3}})\Big|_0^{1/2}.$$

After some simplifications, we get the same answer as before.

**Problem 2.** Determine the radius of convergence and interval of convergence of the power series $\sum_{n\geqslant 1}(1 + 1/n)^{n^2}x^n$.

*Solution*: Since $\lim_{n\to\infty}(1+1/n)^n|x| = e|x|$, we see that the radius of convergence is $1/e$. Thus we must check convergence at the endpoints $1/e$ and $-1/e$. Now

$$\log(1 + 1/n)^{n^2} = n^2(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^4} - \dots),$$

which is $n - 1/2 + O(1/n^2)$. Thus

$$\log((1 + 1/n)^{n^2}(1/e)^n) = -1/2 + O(1/n^2),$$

and consequently $\lim_{n\to\infty}(1+1/n)^{n^2}(1/e)^n = e^{-1/2}$. In particular, at $1/e$ the limit of the $n$th term of the series is nonzero, and at $-1/e$ the limit does not even exist. So the interval of convergence is $(-1/e, 1/e)$.

**Problem 3.** Prove that $\cos x_0 = x_0$ for a unique $x_0 \in [0, 1]$, and show in addition that $\pi/6 < x_0 < \pi/4$.

*Solution*: Consider the function $f(x) = \cos x - x$. We have $f(0) = 1 > 0$, and since $\cos x$ is strictly decreasing on $[0, \pi/2]$ we also have

$$f(1) = \cos(1) - 1 < \cos(\pi/2) - 1 = -1 < 0.$$

So the existence of $x_0$ follows from the Intermediate Value Theorem (or the fact that the continuous image of a connected set is connected). The uniqueness follows from the fact that $f'(x) = -\sin x - 1$ is strictly negative on $[0, 1]$, whence $f$ is strictly decreasing on $[0, 1]$. [*Remark*: One could get the existence and uniqueness simultaneously by appealing to the Contraction Fixed Point Theorem.]

For the second assertion, observe that

$$f(\pi/6) = \sqrt{3}/2 - \pi/6 = \frac{3\sqrt{3} - \pi}{6} > 0$$

while

$$f(\pi/4) = \sqrt{2}/2 - \pi/4 = \frac{2\sqrt{2} - \pi}{4} < 0.$$

Since $f$ is strictly decreasing on $[0, 1]$ it follows that $x_0 \in (\pi/6, \pi/4)$.

**Problem 4.** Using standard techniques of integration, find antiderivatives on some open interval where the integrand is defined and continuous:
(a) $\int \tan(\cos^2 x)\sin(2x)\,dx$.

*Solution*: Let $u = \cos^2 x$, so that $du = -2\sin x\cos x dx = -\sin(2x)dx$. We get

$$\int \tan(\cos^2 x)\sin(2x)\,dx = -\int \tan u\,du = \log|\cos u| + C = \log|\cos(\cos^2 x)| + C$$

(b) $\int \cos(\log x)\,dx$. (Here "log" is understood to be "ln.")

*Solution*: Let $u = \cos(\log x)$ and $v = x$, so that $du = -x\sin(\log x)dx$ and $dv = dx$. We get

$$\int \cos(\log x)\,dx = x\cos(\log x) + \int \sin(\log x)\,dx.$$

A second integration by parts with $u = \sin(\log x)$ and $v = x$ then gives

$$\int \cos(\log x)\,dx = x(\cos\log x + \sin\log x)/2 + C.$$

**Problem 5.** Find all solutions to the differential equation $y'' - y' - 6y = \cos t$ that are bounded on $[0, \infty)$ and satisfy the condition $y(0) = 0$.

*Solution*: Since $x^2 - x - 6 = (x-3)(x+2)$, the general solution to the homogeneous equation is $y(t) = ae^{3t} + be^{-2t}$. Substituting $y(t) = c\cos t + d\sin t$ into the inhomogeneous equation, one finds that $c = -7/50$ and $d = -1/50$, so the general solution to the inhomogeneous equation is

$$y(t) = ae^{3t} + be^{-2t} + (-7/50)\cos t + (-1/50)\sin t.$$

The boundedness on $[0, \infty)$ holds if and only if $a = 0$, and given that $a = 0$, the condition $y(0) = 0$ holds if and only if $b = 7/50$. So the above $y(t)$ with $a = 0$, $b = 7/50$, $c = -7/50$, and $d = -1/50$ is the unique function satisfying the stated conditions.

## Part II.

Solve three of the following six problems.

**Problem 6.** Let $\mathbf{F}(x, y, z) = (2x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j} + z\mathbf{k}$.
(a) Compute $\nabla \times \mathbf{F}$.

*Solution*: Using the definition of $\nabla \times \mathbf{F}$ as a symbolic determinant, we compute

$$\nabla \times \mathbf{F} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 3y & 3x + 2y & z \end{pmatrix},$$

which is $\mathbf{0}$.

(b) Let $C$ be the curve given parametrically by $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}$ for $0 \leqslant t \leqslant 2\pi$. Find the value of the line integral $\int_C \mathbf{F} \cdot ds$.

*Solution*: Since $\nabla \times \mathbf{F}$ is zero and $\mathbb{R}^3$ is simply connected, $\mathbf{F}$ has a potential function, which is easily computed to be

$$\varphi(x, y, z) = x^2 + 3xy + y^2 + z^2/2 + C.$$

Note also that $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}(2\pi) = e^{2\pi}\mathbf{i}$. Hence $\int_C \mathbf{F} \cdot ds = \varphi(e^{2\pi}, 0, 0) - \varphi(1, 0, 0)$, which is $e^{4\pi} - 1$.

**Problem 7.** Fix an element $c \in \mathbb{R}$, and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - c|$. Show that $f$ is uniformly continuous.

*Solution*: By the Triangle Inequality, $||x - c| - |y - c|| \leqslant |x - y|$, or in other words, $|f(x) - f(y)| \leqslant |x - y|$. So given $\varepsilon > 0$ let $\delta = \varepsilon$; then $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

**Problem 8.** The formula $f(x, y, z) = (x + y^2 + z^2, x^2 + y + z^2, x^2 + y^2 + z)$ defines a function $f : \mathbb{R}^3 \to \mathbb{R}^3$.

(a) Explain why there are open neighborhoods $U$ and $V = f(U)$ of $(0, 0, 0) \in \mathbb{R}^3$ and a $C^1$ function $g : V \to U$ such that $g(f(x, y, z)) = (x, y, z)$ for $(x, y, z) \in U$ and $f(g(x, y, z)) = (x, y, z)$ for $(x, y, z) \in V$.

*Solution*: The Jacobian matrix of $f$ at $(0, 0, 0)$ is

$$\begin{pmatrix} 1 & 2y & 2z \\ 2x & 1 & 2z \\ 2x & 2y & 1 \end{pmatrix} \Big|_{(0,0,0)} = I,$$

the $3 \times 3$ identity matrix. In particular, this matrix is invertible, so by the Inverse Function Theorem, there exist $U$, $V$, and $g$ as above.

(b) Now let $h(x, y, z) = (x + e^y + e^z - 2, e^x + y + e^z - 2, e^x + e^y + z - 2)$. Show that if $f$ is replaced by $h$ then no such $U$, $V$, and $g$ exist.

*Solution*: In this case, the Jacobian matrix of $h$ at $(0, 0, 0)$ is

$$\begin{pmatrix} 1 & e^y & e^z \\ e^x & 1 & e^z \\ e^x & e^y & 1 \end{pmatrix} \Big|_{(0,0,0)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

a matrix of row rank 1, not 3, and therefore not invertible. Denote this matrix by $A$. If $U$, $V$, and $g$ as above exist, and if $B$ is the Jacobian matrix of $g$ at $(0, 0, 0)$, then $AB = I$ and hence $\det(A)\det(B) = 1$, a contradiction since $\det(A) = 0$.

**Problem 9.** Let $f_n(x) = nxe^{-nx}$. Show that the sequence $\{f_n\}$ is pointwise convergent on $[0, 1]$ but not uniformly convergent.

*Solution*: If $x = 0$ then $f_n(x) = 0$ for all $n$; otherwise

$$\lim_{n\to\infty} \frac{nx}{e^{nx}} = \lim_{y\to\infty} \frac{y}{e^y} = 0$$

by L'Hôpital's Rule. Thus $\{f_n\}$ converges pointwise to the zero function on $[0, 1]$. But the convergence is not uniform, for take $\varepsilon < 1/e$, let $N \geqslant 1$ be arbitrary, and observe that if $x = 1/(N + 1)$ and $n = N + 1$ then $|f_n(x) - 0| = 1/e$ and thus $|f_n(x) - 0| > \varepsilon$.

**Problem 10.** Let $V$ be the real vector space of $C^\infty$ functions $f : \mathbb{R} \to \mathbb{R}$ with compact support (in other words, $f$ vanishes outside some closed bounded interval).

Define an operator $T : V \to V$ by $T(f) = f''$. Show that $T$ is self-adjoint relative to the $L^2$ inner product on $V$. In other words, letting

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\, dx$$

for $f, g \in V$, show that $\langle T(f), g\rangle = \langle f, T(g)\rangle$.

*Solution*: Using integration by parts, we have

$$\int_{-\infty}^{\infty} f(x)g'(x)\, dx = -\int_{-\infty}^{\infty} f'(x)g(x)\, dx,$$

because both $f$ and $g$ vanish outside a closed bounded interval. Replacing $g$ by $g'$ we obtain an expression on the right-hand which is symmetric in $f$ and $g$. Therefore so is the expression on the left-hand side, which is $\langle f, T(g)\rangle$. In other words, $\langle f, T(g)\rangle$ equals $\langle g, T(f)\rangle$ and hence $\langle T(f), g\rangle$. [Aternatively, just do integration by parts again.]

**Problem 11.** Find the surface area of the torus described parametrically by

$$\mathbf{r}(\theta, \varphi) = \cos\theta(1 + \frac{\cos\varphi}{2})\mathbf{i} + \sin\theta(1 + \frac{\cos\varphi}{2})\mathbf{j} + \frac{\sin\varphi}{2}\mathbf{k} \qquad (0 \leqslant \theta, \varphi \leqslant 2\pi).$$

*Solution*: The surface area is by definition

$$A = \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_\theta \times \mathbf{r}_\varphi|\, d\theta\, d\varphi$$

Now

$$\mathbf{r}_\theta \times \mathbf{r}_\varphi = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta(1 + \frac{\cos\varphi}{2}) & \cos\theta(1 + \frac{\cos\varphi}{2}) & 0 \\ \cos\theta(\frac{-\sin\varphi}{2}) & \sin\theta(\frac{-\sin\varphi}{2}) & \frac{\cos\varphi}{2} \end{pmatrix},$$

which is $(\cos\theta)(1 + \frac{\cos\varphi}{2})\frac{\cos\varphi}{2}\mathbf{i} + (\sin\theta)(1 + \frac{\cos\varphi}{2})\frac{\cos\varphi}{2}\mathbf{j} + (1 + \frac{\cos\varphi}{2})\frac{\sin\varphi}{2}\mathbf{k}$. So

$$|\mathbf{r}_\theta \times \mathbf{r}_\varphi| = (\frac{1}{2} + \frac{\cos\varphi}{4}),$$

and consequently $A = 2\pi^2$. [Alternatively, one could use the theorem of Pappus that the surface area of a surface of revolution is $2\pi r L$, where $2\pi r$ is the distance traveled by the centroid of the curve being rotated and $L$ is its length.]

## Part III.

Solve one of the following three problems.

**Problem 12.** Let $f$ be a $C^{2n}$ function in some neighborhood of a point $a \in \mathbb{R}$, and suppose that $f^{(k)}(a) = 0$ for $1 \leqslant k \leqslant 2n - 1$. Show that if $f^{(2n)}(a) > 0$ then $f$ has a local minimum at $a$.

*Solution*: By Taylor's theorem, $f$ is represented in some neighborhood of $a$ by the relevant Taylor polynomial of order $2n - 1$ plus the remainder term:

$$f(x) = f(a) + f^{(2n)}(c)\frac{(x-a)^{2n}}{(2n)!},$$

where $c$ is strictly between $a$ and $x$. Furthermore, since $f^{(2n)}$ is continuous in some neighborhood of $a$ and $f^{(2n)}(a) > 0$, we see that $f^{(2n)}(c) > 0$ for $x$ near $a$. Of course also $(x-a)^{2n} > 0$ for $x \neq a$. So for $x$ near $a$ but not equal to $a$ it follows that $f(x) > f(a)$.

**Problem 13.** Let $X$ and $Y$ be metric spaces with respective metrics $d_X(*,*)$ and $d_Y(*,*)$, let $x_0$ be a point of $X$, and let $f : X \to Y$ be a function.

(a) Consider the following definitions:

(A) $f$ is continuous at $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \varepsilon$.

(B) $f$ is continuous at $x_0$ if for every sequence $\{x_n\}_{n\geqslant 1}$ in $X$ which converges to $x_0$ the sequence $\{f(x_n)\}_{n\geqslant 1}$ converges to $f(x_0)$.

Show that these definitions are equivalent.

*Solution*: Assume that $f$ is continuous at $x_0$ according to definition (A), and suppose that $\{x_n\}_{n\geqslant 1}$ is a sequence in $X$ which converges to $x_0$. We must show that $\{f(x_n)\}_{n\geqslant 1}$ converges to $f(x_0)$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ as in (A), and then choose $N \geqslant 1$ so that if $n > N$ then $d_X(x_n, x_0) < \delta$. Since (A) is in force, it follows that $d_Y(f(x_n), f(x_0)) < \epsilon$, as desired.

Now assume that $f$ is continuous at $x_0$ according to definition (B), and let $\varepsilon > 0$ be given. Suppose that there does not exist $\delta > 0$ with the property asserted in (A). Then for every positive integer $n$ we cannot take $\delta = 1/n$, so there exists $x_n \in X$ such that $d_X(x_n, x_0) < 1/n$ but $d_Y(f(x_n), f(x_0)) \geqslant \varepsilon$. Then $\{x_n\}_{n\geqslant 1}$ is a sequence in $X$ which converges to $x_0$ but $\{f(x_n)\}_{n\geqslant 1}$ does not converge to $f(x_0)$, contradicting (B). So the required $\delta > 0$ does exist and (A) follows.

(b) Let $I = [0, 2\pi) \subset \mathbb{R}$ and $T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and consider $I$ and $T$ as metric spaces by restricting the standard Euclidean metrics on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. Define $g : I \to T$ by $g(x) = (\cos x, \sin x)$, and put $f = g^{-1}$. Is $f$ continuous at $(1, 0) \in T$? Justify your answer using (B).

*Solution*: To see that $f$ is *not* continuous at $(1, 0)$, consider the sequence $\{x_n\}_{n\geqslant 1}$ in $T$ given by $x_n = (\cos(1/n), \sin(-1/n))$, which converges to the point $x_0 = (1, 0)$. However $f(x_n) = 2\pi - 1/n$, whence $\{f(x_n)\}_{n\geqslant 1}$ does not converge to the point $f(x_0) = 0$ (or in fact to any point of $I$).

**Problem 14.** Define a real-valued function $f$ on $\mathbb{R}$ by setting $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show by induction on $n$ that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ for $x \neq 0$, where $P_n$ is a polynomial.

*Solution*: In the base case $n = 0$ we take $P_0(x) = 1$. Now suppose that for some $n \geqslant 0$ we have $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ for $x \neq 0$. Then

$$f^{(n+1)}(x) = e^{-1/x^2}(2/x^3)P_n(1/x) + e^{-1/x^2} P_n'(1/x)(-1/x^2).$$

Therefore $f^{(n+1)}(x) = e^{-1/x^2} P_{n+1}(1/x)$ with $P_{n+1}(y) = 2y^3 P_n(y) - y^2 P_n'(y)$.

(b) Deduce that $f$ is a $C^\infty$ function on $\mathbb{R}$ and that $f^{(n)}(0) = 0$ for all $n$.

*Solution*: The fact that $f$ is a $C^\infty$ function for $x \neq 0$ is verifiable by inspection. We show by induction on $n$ that $f^{(n)}$ exists and is continuous at 0 and that $f^{(n)}(0) = 0$. For $n = 0$ the continuity is obvious, and $f(0) = 0$ by definition. Now suppose that for some $n \geqslant 0$ we know that $f^{(n)}$ exists and is continuous at 0 and that $f^{(n)}(0) = 0$. To verify that $f^{(n+1)}(0)$ exists, we compute the limit of the relevant difference quotient. By (a), we have

$$\lim_{h\to 0}\frac{f^{(n)}(h)-f^{(n)}(0)}{h}=\lim_{h\to 0}\frac{e^{-1/h^2}P_n(1/h)}{h},$$

and the right-hand side is $\lim_{y\to\infty} e^{-y^2}P_n(y)y$ provided this limit coincides with $\lim_{y\to-\infty} e^{-y^2}P_n(y)y$. It does, because both limits are 0. Thus $f^{(n+1)}(0)$ exists and equals 0, and then $f^{(n+1)}$ is continuous at 0 by (a).