Preliminary Exam 2015
Afternoon Exam (3 hours)

Part I

Solve four of five problems.

**Problem 1.** Let \( T : \mathbb{R}^4 \to \mathbb{R}^3 \) be the linear map
\[
T(x_1, x_2, x_3, x_4) = (2x_1 + x_2 - x_3, x_1 + x_2 + 2x_4, 3x_1 + 2x_2 - x_3 + 2x_4)
\]
(a) Find a basis for \( \text{Ker}(T) \) (the kernel of \( T \))
(b) Find a basis for \( \text{Im}(T) \) (the image of \( T \))
(c) What is the rank of \( T \)? Is \( T \) surjective?

**Problem 2.** Let \( R_{\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \)
(a) Prove that if \( A \) is a \( 2 \times 2 \) matrix \( A \) such that \( AR_{\theta} = R_{\phi}A \) for every \( \theta \in \mathbb{R} \) then \( A = tR_{\phi} \) for some \( t, \phi \in \mathbb{R} \).
(b) Give a geometric interpretation of the result in (a).

**Problem 3.** Let \( p \) be a prime. Up to isomorphism, list all abelian groups of order \( p^4 \). Prove that your list is complete and irredundant.

**Problem 4.** Let \( V \) be the real vector space of continuous functions \( f : [0,1] \to \mathbb{R} \), and \( \langle \ast, \ast \rangle \) an inner product on \( V \) given by
\[
\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)x^2dx
\]
Let \( W \) be the subspace of \( V \) spanned by \( f_1(x) = 1 \) and \( f_2(x) = x^3 \). Find an orthonormal basis for \( W \).

**Problem 5.** Show that the polynomial \( x^6 + 30x^5 - 15x^3 + 6x - 120 \) is irreducible in \( \mathbb{Q}[x] \).

Part II

Solve three out of six problems.

**Problem 6.** Let \( S_n \) denote the symmetric group on \( n \) elements. What is the smallest \( n \) such that \( S_n \) contains a permutation of order 42. Explain and justify your answer.

**Problem 7.** Are the matrices
\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
Similar over \( \mathbb{R} \)? Why or why not?
Problem 8. (a) Let $A$ be a $4 \times 4$ real matrix with characteristic polynomial $(x - 1)x(x + 1)(x + 2)$, and $B$ a $4 \times 4$ real matrix such that $AB = BA$. Prove that $B$ is diagonalizable.

(b) Suppose $A$ is a diagonalizable real matrix with characteristic polynomial $(x+1)^2x(x+2)$, and $B$ a $4 \times 4$ real matrix such that $AB = BA$. Does $B$ have to be diagonalizable? Describe all such matrices $B$.

Problem 9 Let $A_n \subset S_n$ denote the alternating group on $n$ letters (i.e. the subgroup of permutations having even sign). Show that $A_n$ is generated by 3-cycles for $n \geq 3$.

Problem 10 Let $p$ be a prime, and $G$ a group of order $p^n$. Show that $G$ has a non-trivial center.

Problem 11 Let $R_1$ and $R_2$ be commutative rings, and $R_1 \times R_2$ the product ring.

(a) Prove that every ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ where $I_j \subset R_j$ are ideals for $j = 1, 2$.

(b) Which ideals $I_1 \times I_2$ are prime? Which are maximal? Explain.

Part III

Solve one of three problems.

Problem 12 Let $\mathbb{R}[x, y]$ denote the polynomial ring in two variables $x, y$ over $\mathbb{R}$, and let $I = (y^2 - x, y - x)$ be the ideal generated by $y^2 - x$ and $y - x$. Show that \( \mathbb{R}[x, y]/I \) is not an integral domain.

Problem 13 Which of the following rings are integral domains? Which ones are fields?

(a) $\mathbb{Z}[x]/(x^2 + 2x + 3)$

(b) $\mathbb{F}_5[x]/(x^2 + x + 1)$

(c) $\mathbb{R}[x]/(x^4 + 2x^3 + x^2 + 5x + 2)$

Problem 14 Give examples of each of the following, or show that no such can exist. Explain and justify your answer:

(a) A Galois extension $K/\mathbb{Q}$ of degree 4.

(b) A Galois extension $K/\mathbb{Q}$ of degree 6.

(c) A Galois extension $K/\mathbb{Q}$ of degree 7.