**Preliminary Exam 2012**
**Morning Exam (3 hours)**

**PART I:** Solve 4 of the following 5 problems.

(1) For the vector field on $\mathbb{R}^2$, $\mathbf{v}(x,y) := \langle -y, x\rangle$, show that for any piecewise smooth simple closed curve $C$ with the counterclockwise orientation, the line integral $\int_C \mathbf{v}\cdot d\mathbf{r}$, has value equal to twice the area enclosed by $C$.

(2) Let $S = \{\,(x,y)\in\mathbb{R}^2 \mid |x|+|y|\le 1\}$. Evaluate

$$\iint_S (x+y)^{\frac{2}{3}}\,dxdy$$

(3) Consider the function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ given by $\Phi(x,y,z) = (e^x\cos y, e^x \sin y, z^2 e^{xy})$. Show that $\Phi$ is one to one in a neighborhood of any point $(x_0,y_0,z_0)$ in $\mathbb{R}^3$ where $z_0 \neq 0$.

(4) Prove or find a counterexample to the claim that a smooth function that grows faster than any linear function grows faster than $x^{1+\epsilon}$ for some $\epsilon > 0$: i.e. if $g : \mathbb{R}^+ \to \mathbb{R}$ has $\lim_{x\to\infty}\frac{g(x)}{kx} = \infty$ for all constant $k > 0$ then there exists an $\epsilon > 0$ and constant $\ell > 0$ such that $\lim_{x\to\infty}\frac{g(x)}{\ell x^{1+\epsilon}} = \infty$.

(5) Let $b_0 = 0$ and choose a positive number $b_1$ then let $b_{n+1} = b_n + b_{n-1}$ for all $n \ge 1$.
   (a) Show carefully that $b_n \le 2^n b_1$ for all $n \ge 1$.
   (b) Find the smallest $r > 0$ such that the sequence $\{\frac{b_n}{r^n}\}$ is bounded.

**PART II:** Solve 3 of the following 6 problems.

(1) Consider the function $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $d(x,x) = 0$ for all $x$ in $\mathbb{R}$ and $d(x,y) = 1$ if $x \neq y$.
   (a) Prove that $d$ is a metric.
   (b) With respect to this metric, which subsets of $\mathbb{R}$ are closed? Which subsets are compact?

(2) The equation $F(x,y) = 0$ defines a curve $C$ in $\mathbb{R}^2$ consisting of points $(x,y)$ which satisfy this equation. Assume $F$ is continuously differentiable. Also assume that $\nabla F$ is nonzero at every point of $C$. Let $(x_0,y_0)$ belong to $C$.
   (a) Find a formula for the tangent line $T$ at $(x_0,y_0)$ in terms of the above quantities.
   (b) In general, under what conditions do we expect that there will be some neighborhood $N$ of $(x_0,y_0)$ in which the equation $F(x,y) = 0$ is equivalent to a formula of the form $y = f(x)$ for some function $f$? Justify your answer.
   (c) Assume the condition(s) in (b) hold. Consider the line $L$ given by $y = cx - cx_0 + y_0$. Show that there is a neighborhood $N$ of $(x_0,y_0)$ such that $F(x,y)$ is increasing as $x$ increases along $L$, for some choice of $c$. What must $c$ satisfy for this to be true?

(3) Consider the differential equation

$$\frac{dy}{dt} = -3y + b(t) + 7$$

where the function $b(t)$ decreases to zero as $t \to \infty$. Describe carefully the long-term behavior (as $t \to \infty$) of solutions and prove your result.

(4) For all $n \geq 1$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx^2}$.

(a) Show that the sequence of functions $\{ f_n \}$ converges uniformly to some function $f : \mathbb{R} \to \mathbb{R}$.

(b) Identify the points $x$ in $\mathbb{R}$ where $\lim_{n\to\infty} f_n'(x) = f'(x)$.

(5) Suppose $U : \mathbb{R}^n - \{0\} \to \mathbb{R}$ is a smooth function which satisfies for all nonzero $\lambda \in \mathbb{R}$

$$U(\lambda x) = \frac{1}{\lambda} U(x).$$

(a) Compute an expression for

$$\nabla U|_{\lambda x}$$

(b) Suppose $q(t)$ satisfies

$$\ddot{q} = \nabla U|_{q(t)}$$

and suppose $q(t)$ has the form

$$q(t) = \phi(t) q(0)$$

where $\phi(t)$ is a scalar valued function. Derive two equations, one involving only $\phi(t)$ and the other involving only $q(0)$, which must be satisfied.

(6) Prove the Riemann-Lebesgue lemma for the special case of a function $f(x) \in C^1([0,1])$. That is, show that

$$\lim_{n\to\infty} \int_0^1 f(x) \cos(nx)\, dx = 0.$$

**PART III:** Solve 1 of the following 3 problems.

(1) Consider the function

$$f(x) = \begin{cases} 1 + 2x, & \text{if} x \geq 0 \\ 1 - 2x, & \text{if} x < 0 \end{cases}$$

(a) Find the Fourier series of this function on the interval $[-\pi, \pi]$.

(b) Does the series converges uniformly? Justify your result.

(2) Find all possible values of the line integral $\int_C L dx + M dy + N dz$ over a smooth, closed contour $C$ which does not pass through any points of the form $(x, 0, 0)$ if

$$L = x^2, \qquad M = -\frac{z}{y^2 + z^2} + y^2, \qquad N = \frac{y}{y^2 + z^2} + z^2.$$

(3) Let $M(n)$ be the space of $n \times n$ $\mathbb{R}$-matrices.

(a) Let $\mathrm{Tr} : M(n) \to \mathbb{R}$ taking $A \mapsto \mathrm{Tr}(A)$ be the trace of $A$. Find the derivative of Tr at a matrix $A$ in the direction of matrix $B$.

(b) Let $\mathrm{Det} : M(n) \to \mathbb{R}$ taking $A \mapsto \mathrm{Det}(A)$ be the determinant of $A$. Find the derivative of Det at an invertible matrix $A$ in the direction of matrix $B$.