Preliminary Exam 2014
Afternoon Exam (3 hours)

Part I

Solve four of five problems.

Problem 1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 - x_3 - x_4, 2x_1 + 4x_2 + x_3 + 10x_4, x_1 + 2x_2 + x_3 + 7x_4)$$

find a basis for the kernel of T.

Problem 2. Show that the matrices

$$A = \begin{pmatrix} 5 & 1 \\ -6 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 7 & -5 \\ 4 & -2 \end{pmatrix}$$

are similar over $\mathbb{R}$. In other words, show that there is an invertible matrix C with real coefficients such that $A = C^{-1}BC$. However, you do not need to exhibit C explicitly.

Problem 3. Give an example of a polynomial $p(x) \in \mathbb{Z}[x]$ of degree 10 which is reducible modulo $2, 3$ and 5 but irreducible over $\mathbb{Z}$.

Problem 4. Let V be the real vector space of bounded continuous functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and $\langle *, * \rangle$ an inner product on V given by

$$\langle f(x), g(x) \rangle = \int_0^\infty f(x)g(x)e^{-x}dx$$

Let W be the subspace of V spanned by $f_1(x) = 1$ and $f_2(x) = e^{-x}$. Find an orthonormal basis for W.

Problem 5. Let

$$p(x) = x^4 + 7x^3 + 14x^2 + 7x + 1$$
$$q(x) = x^4 + 10x^3 + 23x^2 + 10x + 1$$

Find polynomials $f(x), g(x)$ with rational coefficients such that

$$f(x)p(x) + g(x)q(x) = 2x^2 + 6x + 2$$

Part II

Solve three out of six problems.

Problem 6. Let S_n denote the symmetric group on n elements. Show that S_{12} contains an element of order 35 but no elements of order 33. What is the smallest n such that S_n contains an element of order 33 ?

Problem 7. Let $O(2,\mathbb{R})$ denote the group of orthogonal 2×2 matrices - i.e. 2×2 matrices A such that $AA^T = I$, and $GL(2,\mathbb{R})$ the group of invertible 2×2 matrices. Determine if $O(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$.

Problem 8. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a nonzero linear map such that the image of T is contained in the kernel of T. List the possibilities for the Jordan normal form of T. Be sure that your list is irredundant in the sense that no two matrices on your list are similar.

Problem 9 Let V denote the vector space of polynomials in one variable with coefficients in $\mathbb{R}$, and let

$$T(f(x)) = xf(x).$$

Prove that if $W \subset V$ is a subspace such that $T(W) \subset W$ (i.e. W is stable under T), then V/W is finite-dimensional.

Problem 10 Let $L \subset \mathbb{Z}$ be the subgroup of $\mathbb{Z}^3$ generated by the elements $(-1,-1,4),(2,4,0)$ and $(3,3,8)$. Write $\mathbb{Z}^3/L$ as a direct sum of cyclic groups.

Problem 11 Suppose G and H are finite groups of relatively prime orders. Prove that $Aut(G\times H)$ is isomorphic to the direct product $Aut(G)\times Aut(H)$.

Part III

Solve one of three problems.

Problem 12 (a) Show that for any $n \in \mathbb{N}$, the ring $\mathbb{Z}$ has a chain of ideals

$$\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \cdots \subsetneq I_n \subsetneq \mathbb{Z}.$$

where $\subsetneq$ denotes a *strictly proper* ideal (i.e. $I_n \neq I_{n+1}$)

(b) Does $\mathbb{Z}$ have an infinite strictly increasing chain of ideals

$$\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \cdots \subsetneq I_n \subsetneq \cdots \mathbb{Z}.?$$

If so, exhibit such a chain, and if not, give an example of a commutative ring R with such a chain.

Problem 13 Define a sequence of fields F_n,

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

as follows. $F_1 = \mathbb{Q}$, and for $n \geq 1$, let F_{n+1} be obtained from F_n by adjoining square roots of all elements of F_n. Let $F = \bigcup F_n$. Prove that F does not contain a cube root of 2.

Problem 14 Prove that the additive group of rational numbers $\mathbb{Q}$ is not finitely generated.