Preliminary Exam 2015
Morning Exam (3 hours)

Part I

Solve four of five problems.

Problem 1 Determine, with proof, whether the series

$$\sum_{k=1}^{\infty} \frac{\sin(1/k)}{k}$$

converges,

Problem 2. Calculate the surface integral

$$\iint_S F \cdot dS$$

where $F(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$ oriented by the outwards normal.

Problem 3 Let $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (2xyz + \sin(x))\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$. Evaluate

$$\int_C \mathbf{F} \cdot ds$$

where C is the parametrized curve $c(t) = (\cos^5(t), \sin^3(t), t^4), 0 \leq t \leq \pi$.

Problem 4 Let S be a surface in $\mathbb{R}^3$ with piecewise smooth boundary C, and $\mathbf{F}(x, y, z)$ a smooth vector field defined in a neighborhood of S such that $\mathbf{F}$ is orthogonal to the tangent vectors of the boundary curve C. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot dS$$

.

Problem 5 Prove the following statement, or give a counterexample. "Let $f(x, y)$ be a function of two variables. Then $\lim_{(x,y)\to(0,0)}$ exists if an only if $\lim_{t\to 0} f(tv)$ exists for all vectors $v \in \mathbb{R}^2$."

Part 2

Solve three of the following six problems.

Problem 6. Let a and b be positive constants, and let $u(t)$ be a differentiable function on $[0, \infty)$ satisfying the inequality $u'(t) \leq au(t)$, $u(0) \leq b$. Find an upper bound on $u(t)$, and prove that it is the best possible.

Problem 7 Consider the system

$$\frac{dx}{dt} = -x - 6y$$
$$\frac{dy}{dt} = 3x + 5y$$

(a) Find the general solution of the system.

(b) Sketch a phase portrait of the system.

Problem 8. Prove that there exists an $\epsilon > 0$ with the property that if A is an $n \times n$ real matrix with $|(A - I)_{i,j}| < \epsilon$ for $1 \leq i, j \leq n$, then A is invertible (here $B_{i,j}$ denotes the (i, j) entry of B).

Prolem 9. Let $f, g : [0, 1] \to [0, \infty)$ be continuous, non-negative functions such that $sup_x(f) = sup_x(g)$ for $x \in [0, 1]$. Show that there exists a $t \in [0, 1]$ such that

$$f^2(t) + 5f(t) = g^2 + 5g(t).$$

Problem 10 Compute the volume of intersection of the two solid cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$ in $\mathbb{R}^3$.

Problem 11 Consider the sequence of functions on $(0, \infty)$ given by

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

(a) Determine if the sequence converges pointwise.

(b) Determine if the sequence converges uniformly.

Part 3

Solve one of the remaining three problems.

Problem 12 Let

$$\sum_{k=1}^{\infty} a_k$$

be a conditionally convergent series (i.e. $\sum a_k$ converges, but $\sum |a_k|$ diverges) and L a real number. Show that there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} a_{f(k)}$ converges to L. In other words, show that a conditionally convergent series can be made to converge to an arbitrary number L by rearranging the order of its terms.

Problem 13 Let

$$\exp : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$$

denote the function on the space of 2×2 matrices defined by

$$\exp(A) = I + A + A^2/2! + A^3/3! + \cdots$$

(a) Show that exp is a C^∞ function from $M_{2\times 2}(\mathbb{R})$ to itself. (You may identify $M_{2\times 2}(\mathbb{R})$ with $\mathbb{R}^4$

(b) Show that there exists an open set U of the zero matrix and an open set V of the identity matrix such that $\exp(U) = V$ and such that exp possesses a smooth inverse on V.

(c) Derive an expression for the inverse of exp.

Problem 14 Prove the Arithmetic-Mean-Geometric-Mean inequality, namely, that if $x_1, \cdots, x_n$ are positive real numbers, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_n \cdots x_n}$$

with equality if an only if $x_1 = x_2 = \cdots = x_n$.