Preliminary Exam 2014  
Morning Exam (3 hours)

Part I

Solve four of five problems.

**Problem 1**  Determine whether the sequence \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\} converges, and if so, find the limit.

**Problem 2.**  Consider the vector field \( \mathbf{F}(x, y) = y \mathbf{i} + (x + 2y) \mathbf{j} \).

Compute the line integral \( \int_C \mathbf{F} \cdot ds \), where \( C \) is the curve \( C(t) = (t^4, 2t^6), 0 \leq t \leq 1 \).

**Problem 3**  Show that the system of equations

\[
2 \sin(x) + 3 \sin(y) = a \\
x + 5y^3 = b
\]

has a solution for \((a, b)\) sufficiently close to \((0, 0)\), and that there is a neighborhood of \((0, 0)\) in which this solution is unique.

**Problem 4**  Determine for which real numbers \(x\) the infinite series

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n + 1} - \sqrt{n}}{n^x}
\]

converges.

**Problem 5**  Consider the initial value problem

\[
y' + \tan(x)y = \cos^2(x), \quad y(0) = C
\]

For what values of \(C\) does the solution remain bounded for all values of \(x\)?

Part 2

Solve three of the following six problems.

**Problem 6.**  Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is a differentiable function with bounded derivative (i.e. there exists an \(M \geq 0\) such that \(|f(x)| \leq M\) for all \(x\)). Prove that \(f\) is uniformly continuous.

**Problem 7**  Consider the system

\[
\frac{dx}{dt} = 8x - 11y \\
\frac{dy}{dt} = 6x - 9y
\]
(a) Find the general solution of the system.
(b) Sketch a phase portrait of the system.

**Problem 8.** Let \( a_1, a_2, \cdots, a_n \) be positive real numbers, and \( m \) a positive even integer. For a real number \( b \), let \( S_b \) denote the set of solutions to the equation
\[
a_1x_1^m + a_2x_2^m + \cdots + a_nx_n^m = b
\]
Prove that \( S_b \) is a compact subset of \( \mathbb{R}^n \). Is the conclusion true if the condition that \( m \) be even is relaxed?

**Problem 9.** Let \( n \) be an integer greater than 1. Is there a differentiable function on \([0, \infty)\) which satisfies \( y' = y^n \) and \( y(0) > 0 \)?

**Problem 10** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a twice continuously differentiable function such that \( f''(x) \leq 0 \).
Prove that
\[
 tf(x) + (1-t)f(y) \leq f(tx + (1-t)y)
\]
for any two points \( x, y \in \mathbb{R} \) and \( 0 \leq t \leq 1 \).

**Problem 11** Let \( f : [0, 1] \to \mathbb{R} \) be continuously differentiable with \( f(0) = 0 \). Prove that
\[
\sup_{0 \leq x \leq 1} |f(x)| \leq \left( \int_0^1 (f'(x))^2 \, dx \right)^{1/2}
\]

**Part 3**

Solve one of the remaining three problems.

**Problem 12** Let \( F(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k} \). Compute
\[
\int \int_S (\nabla \times F) \cdot dS
\]
where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies inside the cylinder \( x^2 + y^2 = 1 \) and above the \( xy \)-plane, oriented by the outside normal.

**Problem 13** Consider the series
\[
\sum_{k=0}^{\infty} a_k x^k, \quad a_0 = 1, \quad a_k = \alpha a_{k-1} + \beta, \quad k \geq 1,
\]
where \( \alpha, \beta \geq 0 \). Determine the interval of convergence of the series (which will depend on the values of \( \alpha \) and \( \beta \)).

**Problem 14** Show that there is an \( \epsilon > 0 \) such that if \( A \) is a real \( 2 \times 2 \) matrix satisfying \( |a_{ij}| < \epsilon \), then there is a real \( 2 \times 2 \) matrix \( X \) such that \( X^2 + X^T = A \) (here \( X^T \) denotes the transpose of \( X \)). Is \( X \) unique? explain.