Preliminary Exam 2017
Morning Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Verify that

$$\int_0^{2\pi} \cos^2 x\, dx = 6 \sum_{n \geqslant 0} (-1)^n \frac{3^{-(2n+1)/2}}{2n+1}$$

by computing both sides.

Problem 2. Suppose that $y = y(t)$ is a differentiable function on $\mathbb{R}$ satisfying $y'(t) - \sin(2t)y(t) = e^{\sin^2 t}$. If $y(0) = 0$ what is $y(\pi)$?

Problem 3. Let D be the upper half of the standard unit ball in $\mathbb{R}^3$, defined by the inequalities $x^2 + y^2 + z^2 \leqslant 1$ and $z \geqslant 0$. Assuming that D is of constant density, find the "centroid" or "center of mass" of D. You may use symmetry considerations and a standard volume formula to reduce the amount of calculation.

Problem 4. Let f be a continuous function on $\mathbb{R}$, define $F(x) = \int_0^x f(t)\, dt$, and suppose that a and b are real numbers with $a < b$. Apply the Mean Value Theorem to F on $[a, b]$, simplifying your answer and expressing the result entirely in terms of f. Then interpret the result geometrically.

Problem 5. Let $\varepsilon(n)$ be the nth digit in the decimal expansion of π, so that $\varepsilon(1) = 3$, $\varepsilon(2) = 1$, $\varepsilon(3) = 4$, $\varepsilon(4) = 1$, $\varepsilon(5) = 5$, and so on. Does the infinite series $\sum_{n \geqslant 1} (-1)^{\varepsilon(n)} (\ln(1 + 1/n) - 1/n)$ converge? Why or why not?

Part II.

Solve three of the following six problems.

Problem 6. Find the value of the line integral $\int_C (y + e^x)dx + (x^2 - x + e^y)dy$, where C is the ellipse $x^2/4 + y^2/9 = 1$ in the xy-plane, oriented counterclockwise.

Problem 7. Let f and g be real-valued functions on $\mathbb{R}$. Assume $|f(x)| \leqslant M$ for some constant $M > 0$ and $\lim_{x \to 0} g(x) = 0$.

(a) Using the formal definition of "limit," prove that $\lim_{x \to 0} f(x)g(x) = 0$.

(b) Use (a) to show that the function

$$r(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at 0.

Problem 8. Find the maximum and minimum values of $f(x) = xz + yz$ on the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\}$.

Problem 9. Let $\{x_n\}$ be the sequence of positive real numbers defined by $x_1 = 1$ and, for $n \geqslant 1$,

$$x_{n+1} = \frac{1}{x_n + x_n^{-1}}.$$

Show that $\{x_n\}$ converges. To what number does it converge?

Problem 10. Define functions $f_n : [0,1] \to \mathbb{R}$ for $n \geqslant 1$ by

$$f_n(x) = \begin{cases} x^n \ln x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Is f_n is continuous at 0? Justify your answer.
b) Is $\{f_n\}$ a uniformly convergent sequence of functions? Justify your answer.

Problem 11. Find the value of the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{dS}$, where the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (e^y + xz, e^x - yz, z)$, the surface S is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and the normal vector points outward.

Part III.

Solve one of the following three problems.

Problem 12. Let S be the set of finite sums of the form $\sum_{n=a}^{b} 1/n$, where $1 \leqslant a \leqslant b$. Prove that S is dense in the set of nonnegative real numbers.

Problem 13. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x,y) = (x^3 + e^y, y^5 - e^x)$. Prove that f is an open mapping. In other words, show that if U is an open subset of $\mathbb{R}^2$ then so is $f(U)$.

Problem 14. Let X be a complete metric space with metric d satisfying the following condition: For every $\varepsilon > 0$ there is a collection of finitely many open balls of radius ε which covers X. Prove that X is compact.