Preliminary Exam 2012
Afternoon Exam (3 hours)
Part I

Do four out of five problems.

Problem 1. Find all solutions $(w, x, y, z) \in \mathbb{R}^4$ to the system of equations

$$\begin{cases} w - 2x + 0y - 4z = 2 \\ 3w - 6x + 2y - 8z = 12. \end{cases}$$

Problem 2. Let $c_1, c_2, \ldots, c_n$ be $n \geqslant 1$ distinct real numbers, and define polynomials $f_i \in \mathbb{R}[x]$ $(1 \leqslant i \leqslant n)$ by

$$f_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (x - c_j).$$

Prove that $f_1, f_2, \ldots, f_n$ are linearly independent.

Problem 3. The cyclic group G is generated by x. Show that together, x^{11553} and x^{11513} also generate G.

Problem 4. For which values of the parameter $a \in \mathbb{R}$ does the system

$$\begin{cases} ax + 2y + 3az = 0 \\ 3x + ay + 2z = 0 \\ 3ax + 3y + 2az = 0 \end{cases}$$

have a nontrivial solution?

Problem 5. Let V be the real vector space of polynomials of degree at most two. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be the inner product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Find an orthonormal basis of V.

Part II

Do three out of six problems.

Problem 6. Let L be a subgroup of $\mathbb{Z}^3$ of index 16. What are the possibilities for $\mathbb{Z}^3/L$?

Problem 7. Suppose A is a 5×5 matrix with nullspace of dimension 3. If $A^2 = 0$ then what is the Jordan normal form of A?

Problem 8. Let $U(n)$ denote the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$. In each case, determine whether the two groups are isomorphic or not, giving a reason for your answer:
(a) $U(15)$, $U(20)$.
(b) $U(5)$, $U(12)$.

Problem 9. Let G be a finite group and let $H \subset G$ be a maximal proper subgroup. Assume that H is normal in G. Show that $[G : H]$ is a prime number.

Problem 10. Let A be a 2×2 matrix with real coefficients. If tr(A)=1 and tr(A^2)=5 find tr(A^5).

Problem 11. Let V be a vector space over $\mathbb{R}$, and let S and T be invertible linear transformations from V to itself. Suppose that there is a real number $c > 0$ such that cST=TS.

(a) Show that if $v \in V$ is a nonzero eigenvector of T with eigenvalue λ then $S(v)$ is a nonzero eigenvector of T with eigenvalue $c\lambda$.

(b) Show that if V is finite-dimensional then $c = 1$.

Part III

Do one out of four problems.

Problem 12. An *automorphism* of a finite group G is an isomorphism of G onto itself. A subgroup H of G is a *characteristic* subgroup if $\varphi(H) = H$ for every automorphism φ of G.

a) Prove that a characteristic subgroup is a normal subgroup.

b) Give a counterexample to show that a normal subgroup need not be a characteristic subgroup.

Problem 13. Let p be a prime number, let $f(x) = x^3 + px + p$, and let K be the splitting field of $f(x)$ over $\mathbb{C}$, so that if the factorization of $f(x)$ over $\mathbb{C}$ is

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

then $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Show that $[K : \mathbb{Q}] = 6$.

Problem 14. Let R be a commutative ring, and let $x \in R$ be a *nilpotent* element, i. e. an element such that $x^n = 0$ for some integer $n \geqslant 1$. Show that for all $y \in R$, $1 + xy$ is a unit of R.

Problem 15. Let R be a commutative ring, let I be an ideal of R, and let $\sqrt{I}$ be the set of all $x \in R$ such that $x^m \in I$ for some positive integer m.

a) Show that $\sqrt{I}$ is an ideal of R.

b) If I and J are two ideals of R, prove that $\sqrt{I} + \sqrt{J} \subset \sqrt{I + J}$.

c) If $R = \mathbb{Z}$ and I is the ideal generated by a positive integer b, then what is a generator of $\sqrt{I}$?