Preliminary Exam 2018
Morning Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Consider the series $\sum_{n\geqslant 2}(n\log n)^{-1}$ and $\sum_{n\geqslant 2}(n(\log n)^2)^{-1}$. Show that one converges and one diverges by applying a standard convergence test.

Problem 2. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sqrt{x^2+y^2}}\,dx\,dy = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

by computing both sides.

Problem 3. Prove that if $f(x)$ is $\sin x$ or $\arctan x$ then $|f(b)-f(a)|\leqslant|b-a|$ for all $a,b\in\mathbb{R}$ and that this inequality also holds for $f(x)=\log x$ and $a,b\geqslant 1$.

Problem 4. Let y be a differentiable function and p a continuous function on $(0,\infty)$, and suppose that $y'(t)+p(t)y(t)=p(t)$ for all $t>0$. If $p(t)>c/t$ for some constant $c>0$ prove that $\lim_{t\to\infty}y(t)=1$.

Problem 5. Let $f_n(x)=x^n$ on the interval $I=[0,1]$ in $\mathbb{R}$. Show that the sequence $\{f_n\}_{n\geqslant 1}$ does not converge uniformly on I. You may quote general theorems about uniform convergence.

Part II.

Solve three of the following six problems.

Problem 6. Define $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Show that $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0,0)$ but f is not differentiable at $(0,0)$. You may quote general facts about differentiability.

Problem 7. Let I be any interval in $\mathbb{R}$. Show that if $f:I\to\mathbb{R}$ is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in I then $\{f(x_n)\}$ is also Cauchy. Is the assertion still true if we assume merely that f is continuous? Justify your answer.

Problem 8. Show that

$$\frac{1}{(x-1)(x-2)(x-3)}=\sum_{n\geqslant 0}\left(-\frac{1}{2}+\frac{1}{2^{n+1}}-\frac{1}{2\cdot 3^{n+1}}\right)x^n$$

for $|x|<1$.

Problem 9. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ and $h:\mathbb{R}^2\to\mathbb{R}^2$ be the functions

$$f(x,y)=(e^{2x-y}-e^x, e^{-3x+y}-e^{2y})$$

and

$$h(x,y)=(x^3+x+y, y^2+2x+3y).$$

There is an open neighborhood $\mathcal{U}$ of $(0,0)\in\mathbb{R}^2$ and a differentiable function $g:\mathcal{U}\to\mathbb{R}^2$ such that $g(0,0)=(0,0)$ and $f\circ g=h$. Compute $[g'(0,0)]$, the Jacobian matrix of g at $(0,0)$.

Problem 10. Let $P(x,y) = -y/(x^2+y^2)$ and $Q(x,y) = x/(x^2+y^2)$.

(a) Compute $\partial Q/\partial x - \partial P/\partial y$.

(b) Compute the line integral of $P(x,y)\,dx + Q(x,y)\,dy$ around the unit circle (oriented counterclockwise) $x^2+y^2=1$.

(c) Explain why (a) and (b) do not contradict Green's Theorem (which you should state, of course).

Problem 11. Let C and C' be the circles in $\mathbb{R}^3$ parametrized by $(\cos t, \sin t, 0)$ and $(\cos t, \sin t, 2)$ respectively $(0 \leqslant t \leqslant 2\pi)$. Let $\mathbf{F}(x,y,z)$ be a C^∞ vector field in $\mathbb{R}^3$ such that $\nabla \times \mathbf{F} = \mathbf{0}$. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

where the integrals on the left and right are the line integrals of $\mathbf{F}$ along the oriented circles C and C' respectively.

Part III.

Solve one of the following three problems.

Problem 12. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, put

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and let S denote the unit sphere $||x|| = 1$ in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any linear transformation. Give a reason why the two sides of the equation

$$\max\{x \in S : ||T(x)||\} = \inf\{C \geqslant 0 : ||T(x)|| \leqslant C||x|| \text{ for all } x \in \mathbb{R}^n\}$$

both exist, and then prove the equation.

Problem 13. Let X be a metric space with the following property: For every infinite subset S of X,

$$\inf\{d(x,y) : x \neq y,\ x, y \in S\} = 0.$$

Prove that X is *totally bounded*: In other words, show that for every $\varepsilon > 0$, the space X can be covered by *finitely many* open balls of radius ε.

Problem 14. Let S be the surface area of the sphere $x^2+y^2+z^2 = 1$ and V the volume of the ball $x^2+y^2+z^2 \leqslant 1$. Let S' be the surface area of the portion of the sphere $x^2+y^2+z^2 = 1$ lying above the plane $z = 1/2$, and let V' be the volume of the portion of the ball $x^2+y^2+z^2 \leqslant 1$ lying above the plane $z = 1/2$. Show that $S' = S/4$ and $V' = 5V/32$.