Preliminary Exam 2014
Morning Exam (3 hours)

Part I

Solve four of five problems.

Problem 1 Determine whether the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$ converges, and if so, find the limit.

Problem 2. Consider the vector field

$$\mathbf{F}(x, y) = y\mathbf{i} + (x + 2y)\mathbf{j}.$$

Compute the line integral $\int_C \mathbf{F} \bullet ds$, where C is the curve $C(t) = (t^4, 2t^6), 0 \leq t \leq 1$.

Problem 3 Show that the system of equations

$$\begin{aligned} 2\sin(x) + 3\sin(y) &= a \\ x + 5y^3 &= b \end{aligned}$$

has a solution for (a, b) sufficiently close to $(0, 0)$, and that there is a neighborhood of $(0, 0)$ in which this solution is unique.

Problem 4 Determine for which real numbers x the infinite series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x}$$

converges.

Problem 5 Consider the initial value problem

$$y' + \tan(x)y = \cos^2(x), \quad y(0) = C$$

For what values of C does the solution remain bounded for all values of x?

Part 2

Solve three of the following six problems.

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with bounded derivative (i.e. there exists an $M \geq 0$ such that $|f(x)| \leq M$ for all x). Prove that f is uniformly continuous.

Problem 7 Consider the system

$$\begin{aligned} \frac{dx}{dt} &= 8x - 11y \\ \frac{dy}{dt} &= 6x - 9y \end{aligned}$$

(a) Find the general solution of the system.

(b) Sketch a phase portrait of the system.

Problem 8. Let $a_1, a_2, \cdots a_n$ be positive real numbers, and m a positive even integer. For a real number b, let S_b denote the set of solutions to the equation

$$a_1 x_1^m + a_2 x_2^m + \cdots a_n x_n^m = b$$

Prove that S_b is a compact subset of $\mathbb{R}^n$. Is the conclusion true if the condition that m be even is relaxed ?

Prolem 9. Let n be an integer greater than 1. Is there a differentiable function on $[0, \infty)$ which satisfies $y' = y^n$ and $y(0) > 0$?

Problem 10 Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function such that $f''(x) \leq 0$. Prove that

$$tf(x) + (1-t)f(y) \leq f(tx + (1-t)y)$$

for any two points $x, y \in \mathbb{R}$ and $0 \leq t \leq 1$.

Problem 11 Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Prove that

$$sup_{0 \leq x \leq 1} |f(x)| \leq \left(\int_0^1 (f'(x))^2 dx \right)^{1/2}$$

Part 3

Solve one of the remaining three problems.

Problem 12 Let $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. Compute

$$\iint_S (\nabla \times F) \cdot dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy–plane, oriented by the outside normal.

Problem 13 Consider the series

$$\sum_{k=0}^{\infty} a_k x^k, \qquad a_0 = 1, \qquad a_k = \alpha a_{k-1} + \beta, \quad k \geq 1,$$

where $\alpha, \beta \geq 0$. Determine the interval of convergence of the series (which will depend on the values of α and β.)

Problem 14 Show that there is an $\epsilon > 0$ such that if A is a real 2×2 matrix satisfying $|a_{ij}| < \epsilon$, then there is a real 2×2 matrix X such that $X^2 + X^T = A$ (here X^T denotes the transpose of X). Is X unique ? explain.