Preliminary Exam 2013
Morning Exam (3 hours)

Part I

Solve four of five problems.

Problem 1. Consider the sequence

$$a_n = \frac{n!}{(2n+1)!!},$$

where $(2n+1)!! = (2n+1) \times (2n-1) \times (2n-3) \times ... \times 3 \times 1$. Show that $\{a_n\}$ converges, and find its limit.

Problem 2. Consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{x+y}\mathbf{i} + \frac{1}{x+y}\mathbf{j}.$$

Compute the line integral $\int_C \mathbf{F} \bullet ds$, where C is the segment of the unit circle going from $(1,0)$ to $(0,1)$.

Problem 3. Let $\phi(u,v) : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $\phi(0,0) = 0$ and $\partial_u \phi(0,0) = \partial_v \phi(0,0) = 0$. Show that for $(a,b,c) \in \mathbb{R}^3$, the system of equations

$$\begin{aligned} \sin(x) + \phi(y,z) &= a \\ \sin(y) + \phi(x,z) &= b \\ \sin(z) + \phi(x,y) &= c \end{aligned}$$

has a unique solution for (a,b,c) sufficiently close to $(0,0,0)$.

Problem 4. Give examples of subsets of $\mathbb{R}$ that are:

(a) Neither open nor closed.

(b) Infinite, but not connected.

(c) Bounded and countable.

(d) Bounded and uncountable.

(e) Closed but not compact.

(f) Dense but not complete.

Problem 5. (a) Find the Taylor series around $x = 0$ of

$$f(x) = \int_0^x \frac{dy}{1+y^4}$$

(b) What is the radius of convergence of the series in (a) ? Prove your claim.

Part II

Solve three out of six problems.

Problem 6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Prove that

$$f(x) = \lim_{\epsilon \to 0} \frac{1}{Vol(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) dV,$$

where $B_\epsilon(x)$ denotes the ball of radius ϵ centered at x, and $Vol(B_\epsilon(x))$ denotes its volume.

Problem 7. Let C_a denote the circle of radius $a > 0$ centered at the origin, oriented counterclockwise. Consider the vector field

$$\mathbf{F}(x, y) = (-y + \frac{1}{3}y^3 + x^2 y)\mathbf{i}$$

on $\mathbb{R}^2$. For what values of a is the line integral $\int_{C_a} \mathbf{F} \bullet ds$ equal to 0 ?

Problem 8. (a) Show that $f(x) = x^{1/2}$ is uniformly continuous on $[1, \infty]$.

(b) Show that $f(x) = x^{3/2}$ is not uniformly continuous on $[1, \infty]$.

Problem 9 Consider the system

$$\frac{dx}{dt} = -x + 6y$$
$$\frac{dy}{dt} = x - 2y$$

(a) Find the general solution of the system.

(b) Sketch a phase portrait of the system.

Problem 10 Let $f(x)$ be a real-valued differentiable function on $[1, \infty]$ satisfying $f(1) = 2$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2}.$$

Show that $\lim_{x \to \infty} f(x)$ exists, and that it is less than $2 + \frac{\pi}{4}$.

Problem 11 Find a curve C in the first quadrant in $\mathbb{R}^2$, passing through $(3, 2)$, with the property that if $P = (x_0, y_0)$ lies on C, then P is the midpoint of the tangent line to C at P contained in the first quadrant.

Part III

Solve one of three problems.

Problem 12 Consider the subset of real numbers given by

$$S = \left\{ \frac{(m+n)^2}{2^{mn}} | m, n \in \mathbb{N} \right\}$$

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(a) Find $inf(S)$, i.e. find the greatest real number $A \in \mathbb{R}$ such that $x \geq A$ for all $x \in S$.

(b) Find $sup(S)$, i.e. find the least real number $B \in \mathbb{R}$ such that $x \leq B$ for all $x \in S$.

Problem 13 Let $[a, b]$ denote a finite interval. Consider a sequence $\{f_n(x)\}_{n=0}^{\infty} \subset C^1([a, b])$. If $\{f_n(x)\}$ converges uniformly on $[a, b]$ to a function $f(x) \in C^1([a, b])$, does $\{f_n^{'}(x)\}$ converge uniformly to $f'(x)$? If yes, give a proof, if not, give a counter-example, and strengthen the assumptions so that $f_n'(x) \to f'(x)$ uniformly on $[a, b]$.

Problem 14 (a) Explicitly construct a function which is twice continuously differentiable on $[-1, 1]$, but not (everywhere) three times differentiable there. Justify your claims.

(b) Assume that $f(x)$ is a function in $C^3([-1, 1])$ such that $f(0) = 1$, $f'(0) = f''(0) = 0$, and $f'''(0) = A \neq 0$. Fix $t \in [-1, 1]$, and find the limit of the sequence

$$\left\{ \left(f\left(\frac{t}{n^{1/3}} \right) \right)^n \right\}_{n=1}^{\infty} .$$