Preliminary Exam 2015  
Morning Exam (3 hours)

Part I

Solve four of five problems.

**Problem 1** Determine, with proof, whether the series
\[ \sum_{k=1}^{\infty} \frac{\sin(1/k)}{k} \]
converges,

**Problem 2.** Calculate the surface integral
\[ \iint_S \mathbf{F} \cdot dS \]
where \( \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k} \) and \( S \) is the surface of the cylinder \( x^2 + y^2 \leq 1, 0 \leq z \leq 1 \) oriented by the outwards normal.

**Problem 3** Let \( \mathbf{F}(x, y, z) = (2xyz + \sin(x))\mathbf{i} + x^2\mathbf{j} + x^2y\mathbf{k} \). Evaluate
\[ \int_C \mathbf{F} \cdot ds \]
where \( C \) is the parametrized curve \( c(t) = (\cos^5(t), \sin^3(t), t^4), 0 \leq t \leq \pi. \)

**Problem 4** Let \( S \) be a surface in \( \mathbb{R}^3 \) with piecewise smooth boundary \( C \), and \( \mathbf{F}(x, y, z) \) a smooth vector field defined in a neighborhood of \( S \) such that \( \mathbf{F} \) is orthogonal to the tangent vectors of the boundary curve \( C \). Compute
\[ \iint_S (\nabla \times \mathbf{F}) \cdot dS \].

**Problem 5** Prove the following statement, or give a counterexample. "Let \( f(x, y) \) be a function of two variables. Then \( \lim_{(x,y) \to (0,0)} f \) exists if and only if \( \lim_{t \to 0} f(tv) \) exists for all vectors \( v \in \mathbb{R}^2 \)."

Part 2

Solve three of the following six problems.

**Problem 6.** Let \( a \) and \( b \) be positive constants, and let \( u(t) \) be a differentiable function on \( [0, \infty) \) satisfying the inequality \( u'(t) \leq au(t), u(0) \leq b \). Find an upper bound on \( u(t) \), and prove that it is the best possible.
Problem 7 Consider the system
\[
\begin{align*}
\frac{dx}{dt} &= -x - 6y \\
\frac{dy}{dt} &= 3x + 5y
\end{align*}
\]
(a) Find the general solution of the system.
(b) Sketch a phase portrait of the system.

Problem 8. Prove that there exists an \( \epsilon > 0 \) with the property that if \( A \) is an \( n \times n \) real matrix with \( |(A - I)_{i,j}| < \epsilon \) for \( 1 \leq i, j \leq n \), then \( A \) is invertible (here \( B_{i,j} \) denotes the \( (i,j) \) entry of \( B \)).

Problem 9. Let \( f,g : [0,1] \to [0, \infty) \) be continuous, non-negative functions such that \( \sup_x(f) = \sup_x(g) \) for \( x \in [0,1] \). Show that there exists a \( t \in [0,1] \) such that
\[
f^2(t) + 5f(t) = g^2 + 5g(t).
\]

Problem 10 Compute the volume of intersection of the two solid cylinders \( x^2 + y^2 \leq 1 \) and \( x^2 + z^2 \leq 1 \) in \( \mathbb{R}^3 \).

Problem 11 Consider the sequence of functions on \( (0, \infty) \) given by
\[
f_n(x) = \frac{nx}{1 + nx^2}
\]
(a) Determine if the sequence converges pointwise.
(b) Determine if the sequence converges uniformly.

Part 3
Solve one of the remaining three problems.

Problem 12 Let
\[
\sum_{k=1}^{\infty} a_k
\]
be a conditionally convergent series (i.e. \( \sum a_k \) converges, but \( \sum |a_k| \) diverges ) and \( L \) a real number. Show that there exists a bijection \( f : \mathbb{N} \to \mathbb{N} \) such that the series \( \sum_{k=1}^{\infty} a_{f(k)} \) converges to \( L \). In other words, show that a conditionally convergent series can be made to converge to an arbitrary number \( L \) by rearranging the order of its terms.

Problem 13 Let
\[
\exp : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})
\]
denote the function on the space of \( 2 \times 2 \) matrices defined by
\[
\exp(A) = I + A + A^2/2! + A^3/3! + \cdots
\]
(a) Show that \( \exp \) is a \( C^\infty \) function from \( M_{2 \times 2}(\mathbb{R}) \) to itself. (You may identify \( M_{2 \times 2}(\mathbb{R}) \) with \( \mathbb{R}^4 \).

(b) Show that there exists an open set \( U \) of the zero matrix and an open set \( V \) of the identity matrix such that \( \exp(U) = V \) and such that \( \exp \) possesses a smooth inverse on \( V \).

(c) Derive an expression for the inverse of \( \exp \).

**Problem 14** Prove the Arithmetic-Mean-Geometric-Mean inequality, namely, that if \( x_1, \ldots, x_n \) are positive real numbers, then

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}
\]

with equality if and only if \( x_1 = x_2 = \cdots = x_n \).