Preliminary Exam 2017
Solutions to Afternoon Exam

Part I.

Solve four of the following five problems.

Problem 1. Find a basis for the solution space of the system of equations
\[
\begin{align*}
    w + 5x + y + 2z &= 0 \\
    3w + 15x + 4y - 4z &= 0.
\end{align*}
\]

Solution: The matrix associated to this homogeneous system is
\[
\begin{pmatrix}
    1 & 5 & 1 & 2 \\
    3 & 15 & 4 & -4
\end{pmatrix}.
\]
Subtracting 3 times the first row from the second, and then the second from the first, we obtain
\[
\begin{pmatrix}
    1 & 5 & 0 & 12 \\
    0 & 0 & 1 & -10
\end{pmatrix},
\]
a matrix in row-reduced upper-echelon form. Hence a basis for the solution space is \{\((-5, 1, 0, 0), (-12, 0, 10, 1)\).

Problem 2. Let
\[
A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.
\]
Find an invertible 2×2 matrix \(U\) with coefficients in \(\mathbb{C}\) such that \(U^{-1}AU\) is diagonal.

Solution: The characteristic polynomial of \(A\) is \(x^2 - 6x + 25\). Thus the eigenvalues of \(A\) are
\[
\lambda_{\pm} = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm 4i.
\]
The corresponding eigenvectors span the null space of
\[
A - \lambda_{\pm} I = \begin{pmatrix} 3+4i & -4 \\ 4 & 3+4i \end{pmatrix}
\]
By a row reduction – or simply by inspection – we see that the vectors \((\pm i, 1)\) span the null space. Thus putting
\[
U = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix},
\]
we see that \(U^{-1}AU\) is diagonal.

Problem 3. Find an orthonormal basis (relative to the dot product) for the subspace of \(\mathbb{R}^4\) spanned by the vectors \((-1/2, 1/2, 1/2, 1/2), (1/2, 1/2, -1/2, 1/2),\) and \((1, 1, 2, 2)\).

Solution: Put \(v_1 = (-1/2, 1/2, 1/2, 1/2), v_2 = (1/2, 1/2, -1/2, 1/2),\) and \(v_3 = (1, 1, 2, 2)\). Apply the Gram-Schmidt process to the vectors \(v_1, v_2, v_3\). The vectors \(v_1\) and \(v_2\) are already orthonormal, so put \(u_1 = v_1\) and \(u_2 = v_2\). So the vector
\[
w = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2
\]
is orthogonal to both \(u_1\) and \(u_2\). A calculation shows that \(w = (3/2, -1/2, 3/2, 1/2)\). Putting

\[
u_3 = \frac{1}{\sqrt{5}}(3/2, -1/2, 3/2, 1/2),
\]

we conclude that \(\{u_1, u_2, u_3\}\) is the desired orthonormal set.

**Problem 4.** If \(A\) and \(B\) are \(3 \times 3\) matrices with coefficients in \(\mathbb{C}\), then \(A\) and \(B\) are *similar* if there is an invertible matrix \(U\) such that \(UBU^{-1} = A\). Are the matrices

\[
A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}
\]

similar? Why or why not?

**Solution:** No, \(A\) and \(B\) are not similar. For suppose on the contrary that \(UBU^{-1} = A\) with an invertible matrix \(U\). Then \(U(B - I)U^{-1} = A - I\), so

\[
(A - I)^2 = U(B - I)^2U^{-1} = UOU^{-1} = O,
\]

where \(I\) and \(O\) are the \(3 \times 3\) identity matrix and zero matrix respectively. But \((A - I)^2 \neq 0\), a contradiction.

Alternatively, the fact that \(A\) has characteristic polynomial \((x - 1)^3\) but \((A - I)^2 \neq 0\) means that the Jordan normal form of \(A\) is a single \(3 \times 3\) Jordan block with eigenvalue 1, whereas the fact \(B\) has characteristic polynomial \((x - 1)^3\) but \((B - I)^2 = O\) means that the Jordan normal form of \(B\) consists of a \(1 \times 1\) Jordan block and a \(2 \times 2\) Jordan block, both with eigenvalue 1. Since the Jordan normal forms of \(A\) and \(B\) differ, \(A\) and \(B\) are not similar.

**Problem 5.** Let \(I\) be the ideal of \(\mathbb{Z}\) generated by 6670 and 14007. Find the positive integer \(c\) such that \(I\) is the principal ideal generated by \(c\).

**Solution:** We use the Euclidean algorithm to determine \(c = \gcd(6670, 14007)\). Multiplying 6670 by 2 and subtracting from 14007, we obtain 667, so that

\[c = \gcd(6670, 14007) = \gcd(6670, 667) = 667.\]

**Part II.**

Solve three of the following six problems.

**Problem 6.** Let \(L\) be the subgroup of \(\mathbb{Z}^3\) generated by \((1, 0, 1)\), \((6, 2, 0)\), and \((7, 2, 5)\). Find a direct sum of cyclic groups isomorphic to \(\mathbb{Z}^3/L\).

**Solution:** By row and column operations over \(\mathbb{Z}\) we find that \(UAV = B\), where

\[
A = \begin{pmatrix} 1 & 6 & 7 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix},
\]

\[
B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},
\]

and \(U\) and \(V\) are invertible matrices over \(\mathbb{Z}\). Therefore \(\mathbb{Z}^3/L \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})\).
**Problem 7.** Let $A_n$ be the $n \times n$ matrix with the integers 1, 2, 3, ..., $n$ along the first row and column, 1’s down the diagonal, and 0’s elsewhere, so that

$$A_n = \begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & n \\
2 & 1 & 0 & 0 & \cdots & 0 \\
3 & 0 & 1 & 0 & \cdots & 0 \\
4 & 0 & 0 & 1 & \cdots & 0 \\
& & & & \ddots & \cdots & \cdots \\
n & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}. $$

Prove that

$$\det(A_n) = 2 - \frac{n(n+1)(2n+1)}{6}. $$

*Solution:* By direct calculation, $\det(A_1) = 1$ and $\det(A_2) = -3$, proving the formula in these cases. Now suppose that the formula holds for some $n \geq 2$. Expanding $\det(A_{n+1})$ along the last column, we obtain

$$\det(A_{n+1}) = (-1)^{n+2}(n+1)\det(B) + \det(A_n), $$

where

$$B = \begin{pmatrix}
2 & 1 & 0 & 0 & \cdots & 0 \\
3 & 0 & 1 & 0 & \cdots & 0 \\
4 & 0 & 0 & 1 & \cdots & 0 \\
& & & & \ddots & \cdots & \cdots \\
n & 0 & 0 & 0 & \cdots & 1 \\
(n+1) & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}. $$

By expansion along the bottom row, we see that $\det(B) = (-1)^{n+1}(n+1)$, so

$$\det(A_{n+1}) = -(n+1)^2 + \det(A_n) = 2 - \frac{n(n+1)(2n+1)}{6} - (n+1)^2 $$

by inductive hypothesis. Doing the arithmetic, we obtain

$$\det(A_{n+1}) = 2 - (n+1)(n+2)(2n+3)/6, $$

as desired.

**Problem 8.** Prove or give a counterexample: If $A$ and $B$ are $n \times n$ diagonalizable matrices over $\mathbb{C}$ then $AB$ is also diagonalizable.

*Solution:* Without the assumption that $AB = BA$ the statement is false. To get a counterexample, let

$$A = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix} $$

and

$$C = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, $$

and put $B = A^{-1}C$. Then $A$ is diagonal, hence diagonalizable, and $B$ is diagonalizable because its eigenvalues 1 and 1/2 are distinct. But $AB = C$, which is not diagonalizable because it is a nondiagonal Jordan block.

**Problem 9.** Let $A$ be an $n \times n$ matrix with coefficients in $\mathbb{R}$. If the minimal polynomial of $A$ is $(x+1)^2$ then what is the minimal polynomial of $A^2 + A$? Why?
Solution: Write $A = UJU^{-1}$, where $J$ is an $n \times n$ matrix in Jordan normal form and $U$ is an $n \times n$ invertible matrix. Then the minimal polynomials of $A$ and $J$ are equal, and as 
\[ A^2 + A = U(J^2 + J)U^{-1}, \]
so are the minimal polynomials of $A^2 + A$ and $J^2 + J$. Now given that the minimal polynomial of $A$ is $(x + 1)^2$, we see that $J$ is a diagonal array of one or more $2 \times 2$ Jordan blocks of eigenvalue $-1$ and zero or more $1 \times 1$ Jordan blocks of eigenvalue $-1$. But if
\[ B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \]
is one of the $2 \times 2$ blocks, then
\[ B^2 + B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \]
which has minimal polynomial $x^2$, and if $B = (-1)$ is a $1 \times 1$ block then $B^2 + B = (0)$, which has minimal polynomial $x$. Since $J$ has at least one $2 \times 2$ block, we conclude that the minimal polynomial of $J^2 + J$, and hence the minimal polynomial of $A^2 + A$, is $x^2$.

Problem 10. Let $A$ be the $n \times n$ matrix over $\mathbb{R}$ with 1’s on the diagonal and $1/n!$ everywhere else. Show that $\det(A) \neq 0$.

Solution: Let $a_{i,j}$ be the entry in the $i$th row and $j$th column of $A$. By definition,
\[ \det(A) = \sum_{\sigma} \text{sign}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}, \]
where $\sigma$ runs over all permutations of $\{1, 2, \ldots, n\}$. Since $a_{i,i} = 1$ for all $i$,
\[ \det(A) = 1 + \sum_{\sigma \neq 1} \text{sign}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}, \]
where $\sigma$ now runs over the nontrivial permutations. Each summand in the sum is a product of certain number of factors equal to 1 and a certain number of factors equal to $1/n!$, with at least one factor equal to $1/n!$. Since there are a total of $n! - 1$ summands in the sum, we get $\det(A) = 1 + c$, where $|c| \leq (n! - 1)/n!$. Therefore
\[ \det(A) \geq 1 - |c| \geq 1 - ((n! - 1)/n!) \geq 1/n!, \]
whence $\det(A) > 0$.

Problem 11. Let $R$ and $S$ be commutative rings, let $f : R \to S$ be a ring homomorphism, and let $I$ be an ideal of $R$. Prove that if $f$ is surjective (or “onto”) then $f(I)$ is an ideal of $S$.

Solution: First we show that $f(I)$ is an additive subgroup of $S$. Certainly $0 = f(0) \in f(I)$. Now suppose that $j, j' \in f(I)$, and write $j = f(i)$, $j' = f(i')$ with $i, i' \in I$. Then $j - j' = f(i) - f(i') = f(i - i')$, and since $i - i' \in I$ we deduce that $j - j' \in f(I)$.

To complete the proof that $f(I)$ is an ideal of $S$, suppose that $j \in f(I)$ and $s \in S$. As before we can write $j = f(i)$ with $i \in I$, and also, because $f$ is surjective, $s = f(r)$ with $r \in R$. Since $I$ is an ideal we have $ri \in I$; then $sj = f(r)f(i) = f(ri) \in f(I)$.

Part III.

Solve one of the following three problems.
Problem 12. Let $p$ and $q$ be primes. Show that a nonabelian group of order $pq$
has trivial center.

Solution: If the center $Z(G)$ of $G$ is nontrivial then it has order $p$ or $q$, because
if it has order $pq$ then $G$ is abelian. Thus the quotient $G/Z(G)$ is of prime order
and is therefore cyclic. Let $cZ(G)$ be a generator of $G/Z(G)$. We will show that $G$
is abelian and hence obtain a contradiction. Given $g, g' \in G$, we can write $g = c^i z$
and $g' = c^j z'$ with integers $i$ and $j$ and $z, z' \in Z(G)$. Since $z$ and $z'$ commute with
c and with each other, we get

$$gg' = (c^i z)(c^j z') = c^{i+j} zz' = c^{i+j} z' z = (c^i z')(c^j z) = g'g,$$

proving that $G$ is abelian.

Problem 13. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map and $A^t : \mathbb{R}^n \to \mathbb{R}^n$
the transpose or adjoint relative to the dot product. Put $B = A^t A$, let $S$ be the unit
sphere in $\mathbb{R}^n$ centered at the origin, and define $f : S \to S$ by $f(x) = B(x)/||B(x)||$,
where $||x|| = \sqrt{x \cdot x}$. Show that there are at least $n$ points $u \in S$ such that $f(u) = u$.

Solution: Since $B = B^t$, we see that $B$ is a symmetric (or self-adjoint) operator,
whence $\mathbb{R}^n$ has an orthonormal basis $u_1, u_2, \ldots, u_n$ consisting of eigenvectors of $B$.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ be the corresponding eigenvalues. For nonzero $x \in \mathbb{R}^n$ we
have $A(x) \neq 0$, because $A$ is invertible. Consequently

$$B(x) \cdot x = A(x) \cdot A(x) > 0.$$  

If $x = u_j$ then $B(x) \cdot x = \lambda_j$, so we deduce that $\lambda_j > 0$. So

$$f(u_j) = \frac{B(u_j)}{||B(u_j)||} = \frac{\lambda_j u_j}{||\lambda_j \cdot u_j||} = u_j.$$

Thus $u_1, u_2, \ldots, u_n$ are the desired $n$ fixed points.

Problem 14. Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbb{R}$.

(a) Find the irreducible monic polynomial of $\alpha$ over $\mathbb{Q}$. Be sure to explain how
you know it is irreducible.

Solution: Let $f(x) = x^4 - 2x^2 - 1$. Then $f$ is the irreducible monic polynomial
of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Indeed $\alpha^2$ satisfies the equation $x^2 - 2x - 1 = 0$, so $\alpha$ satisfies
$f(x) = 0$. Thus it suffices to see that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Write

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$$

and observe that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ because $2$ is irrational. On the other hand,
let $\sigma$ be the field embedding $\mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ satisfying $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $\sigma$
has an extension to an embedding (which we will also denote $\sigma$) of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$, and
$\sigma(\alpha) \notin \mathbb{R}$ because $1 - \sqrt{2} < 0$. But $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, so $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2})] = 2$. Hence
$[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ and $f$ is irreducible.

(b) Is $\mathbb{Q}(\alpha)$ Galois over $\mathbb{Q}$? Why or why not?

Solution: No, $\mathbb{Q}(\alpha)$ is not Galois over $\mathbb{Q}$. For let $\sigma$ be as above. Then $\mathbb{Q}(\alpha) \subseteq \mathbb{R}$
but $\sigma(\mathbb{Q}(\alpha)) \not\subseteq \mathbb{R}$, so $\sigma(\mathbb{Q}(\alpha)) \neq \mathbb{Q}(\alpha)$. Hence $\mathbb{Q}(\alpha)$ is not normal over $\mathbb{Q}$ and	herefore not Galois.