Preliminary Exam 2017
Afternoon Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Find a basis for the solution space of the system of equations

$$\begin{cases} w + 5x + y + 2z = 0 \\ 3w + 15x + 4y - 4z = 0. \end{cases}$$

Problem 2. Let

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

Find an invertible 2×2 matrix U with coefficients in $\mathbb{C}$ such that $U^{-1}AU$ is diagonal.

Problem 3. Find an orthonormal basis (relative to the dot product) for the subspace of $\mathbb{R}^4$ spanned by the vectors $(-1/2, 1/2, 1/2, 1/2)$, $(1/2, 1/2, -1/2, 1/2)$, and $(1, 1, 2, 2)$.

Problem 4. If A and B are 3×3 matrices with coefficients in $\mathbb{C}$, then A and B are *similar* if there is an invertible matrix U such that $UBU^{-1} = A$. Are the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

similar? Why or why not?

Problem 5. Let I be the ideal of $\mathbb{Z}$ generated by 6670 and 14007. Find the positive integer c such that I is the principal ideal generated by c.

Part II.

Solve three of the following six problems.

Problem 6. Let L be the subgroup of $\mathbb{Z}^3$ generated by $(1, 0, 1)$, $(6, 2, 0)$, and $(7, 2, 5)$. Find a direct sum of cyclic groups isomorphic to $\mathbb{Z}^3/L$.

Problem 7. Let A_n be the $n \times n$ matrix with the integers $1, 2, 3, \ldots, n$ along the first row and column, 1's down the diagonal, and 0's elsewhere, so that

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ 2 & 1 & 0 & 0 & \ldots & 0 \\ 3 & 0 & 1 & 0 & \ldots & 0 \\ 4 & 0 & 0 & 1 & \ldots & 0 \\ & & & & \ldots & \\ n & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Prove that

$$\det(A_n) = 2 - \frac{n(n+1)(2n+1)}{6}.$$

Problem 8. Prove or give a counterexample: If A and B are $n\times n$ diagonalizable matrices over $\mathbb{C}$ then AB is also diagonalizable.

Problem 9. Let A be an $n \times n$ matrix with coefficients in $\mathbb{R}$. If the minimal polynomial of A is $(x+1)^2$ then what is the minimal polynomial of $A^2 + A$? Why?

Problem 10. Let A be the $n \times n$ matrix over $\mathbb{R}$ with 1's on the diagonal and $1/n!$ everywhere else. Show that $\det(A) \neq 0$.

Problem 11. Let R and S be commutative rings, let $f : R \to S$ be a ring homomorphism, and let I be an ideal of R. Prove that if f is surjective (or "onto") then $f(I)$ is an ideal of S.

Part III.

Solve one of the following three problems.

Problem 12. Let p and q be primes. Show that a nonabelian group of order pq has trivial center.

Problem 13. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map and $A^{\mathrm{t}} : \mathbb{R}^n \to \mathbb{R}^n$ the transpose or adjoint relative to the dot product. Put $B = A^{\mathrm{t}}A$, let S be the unit sphere in $\mathbb{R}^n$ centered at the origin, and define $f : S \to S$ by $f(x) = B(x)/||B(x)||$, where $||x|| = \sqrt{x \cdot x}$. Show that there are at least n points $u \in S$ such that $f(u) = u$.

Problem 14. Let $\alpha = \sqrt{1 + \sqrt{2}} \in \mathbb{R}$.

(a) Find the irreducible monic polynomial of α over $\mathbb{Q}$. Be sure to explain how you know it is irreducible.

(a) Is $\mathbb{Q}(\alpha)$ Galois over $\mathbb{Q}$? Why or why not?