Preliminary Exam 2017
Solutions to Afternoon Exam

Part I.

Solve four of the following five problems.

Problem 1. Find a basis for the solution space of the system of equations

$$\begin{cases} w + 5x + y + 2z = 0 \\ 3w + 15x + 4y - 4z = 0. \end{cases}$$

Solution: The matrix associated to this homogeneous system is

$$\begin{pmatrix} 1 & 5 & 1 & 2 \\ 3 & 15 & 4 & -4 \end{pmatrix}.$$

Subtracting 3 times the first row from the second, and then the second from the first, we obtain

$$\begin{pmatrix} 1 & 5 & 0 & 12 \\ 0 & 0 & 1 & -10 \end{pmatrix},$$

a matrix in row-reduced upper-echelon form. Hence a basis for the solution space is $\{(-5, 1, 0, 0), (-12, 0, 10, 1)\}$.

Problem 2. Let

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

Find an invertible 2×2 matrix U with coefficients in $\mathbb{C}$ such that $U^{-1}AU$ is diagonal.

Solution: The characteristic polynomial of A is $x^2-6x+25$. Thus the eigenvalues of A are

$$\lambda_\pm = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm 4i.$$

The corresponding eigenvectors span the null space of

$$A - \lambda_\pm I = \begin{pmatrix} \mp 4i & -4 \\ 4 & \mp 4i \end{pmatrix}$$

By a row reduction – or simply by inspection – we see that the vectors $(\pm i, 1)$ span the null space. Thus putting

$$U = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix},$$

we see that $U^{-1}AU$ is diagonal.

Problem 3. Find an orthonormal basis (relative to the dot product) for the subspace of $\mathbb{R}^4$ spanned by the vectors $(-1/2, 1/2, 1/2, 1/2)$, $(1/2, 1/2, -1/2, 1/2)$, and $(1, 1, 2, 2)$.

Solution: Put $v_1 = (-1/2, 1/2, 1/2, 1/2)$, $v_2 = (1/2, 1/2, -1/2, 1/2)$, and $v_3 = (1, 1, 2, 2)$. Apply the Gram-Schmidt process to the vectors v_1, v_2, v_3. The vectors v_1 and v_2 are already orthonormal, so put $u_1 = v_1$ and $u_2 = v_2$. So the vector

$$w = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$$

is orthogonal to both u_1 and u_2. A calculation shows that $w = (3/2, -1/2, 3/2, 1/2)$. Putting

$$u_3 = \frac{1}{\sqrt{5}}(3/2, -1/2, 3/2, 1/2),$$

we conclude that $\{u_1, u_2, u_3\}$ is the desired orthonormal set.

Problem 4. If A and B are 3×3 matrices with coefficients in $\mathbb{C}$, then A and B are *similar* if there is an invertible matrix U such that $UBU^{-1} = A$. Are the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

similar? Why or why not?

Solution: No, A and B are not similar. For suppose on the contrary that $UBU^{-1} = A$ with an invertible matrix U. Then $U(B-I)U^{-1} = A - I$, so

$$(A-I)^2 = U(B-I)^2U^{-1} = UOU^{-1} = O,$$

where I and O are the 3×3 identity matrix and zero matrix respectively. But $(A-I)^2 \neq 0$, a contradiction.

Alternatively, the fact that A has characteristic polynomial $(x-1)^3$ but $(A-I)^2 \neq 0$ means that the Jordan normal form of A is a single 3×3 Jordan block with eigenvalue 1, whereas the fact B has characteristic polynomial $(x-1)^3$ but $(B-I)^2 = O$ means that the Jordan normal form of B consists of a 1×1 Jordan block and a 2×2 Jordan block, both with eigenvalue 1. Since the Jordan normal forms of A and B differ, A and B are not similar.

Problem 5. Let I be the ideal of $\mathbb{Z}$ generated by 6670 and 14007. Find the positive integer c such that I is the principal ideal generated by c.

Solution: We use the Euclidean algrorithm to determine $c = \gcd(6670, 14007)$. Multiplying 6670 by 2 and subtracting from 14007, we obtain 667, so that

$$c = \gcd(6670, 14007) = \gcd(6670, 667) = 667.$$

Part II.

Solve three of the following six problems.

Problem 6. Let L be the subgroup of $\mathbb{Z}^3$ generated by $(1,0,1)$, $(6,2,0)$, and $(7,2,5)$. Find a direct sum of cyclic groups isomorphic to $\mathbb{Z}^3/L$.

Solution: By row and column operations over $\mathbb{Z}$ we find that $UAV = B$, where

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

and U and V are invertible matrices over $\mathbb{Z}$. Therefore $\mathbb{Z}^3/L \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$.

Problem 7. Let A_n be the $n \times n$ matrix with the integers $1, 2, 3, \ldots, n$ along the first row and column, 1's down the diagonal, and 0's elsewhere, so that

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ 2 & 1 & 0 & 0 & \ldots & 0 \\ 3 & 0 & 1 & 0 & \ldots & 0 \\ 4 & 0 & 0 & 1 & \ldots & 0 \\ & & & & \ldots & \\ n & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Prove that

$$\det(A_n) = 2 - \frac{n(n+1)(2n+1)}{6}.$$

Solution: By direct calculation, $\det(A_1) = 1$ and $\det(A_2) = -3$, proving the formula in these cases. Now suppose that the formula holds for some $n \geqslant 2$. Expanding $\det(A_{n+1})$ along the last column, we obtain

$$\det(A_{n+1}) = (-1)^{n+2}(n+1)\det(B) + \det(A_n),$$

where

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & \ldots & 0 \\ 3 & 0 & 1 & 0 & \ldots & 0 \\ 4 & 0 & 0 & 1 & \ldots & 0 \\ & & & & \ldots & \\ n & 0 & 0 & 0 & \ldots & 1 \\ n+1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

By expansion along the bottom row, we see that $\det(B) = (-1)^{n+1}(n+1)$, so

$$\det(A_{n+1}) = -(n+1)^2 + \det(A_n) = 2 - \frac{n(n+1)(2n+1)}{6} - (n+1)^2$$

by inductive hypothesis. Doing the arithmetic, we obtain

$$\det(A_{n+1}) = 2 - (n+1)(n+2)(2n+3)/6,$$

as desired.

Problem 8. Prove or give a counterexample: If A and B are $n \times n$ diagonalizable matrices over $\mathbb{C}$ then AB is also diagonalizable.

Solution: Without the assumption that $AB = BA$ the statement is false. To get a counterexample, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and put $B = A^{-1}C$. Then A is diagonal, hence diagonalizable, and B is diagonalizable because its eigenvalues 1 and 1/2 are distinct. But $AB = C$, which is not diagonalizable because it is a nondiagonal Jordan block.

Problem 9. Let A be an $n \times n$ matrix with coefficients in $\mathbb{R}$. If the minimal polynomial of A is $(x+1)^2$ then what is the minimal polynomial of $A^2 + A$? Why?

Solution: Write $A = UJU^{-1}$, where J is an $n \times n$ matrix in Jordan normal form and U is an $n \times n$ invertible matrix. Then the minimal polynomials of A and J are equal, and as

$$A^2 + A = U(J^2 + J)U^{-1},$$

so are the minimal polynomials of $A^2 + A$ and $J^2 + J$. Now given that the minimal polynomial of A is $(x+1)^2$, we see that J is a diagonal array of one or more 2×2 Jordan blocks of eigenvalue -1 and zero or more 1×1 Jordan blocks of eigenvalue -1. But if

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is one of the 2×2 blocks, then

$$B^2 + B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

which has minimal polynomial x^2, and if $B = (-1)$ is a 1×1 block then $B^2 + B = (0)$, which has minimal polynomial x. Since J has at least one 2×2 block, we conclude that the minimal polynomial of $J^2 + J$, and hence the minimal polynomial of $A^2 + A$, is x^2.

Problem 10. Let A be the $n \times n$ matrix over $\mathbb{R}$ with 1's on the diagonal and $1/n!$ everywhere else. Show that $\det(A) \neq 0$.

Solution: Let $a_{i,j}$ be the entry in the ith row and jth column of A. By definition,

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where σ runs over all permutations of $\{1, 2, \ldots, n\}$. Since $a_{i,i} = 1$ for all i,

$$\det(A) = 1 + \sum_{\sigma \neq 1} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where σ now runs over the nontrivial permutations. Each summand in the sum is a product of certain number of factors equal to 1 and a certain number of factors equal to $1/n!$, with at least one factor equal to $1/n!$. Since there are a total of $n! - 1$ summands in the sum, we get $\det(A) = 1 + c$, where $|c| \leqslant (n! - 1)/n!$. Therefore

$$\det(A) \geqslant 1 - |c| \geqslant 1 - ((n! - 1)/n!) \geqslant 1/n!,$$

whence $\det(A) > 0$.

Problem 11. Let R and S be commutative rings, let $f : R \to S$ be a ring homomorphism, and let I be an ideal of R. Prove that if f is surjective (or "onto") then $f(I)$ is an ideal of S.

Solution: First we show that $f(I)$ is an additive subgroup of S. Certainly $0 = f(0) \in f(I)$. Now suppose that $j, j' \in f(I)$, and write $j = f(i)$, $j' = f(i')$ with $i, i' \in I$. Then $j - j' = f(i) - f(i') = f(i - i')$, and since $i - i' \in I$ we deduce that $j - j' \in f(I)$.

To complete the proof that $f(I)$ is an ideal of S, suppse that $j \in f(I)$ and $s \in S$. As before we can write $j = f(i)$ with $i \in I$, and also, because f is surjective, $s = f(r)$ with $r \in R$. Since I is an ideal we have $ri \in I$; then $sj = f(r)f(i) = f(ri) \in f(I)$.

Part III.

Solve one of the following three problems.

Problem 12. Let p and q be primes. Show that a nonabelian group of order pq has trivial center.

Solution: If the center $Z(G)$ of G is nontrivial then it has order p or q, because if it has order pq then G is abelian. Thus the quotient $G/Z(G)$ is of prime order and is therefore cyclic. Let $cZ(G)$ be a generator of $G/Z(G)$. We will show that G is abelian and hence obtain a contradiction. Given $g, g' \in G$, we can write $g = c^i z$ and $g' = c^j z'$ with integers i and j and $z, z' \in Z(G)$. Since z and z' commute with c and with each other, we get

$$gg' = (c^i z)(c^j z') = c^{i+j} zz' = c^{j+i} z' z = (c^j z')(c^i z) = g' g,$$

proving that G is abelian.

Problem 13. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map and $A^{\mathrm{t}} : \mathbb{R}^n \to \mathbb{R}^n$ the transpose or adjoint relative to the dot product. Put $B = A^{\mathrm{t}} A$, let S be the unit sphere in $\mathbb{R}^n$ centered at the origin, and define $f : S \to S$ by $f(x) = B(x)/||B(x)||$, where $||x|| = \sqrt{x \cdot x}$. Show that there are at least n points $u \in S$ such that $f(u) = u$.

Solution: Since $B = B^{\mathrm{t}}$, we see that B is a symmetric (or self-adjoint) operator, whence $\mathbb{R}^n$ has an orthonormal basis $u_1, u_2, \cdots, u_n$ consisting of eigenvectors of B. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ be the corresponding eigenvalues. For nonzero $x \in \mathbb{R}^n$ we have $A(x) \neq 0$, because A is invertible. Consequently

$$B(x) \cdot x = A(x) \cdot A(x) > 0.$$

If $x = u_j$ then $B(x) \cdot x = \lambda_j$, so we deduce that $\lambda_j > 0$. So

$$f(u_j) = \frac{B(u_j)}{||B(u_j)||} = \frac{\lambda_j u_j}{|\lambda_j| \cdot ||u_j||} = u_j.$$

Thus $u_1, u_2, \ldots, u_n$ are the desired n fixed points.

Problem 14. Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbb{R}$.

(a) Find the irreducible monic polynomial of α over $\mathbb{Q}$. Be sure to explain how you know it is irreducible.

Solution: Let $f(x) = x^4 - 2x^2 - 1$. Then f is the irreducible monic polynomial of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Indeed α^2 satisfies the equation $x^2 - 2x - 1 = 0$, so α satisfies $f(x) = 0$. Thus it suffices to see that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Write

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$$

and observe that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ because 2 is irrational. On the other hand, let σ be the field embedding $\mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ satisfying $\sigma(\sqrt{2}) = -\sqrt{2}$. Then σ has an extension to an embedding (which we will also denote σ) of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$, and $\sigma(\alpha) \notin \mathbb{R}$ because $1 - \sqrt{2} < 0$. But $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$, so $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{2})] = 2$. Hence $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ and f is irreducible.

(b) Is $\mathbb{Q}(\alpha)$ Galois over $\mathbb{Q}$? Why or why not?

Solution: No, $\mathbb{Q}(\alpha)$ is not Galois over $\mathbb{Q}$. For let σ be as above. Then $\mathbb{Q}(\alpha) \subset \mathbb{R}$ but $\sigma(\mathbb{Q}(\alpha)) \not\subset \mathbb{R}$, so $\sigma(\mathbb{Q}(\alpha)) \neq \mathbb{Q}(\alpha)$. Hence $\mathbb{Q}(\alpha)$ is not normal over $\mathbb{Q}$ and therefore not Galois.