Preliminary Exam 2013
Afternoon Exam (3 hours)

Part I

Solve four of five problems.

Problem 1. Let P_3 denote the real subspace of $\mathbb{R}[x]$ of polynomials of degree at most 3. Let

$$T : P_3 \to P_3$$

denote the linear transformation

$$T(p(x)) = x(\frac{d}{dx}p(x)) - p(x).$$

(a) Find a basis for $Ker(T)$ and $Im(T)$. What is the rank of T ?

(b) What are the eigenvalues of T ?

Problem 2. Let $v_1, v_2, \cdots, v_n$ be vectors in a finite-dimensional real vector space V, and suppose that for every choice of scalars $c_1, c_2, \cdots, c_n \in \mathbb{R}$, there exists a linear map $\phi : V \to \mathbb{R}$ such that $\phi(v_j) = c_j$ for $1 \leq j \leq n$. Are $v_1, v_2, \cdots, v_n$ linearly independent ? Give a proof or counterexample.

Problem 3. Let A_1, A_2 and A_3 denote the columns of a 3×3 matrix. If $det(A_1, A_2, A_3) = 5$, find

$$det(A_3 - 2A_2, 4A_1 + A_3, 7A_1).$$

Problem 4. Let p be a prime number. Suppose that G is a finite group which contains exactly m subgroups of order p. Find the number of elements of G which have order p.

Problem 5. Let P_2 denote the real vector space of polynomials of degree at most 2. Let $\langle , \rangle : P_2 \times P_2 \to \mathbb{R}$ be the inner product defined by

$$\langle f, g \rangle = 2\int_0^1 xf(x)g(x)dx.$$

Find an orthonormal basis for P_2.

Part II

Solve three out of six problems.

Problem 6. In each case give a justification or a counterexample:

(a) Two 4×4 matrices over $\mathbb{R}$ with minimal polynomial $x^2(x-1)(x-2)$ are similar.

(b) Two 6×6 matrices over $\mathbb{R}$ with minimal polynomial $x^2(x-1)(x-2)$ are similar.

Problem 7. In each case, give an example of the stated type or say why none exists:

(a) Fields F and K with $F \subset K$ and a polynomial $p(x) \in F[x]$ which generates a maximal ideal of $F[x]$ but not of $K[x]$.

(b) Fields F and K with $F \subset K$ and a polynomial $p(x) \in F[x]$ which generates a maximal ideal of $K[x]$ but not of $F[x]$.

Problem 8. Give an example of two *non-isomorphic* abelian groups of order 32 which both have exactly 16 elements of order 8.

Problem 9 Give an example of each of the following or prove that no such example exists.

(a) A group of order 81 with trivial center.

(b) A group of order 40 which is not isomorphic to a subgroup of S_{40}, where the latter denotes the symmetric group on 40 elements.

Problem 10 Let $f : R \to S$ be a ring homomorphism. For each of following statements either prove them or give an explicit counter-example.

(a) If f is one-to-one and R is an integral domain, then S is an integral domain.

(b) If f is onto and R is an integral domain, then S is an integral domain.

(c) If f is one-to-one and R is a field, then S is a field.

(d) If f is onto and R is a field, then S is a field.

Problem 11 Let A and B be $n \times n$ complex matrices such that $AB - BA = A$.

(a) Show that if B has an eigenvector with eigenvalue λ, then Av is either zero, or an eigenvector of B. Find the eigenvalue.

(b) Prove that A is nilpotent, i.e. that $A^n = 0$ for some $n > 0$.

Part III

Solve one of three problems.

Problem 12 Show that there are no simple groups of order 12.

Problem 13 Find all subfields of $\mathbb{Q}(\sqrt[3]{5}, \exp 2\pi i/3)$ which are Galois over $\mathbb{Q}$.

Problem 14 Let R be a commutative ring with 1. Show that if there exists a monic polynomial $p(x) \in R[x]$ of degree at least one such that the ideal $(p(x)) \subset R[x]$ is maximal, then R is a field.