Preliminary Exam 2015
Afternoon Exam (3 hours)

Part I

Solve four of five problems.

Problem 1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map

$$T(x_1, x_2, x_3, x_4) = (2x_1 + x_2 - x_3, x_1 + x_2 + 2x_4, 3x_1 + 2x_2 - x_3 + 2x_4)$$

(a) Find a basis for $Ker(T)$ (the kernel of T)
(b) Find a basis for $Im(T)$ (the image of T)
(c) What is the rank of T ? Is T surjective ?

Problem 2. Let

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

(a) Prove that if A is a 2×2 matrix A such that $AR_\theta = R_\theta A$ for every $\theta \in \mathbb{R}$ then $A = tR_\phi$ for some $t, \phi \in \mathbb{R}$.
(b) Give a geometric interpretation of the result in (a).

Problem 3. Let p be a prime. Up to isomorphism, list all abelian groups of order p^4. Prove that your list is complete and irredundant.

Problem 4. Let V be the real vector space of continuous functions $f : [0, 1] \to \mathbb{R}$, and $\langle *, * \rangle$ an inner product on V given by

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)x^2 dx$$

Let W be the subspace of V spanned by $f_1(x) = 1$ and $f_2(x) = x^3$. Find an orthonormal basis for W.

Problem 5. Show that the polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ is irreducible in $\mathbb{Q}[x]$.

Part II

Solve three out of six problems.

Problem 6. Let S_n denote the symmetric group on n elements. What is the smallest n such that S_n contains a permutation of order 42. Explain and justify your answer.

Problem 7. Are the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Similar over $\mathbb{R}$? Why or why not ?

Problem 8. (a) Let A be a 4×4 real matrix with characteristic polynomial $(x-1)x(x+1)(x+2)$, and B a 4×4 real matrix such that $AB = BA$. Prove that B is diagonalizable.

(b) Suppose A is a diagonalizable real matrix with characteristic polynomial $(x+1)^2x(x+2)$, and B a 4×4 real matrix such that $AB = BA$. Does B have to be diagonzalizable ? Describe all such matrices B.

Problem 9 Let $A_n \subset S_n$ denote the alternating group on n letters (i.e. the subgroup of permutations having even sign). Show that A_n is generated by 3-cycles for $n \geq 3$.

Problem 10 Let p be a prime, and G a group of order p^n. Show that G has a non-trivial center.

Problem 11 Let R_1 and R_2 be commutative rings, and $R_1 \times R_2$ the product ring.

(a) Prove that every ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ where $I_j \subset R_j$ are ideals for $j = 1, 2$.

(b) Which ideals $I_1 \times I_2$ are prime ? Which are maximal ? Explain.

Part III

Solve one of three problems.

Problem 12 Let $\mathbb{R}[x, y]$ denote the polynomial ring in two variables x, y over $\mathbb{R}$, and let $I = (y^2 - x, y - x)$ be the ideal generated by $y^2 - x$ and $y - x$. Show that

$$\mathbb{R}[x, y]/I$$

is not an integral domain.

Problem 13 Which of the following rings are integral domains ? Which ones are fields ?

(a) $\mathbb{Z}[x]/(x^2 + 2x + 3)$

(b) $\mathbb{F}_5[x]/(x^2 + x + 1)$

(c) $\mathbb{R}[x]/(x^4 + 2x^3 + x^2 + 5x + 2)$

Problem 14 Give examples of each of the following, or show that no such can exist. Explain and justify your answer:

(a) A Galois extension $K/\mathbb{Q}$ of degree 4.

(b) A Galois extension $K/\mathbb{Q}$ of degree 6.

(c) A Galois extension $K/\mathbb{Q}$ of degree 7.