Preliminary Exam 2016
Solutions to Afternoon Exam

Part I.

Solve four of the following five problems.

Problem 1. Let W be the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$. Find a basis for W which is orthonomal relative to the usual dot product.

Solution: The vectors $w_1 = (2, -1, 0)$ and $w_2 = (3, 0, -1)$ are a basis for W. We use the Gram-Schmidt process to obtain an orthogonal basis $\{v_1, v_2\}$ for W: Put $v_1 = w_1$ and

$$v_2 = w_2 - \frac{w_1 \cdot w_2}{w_1 \cdot w_1} w_1 = w_2 - (6/5)w_1 = (3/5, 6/5, -1).$$

Finally, an orthonormal basis is $\{u_1, u_2\}$, where

$$u_1 = v_1/|v_1| = (2/\sqrt{5}, -1/\sqrt{5}, 0)$$

and

$$u_2 = v_2/|v_2| = (3/\sqrt{70}, 6/\sqrt{70}, -5/\sqrt{70}).$$

Of course there are infinitely many correct answers.

Problem 2. Consider the matrix (with real coefficients)

$$A = \begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}.$$

Find an invertible matrix C such that $C^{-1}AC$ is diagonal.

Solution: The characteristic polynomial of A is $x^2 - 5x + 6 = (x - 2)(x - 3)$, and one easily computes that $(2, 3)$ and $(1, 1)$ are eigenvectors corresponding to the eigenvalues 2 and 3. So the matrix

$$C = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

has the property that $C^{-1}AC$ is the diagonal matrix with diagonal entries 2 and 3.

Problem 3. Let A be a 2×2 matrix with real coefficients, and suppose that $A^4 = I$ but $A^2 \neq I$, where I is the 2×2 identity matrix.

(a) Find $\operatorname{tr}(A)$.

Solution: Since $A^4 = I$ the eigenvalues of A are contained in the set $\{\pm 1, \pm i\}$, and since the equation $x^4 - 1 = 0$ has 4 distinct roots A is diagonalizable. But since $A^2 \neq I$, either i or $-i$ is an eigenvalue. In fact if one i or $-i$ is an eigenvalue then both are eigenvalues, because A – and therefore also the characteristic polynomial of A – has real coefficients. Therefore the eigenvalues are precisely i and $-i$, and $\operatorname{tr}(A) = i + (-i) = 0$.

(b) Give an example of a 2×2 matrix B with *complex* coefficients such that $B^4 = I$ and $B^2 \neq I$ but $\operatorname{tr}(B) \neq \operatorname{tr}(A)$.

Solution: Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

so that $\operatorname{tr}(B) = 1 + i \neq \operatorname{tr}(A)$.

Problem 4. Let $\varphi : \mathbb{Q} \to \mathbb{Z}$ be a group homomorphism. Show that $\varphi(x) = 0$ for all $x \in \mathbb{Q}$.

Solution: Suppose that for some $x \in \mathbb{Q}$ we have $\varphi(x) \neq 0$. Then there is a prime number p not dividing $\varphi(x)$. Since φ is a homomorphism, we find

$$\varphi(x) = \varphi(p(x/p)) = p\varphi(x/p) = pn$$

where $n = \varphi(x/p) \in \mathbb{Z}$, a contradiction.

Problem 5. A sequence of row operations transforms the matrix

$$A = \begin{pmatrix} 1 & 4 & * & * \\ 3 & 2 & * & * \\ 5 & 3 & * & * \end{pmatrix} \qquad \text{into the matrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation which has A as its matrix relative to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$. Find a basis for the kernel and image of T, and determine the third and fourth columns of A.

Solution: Because row operations correspond to left-multiplication by an invertible matrix, B is the matrix of T relative to the *standard basis* $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$ and a new basis $\{v_1, v_2, v_3, \}$ for $\mathbb{R}^3$. By inspection, $T(e_3) = T(e_1) + 2T(e_2)$ and $T(e_4) = -3T(e_1) + 5T(e_2)$, and consequently the vectors

$$e_1 + 2e_2 - e_3 = (1, 2, -1, 0)$$

and

$$-3e_1 + 5e_2 - e_4 = (-3, 5, 0, -1)$$

are in the kernel of T. Since the nullity of T is visibly 2 and these vectors are linearly independent, they form a basis for the kernel of T. Using B and A one can also read off a basis for the image of T: A basis is given by

$$v_1 = T(e_1) = (1, 3, 5)$$

and

$$v_2 = T(e_2) = (4, 2, 3)$$

Finally, the third and fourth columns of A are the transposes of

$$v_1 + 2v_2 = T(e_1) + 2T(e_2) = (1, 3, 5) + 2(4, 2, 3) = (9, 7, 11)$$

and

$$-3v_1 + 5v_2 = -3T(e_1) + 5T(e_2) = -3(1, 3, 5) + 5(4, 2, 3) = (17, 1, 0)$$

respectively.

Part II.

Solve three of the following six problems.

Problem 6. Let A be a 3×3 matrix such that $(A^2 - I)(A - I) = 0$ but $A^2 - I \neq 0$. What are the possibilites for the Jordan normal form of A? (You may take A to have complex coefficients.)

Solution: Let $f(x)$ be the minimal monic polynomial of A. As $(A-I)^2(A+I) = 0$ but $(A - I)(A + I) \neq 0$ we see that $f(x)$ divides $(x - 1)^2(x + 1)$ but does not divide

$(x-1)(x+1)$. So the possibilities for $f(x)$ are $(x-1)^2(x+1)$ and $(x-1)^2$. Given that A is a 3×3 matrix, the corresponding Jordan normal forms are

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 7. Make a list of abelian groups of order 32 so that every abelian group of order 32 is isomorphic to exactly one group on your list.

Solution: By the Elementary Divisor Theorem, there are 7 possibilities:

$$\mathbb{Z}/32\mathbb{Z},$$

$$(\mathbb{Z}/16\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

$$(\mathbb{Z}/8\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}),$$

$$(\mathbb{Z}/8\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

$$(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

$$(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

$$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

Problem 8. Let R be a commutative ring, and let I be the set of all $r \in R$ such that $r^k = 0$ for some positive integer k (which may depend on r). Show that I is an ideal of R.

Solution: If $r, s \in I$ then there exist integrers $k, l \geqslant 1$ such that $r^k = s^l = 0$. Since R is commutative, the Binomial Theorem shows that $(r+s)^{k+l}$ is a sum of integer multiples of expressions of the form $r^j s^{k+l-j}$ with $0 \leqslant j \leqslant k+l$, and either $j \geqslant k$ or $k+l-j \geqslant l$. Therefore all such terms are 0 and $r+s \in I$. On the other hand, if $a \in R$ then $(ar)^k = a^k r^k = a^k \cdot 0 = 0$. So I is an ideal.

Problem 9. Write $\mathbb{Z}^3/L$ as a direct sum (or a direct product) of cyclic groups, where L is the subgroup of $\mathbb{Z}^3$ generated by $(1,1,-2)$, $(7,9,-14)$, and $(5,9,-6)$.

Solution: Let A be the matrix with these vectors as columns:

$$A = \begin{pmatrix} 1 & 7 & 5 \\ 1 & 9 & 9 \\ -2 & -14 & -6 \end{pmatrix}.$$

A series of row and column operations over $\mathbb{Z}$ converts A into a diagonal matrix with diagonal entries 1, 2, and 4. So $\mathbb{Z}^3/L \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

Problem 10. Find an integer $n \geqslant 1$ such that the Galois group over $\mathbb{Q}$ of the polynomial $x^n - n$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Solution: Take $n = 4$. Since $x^4 - 4 = (x^2+2)(x^2-2)$, the corresponding Galois extension of $\mathbb{Q}$ is the compositum $K = K_1K_2$, where $K_1 = \mathbb{Q}(\sqrt{-2})$ and $K_2 = \mathbb{Q}(\sqrt{2})$. Note that $\mathrm{Gal}(K_1/\mathbb{Q}) \cong \mathrm{Gal}(K_2/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$. Also $K_1 \cap K_2 = \mathbb{Q}$: Indeed $K_1 \neq K_2$ (since $K_2 \subset \mathbb{R}$ and $K_1 \not\subset \mathbb{R}$) and the only subfields of K_2 are K_2 itself and $\mathbb{Q}$ (since $[K_2 : \mathbb{Q}] = 2$). Hence the formal properties of the Galois correspondence give $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Gal}(K_1/\mathbb{Q}) \times \mathrm{Gal}(K_2/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Problem 11. Let A be an $n \times n$ matrix, let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of A, and let a_{ij} be the entry in the ith row and jth column of A. For $1 \leqslant h, i \leqslant n$, what is $\sum_{j=1}^{n} a_{hj}(-1)^{i+j}\det(A_{ij})$? Why? Your answer should depend on whether $h = i$ or $h \neq i$.

Solution: If $h = i$ then the given sum is just the expansion of $\det(A)$ along the ith row, so the value of the sum is $\det(A)$. If $h \neq i$ then the given sum is the expansion of $\det(A')$ along the ith row, where A' is obtained from A by changing the ith row so that it is identical to the hth row and leaving all other rows the same. Thus in A' the ith row and hth row are equal, and so $\det(A') = 0$. So the given sum is 0.

Part III.

Solve one of the following three problems.

Problem 12. Factor the polynomial $x^3 - 2$ into irreducible polynomials in $\mathbb{F}_p[x]$ for $p = 3, 5$, and 7, where $\mathbb{F}_p$ is the field with p elements.

Solution: If $p = 3$ then $x^3 - 2 = x^3 + 1 = (x+1)^3$.

If $p = 5$ then $|\mathbb{F}_p^\times| = 4$, so the map $u \mapsto u^3$ is an isomorphism and $2 = \alpha^3$ for some $\alpha \in \mathbb{F}_5$. In fact $3^3 = 27 = 2$ in $\mathbb{F}_5$, so $x - 3$ is a factor of $x^3 - 2$ in $\mathbb{F}_5[x]$. Using synthetic division or otherwise, one finds that $x^3 - 2 = (x-3)(x^2 - 2x - 1)$, and $f(x) = x^2 - 2x - 1$ is irreducible because all the values of f are nonzero: $f(0) = 4$, $f(1) = 3$, $f(2) = 4$, $f(3) = 2$, and $f(4) = 2$.

Finally, if $p = 7$ then $f(x) = x^3 - 2$ is irreducible, either because the values of f are all nonzero or because 2 has order 3 in $\mathbb{F}_7^\times$ (indeed $2^3 = 8 = 1$ in $\mathbb{F}_7$) and any cube in a group of order 6 has order dividing 2.

Problem 13. Let G be a finite group and H a subgroup, and let $\mathcal{S}$ be the group of permutations of G/H, the set of left cosets of H in G. Define a homomorphism $\varphi : G \to \mathcal{S}$ by setting $\varphi(a)(bH) = abH$ for $a, b \in G$. Show that the kernel of φ is the largest normal subgroup of G contained in H.

Solution: We have $a \in \ker\varphi$ if and only if $bH = abH$ for all $b \in G$, or in other words, $b^{-1}ab \in H$ for all $b \in G$. Equivalently, $\ker\varphi = \bigcap_{b\in G} bHb^{-1}$. This intersection is contained in H (take $b = 1$), and it is a subgroup of G because it is an intersection of subgroups of G. Furthermore it is normal in G because conjugation by a fixed element of G merely permutes the terms in the intersection. Finally, if N is a normal subgroup of G and $N \subset H$ then for any $b \in G$ we have $N = bNb^{-1} \subset bHb^{-1}$. Therefore N is contained in the intersection $\bigcap_{b\in G} bHb^{-1}$.

Problem 14. Let A be a square matrix with coefficients in $\mathbb{C}$, and suppose that A is not diagonalizable but that A^n is diagonalizable for some $n \geqslant 2$. Show that $\det(A) = 0$.

Solution: A square matrix over $\mathbb{C}$ is diagonalizable if and only if its minimal polynomial can be factored into distinct monic linear factors. Thus $f(A^n) = 0$ for some polynomial $f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are all distinct. Equivalently, $g(A) = 0$, where $g(x) = \prod_{j=1}^{m}\prod_{k=1}^{n}(x - \mu_j e^{2\pi i k/n})$ where $\mu_j^n = \lambda_j$. If every λ_j is nonzero then $g(x)$ is a product of distinct monic linear factors and therefore the same assertion holds for the minimal polynomial of A, which divides $g(x)$ since $g(A) = 0$. Then A is diagonalizable, contrary to hypothesis. So one of the $\lambda_j' s$ is zero and therefore $\det(A^n) = 0$, whence also $\det(A) = 0$.