Part I.
Solve four of the following five problems.

**Problem 1.** Verify that
\[
\int_0^{2\pi} \cos^2 x \, dx = 6 \sum_{n \geq 0} (-1)^n \frac{3^{-(2n+1)/2}}{2n+1}
\]
by computing both sides.

**Problem 2.** Suppose that \( y = y(t) \) is a differentiable function on \( \mathbb{R} \) satisfying
\[
y'(t) - \sin(2t)y(t) = e^{\sin^2 t}.
\]
If \( y(0) = 0 \) what is \( y(\pi) \)?

**Problem 3.** Let \( D \) be the upper half of the standard unit ball in \( \mathbb{R}^3 \), defined by the inequalities
\[
x^2 + y^2 + z^2 \leq 1 \quad \text{and} \quad z \geq 0.
\]
Assuming that \( D \) is of constant density, find the “centroid” or “center of mass” of \( D \). You may use symmetry considerations and a standard volume formula to reduce the amount of calculation.

**Problem 4.** Let \( f \) be a continuous function on \( \mathbb{R} \), define
\[
F(x) = \int_0^x f(t) \, dt,
\]
and suppose that \( a \) and \( b \) are real numbers with \( a < b \). Apply the Mean Value Theorem to \( F \) on \([a, b]\), simplifying your answer and expressing the result entirely in terms of \( f \). Then interpret the result geometrically.

**Problem 5.** Let \( \varepsilon(n) \) be the \( n \)th digit in the decimal expansion of \( \pi \), so that \( \varepsilon(1) = 3, \varepsilon(2) = 1, \varepsilon(3) = 4, \varepsilon(4) = 1, \varepsilon(5) = 5 \), and so on. Does the infinite series
\[
\sum_{n \geq 1} (-1)^{\varepsilon(n)}(\ln(1 + 1/n) - 1/n)
\]
converge? Why or why not?

Part II.
Solve three of the following six problems.

**Problem 6.** Find the value of the line integral
\[
\int_C (y + e^x) \, dx + (x^2 - x + e^y) \, dy,
\]
where \( C \) is the ellipse \( x^2/4 + y^2/9 = 1 \) in the \( xy \)-plane, oriented counterclockwise.

**Problem 7.** Let \( f \) and \( g \) be real-valued functions on \( \mathbb{R} \). Assume \( |f(x)| \leq M \) for some constant \( M > 0 \) and \( \lim_{x \to 0} g(x) = 0 \).

(a) Using the formal definition of “limit,” prove that \( \lim_{x \to 0} f(x)g(x) = 0 \).

(b) Use (a) to show that the function
\[
r(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}
\]
is differentiable at 0.

**Problem 8.** Find the maximum and minimum values of \( f(x) = xz + yz \) on the sphere \( S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4 \} \).

**Problem 9.** Let \( \{x_n\} \) be the sequence of positive real numbers defined by \( x_1 = 1 \) and, for \( n \geq 1 \),
\[
x_{n+1} = \frac{1}{x_n + x_n^{-1}}.
\]
Show that \( \{x_n\} \) converges. To what number does it converge?
Problem 10. Define functions $f_n : [0, 1] \to \mathbb{R}$ for $n \geq 1$ by

$$f_n(x) = \begin{cases} x^n \ln x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Is $f_n$ is continuous at 0? Justify your answer.

b) Is $\{f_n\}$ a uniformly convergent sequence of functions? Justify your answer.

Problem 11. Find the value of the surface integral $\int_S \mathbf{F} \cdot d\mathbf{S}$, where the vector field $\mathbf{F}$ is given by $\mathbf{F}(x, y, z) = (e^y + xz, e^x - yz, z)$, the surface $S$ is the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$, and the normal vector points outward.

Part III.

Solve one of the following three problems.

Problem 12. Let $S$ be the set of finite sums of the form $\sum_{n=a}^{b} 1/n$, where $1 \leq a \leq b$. Prove that $S$ is dense in the set of nonnegative real numbers.

Problem 13. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) = (x^3 + e^y, y^5 - e^x)$. Prove that $f$ is an open mapping. In other words, show that if $U$ is an open subset of $\mathbb{R}^2$ then so is $f(U)$.

Problem 14. Let $X$ be a complete metric space with metric $d$ satisfying the following condition: For every $\varepsilon > 0$ there is a collection of finitely many open balls of radius $\varepsilon$ which covers $X$. Prove that $X$ is compact.