Preliminary Exam 2016
Morning Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Find the volume of the "ice cream cone" defined by the inequalities $x^2 + y^2 + z^2 \leqslant 1$ and $x^2 + y^2 \leqslant z^2/3$ for $z \geqslant 0$.

Problem 2. Determine the radius of convergence and interval of convergence of the power series $\sum_{n\geqslant 1}(1 + 1/n)^{n^2} x^n$.

Problem 3. Prove that $\cos x_0 = x_0$ for a unique $x_0 \in [0, 1]$, and show in addition that $\pi/6 < x_0 < \pi/4$.

Problem 4. Using standard techniques of integration, find antiderivatives on some open interval where the integrand is defined and continuous:
(a) $\int \tan(\cos^2 x)\sin(2x)\,dx$.
(b) $\int \cos(\log x)\,dx$. (Here "log" is understood to be "ln.")

Problem 5. Find all solutions to the differential equation $y'' - y' - 6y = \cos t$ that are bounded on $[0, \infty)$ and satisfy the condition $y(0) = 0$.

Part II.

Solve three of the following six problems.

Problem 6. Let $\mathbf{F}(x, y, z) = (2x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j} + z\mathbf{k}$.
(a) Compute $\nabla \times \mathbf{F}$.
(b) Let C be the curve given parametrically by $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j}$ for $0 \leqslant t \leqslant 2\pi$. Find the value of the line integral $\int_C \mathbf{F} \cdot ds$.

Problem 7. Fix an element $c \in \mathbb{R}$, and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x - c|$. Show that f is uniformly continuous.

Problem 8. The formula $f(x, y, z) = (x + y^2 + z^2, x^2 + y + z^2, x^2 + y^2 + z)$ defines a function $f : \mathbb{R}^3 \to \mathbb{R}^3$.
(a) Explain why there are open neighborhoods U and $V = f(U)$ of $(0, 0, 0) \in \mathbb{R}^3$ and a C^1 function $g : V \to U$ such that $g(f(x, y, z)) = (x, y, z)$ for $(x, y, z) \in U$ and $f(g(x, y, z)) = (x, y, z)$ for $(x, y, z) \in V$.
(b) Now let $h(x, y, z) = (x + e^y + e^z - 2, e^x + y + e^z - 2, e^x + e^y + z - 2)$. Show that if f is replaced by h then no such U, V, and g exist.

Problem 9. Let $f_n(x) = nxe^{-nx}$. Show that the sequence $\{f_n\}$ is pointwise convergent on $[0, 1]$ but not uniformly convergent.

Problem 10. Let V be the real vector space of C^∞ functions $f : \mathbb{R} \to \mathbb{R}$ with compact support (in other words, f vanishes outside some closed bounded interval). Define an operator $T : V \to V$ by $T(f) = f''$. Show that T is self-adjoint relative to the L^2 inner product on V. In other words, letting

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

for $f, g \in V$, show that $\langle T(f), g\rangle = \langle f, T(g)\rangle$.

Problem 11. Find the surface area of the torus described parametrically by

$$\mathbf{r}(\theta, \varphi) = \cos\theta(1 + \frac{\cos\varphi}{2})\mathbf{i} + \sin\theta(1 + \frac{\cos\varphi}{2})\mathbf{j} + \frac{\sin\varphi}{2}\mathbf{k} \qquad (0 \leqslant \theta, \varphi \leqslant 2\pi).$$

Part III.

Solve one of the following three problems.

Problem 12. Let f be a C^{2n} function in some neighborhood of a point $a \in \mathbb{R}$, and suppose that $f^{(k)}(a) = 0$ for $1 \leqslant k \leqslant 2n-1$. Show that if $f^{(2n)}(a) > 0$ then f has a local minimum at a.

Problem 13. Let X and Y be metric spaces with respective metrics $d_X(*,*)$ and $d_Y(*,*)$, let x_0 be a point of X, and let $f : X \to Y$ be a function.

(a) Consider the following definitions:

(A) f is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \varepsilon$.

(B) f is continuous at x_0 if for every sequence $\{x_n\}_{n \geqslant 1}$ in X which converges to x_0 the sequence $\{f(x_n)\}_{n \geqslant 1}$ converges to $f(x_0)$.

Show that these definitions are equivalent.

(b) Let $I = [0, 2\pi) \subset \mathbb{R}$ and $T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and consider I and T as metric spaces by restricting the standard Euclidean metrics on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. Define $g : I \to T$ by $g(x) = (\cos x, \sin x)$, and put $f = g^{-1}$. Is f continuous at $(1, 0) \in T$? Justify your answer using (B).

Problem 14. Define a real-valued function f on $\mathbb{R}$ by setting $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show by induction on n that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ for $x \neq 0$, where P_n is a polynomial.

(b) Deduce that f is a C^∞ function on $\mathbb{R}$ and that $f^{(n)}(0) = 0$ for all n.