Preliminary Exam 2012
Afternoon Exam (3 hours)

Part I

Do four out of five problems.

Problem 1. Find all solutions \((w, x, y, z) \in \mathbb{R}^4\) to the system of equations

\[
\begin{align*}
    w - 2x + 0y - 4z &= 2 \\
    3w - 6x + 2y - 8z &= 12.
\end{align*}
\]

Problem 2. Let \(c_1, c_2, \ldots, c_n\) be \(n \geq 1\) distinct real numbers, and define polynomials \(f_i \in \mathbb{R}[x]\) \((1 \leq i \leq n)\) by

\[f_i(x) = \prod_{\substack{j=1 \atop j \neq i}}^{n} (x - c_j).\]

Prove that \(f_1, f_2, \ldots, f_n\) are linearly independent.

Problem 3. The cyclic group \(G\) is generated by \(x\). Show that together, \(x^{11553}\) and \(x^{11513}\) also generate \(G\).

Problem 4. For which values of the parameter \(a \in \mathbb{R}\) does the system

\[
\begin{align*}
    ax + 2y + 3az &= 0 \\
    3x + ay + 2z &= 0 \\
    3ax + 3y + 2az &= 0
\end{align*}
\]

have a nontrivial solution?

Problem 5. Let \(V\) be the real vector space of polynomials of degree at most two. Let \(\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}\) be the inner product defined by

\[\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx.\]

Find an orthonormal basis of \(V\).

Part II

Do three out of six problems.

Problem 6. Let \(L\) be a subgroup of \(\mathbb{Z}^3\) of index 16. What are the possibilities for \(\mathbb{Z}^3/L\)?

Problem 7. Suppose \(A\) is a 5 \times 5 matrix with nullspace of dimension 3. If \(A^2 = 0\) then what is the Jordan normal form of \(A\)?

Problem 8. Let \(U(n)\) denote the group of units of the ring \(\mathbb{Z}/n\mathbb{Z}\). In each case, determine whether the two groups are isomorphic or not, giving a reason for your answer:

(a) \(U(15), U(20)\).
(b) \(U(5), U(12)\).

Problem 9. Let \(G\) be a finite group and let \(H \subset G\) be a maximal proper subgroup. Assume that \(H\) is normal in \(G\). Show that \([G : H]\) is a prime number.
**Problem 10.** Let $A$ be a $2 \times 2$ matrix with real coefficients. If $\text{tr}(A)=1$ and $\text{tr}(A^2)=5$ find $\text{tr}(A^5)$.

**Problem 11.** Let $V$ be a vector space over $\mathbb{R}$, and let $S$ and $T$ be invertible linear transformations from $V$ to itself. Suppose that there is a real number $c > 0$ such that $cST=TS$.

(a) Show that if $v \in V$ is a nonzero eigenvector of $T$ with eigenvalue $\lambda$ then $S(v)$ is a nonzero eigenvector of $T$ with eigenvalue $c\lambda$.

(b) Show that if $V$ is finite-dimensional then $c = 1$.

**Part III**

Do one out of four problems.

**Problem 12.** An automorphism of a finite group $G$ is an isomorphism of $G$ onto itself. A subgroup $H$ of $G$ is a characteristic subgroup if $\varphi(H) = H$ for every automorphism $\varphi$ of $G$.

a) Prove that a characteristic subgroup is a normal subgroup.

b) Give a counterexample to show that a normal subgroup need not be a characteristic subgroup.

**Problem 13.** Let $p$ be a prime number, let $f(x) = x^3 + px + p$, and let $K$ be the splitting field of $f(x)$ over $\mathbb{C}$, so that if the factorization of $f(x)$ over $\mathbb{C}$ is

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

then $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Show that $[K : \mathbb{Q}] = 6$.

**Problem 14.** Let $R$ be a commutative ring, and let $x \in R$ be a nilpotent element, i.e., an element such that $x^n = 0$ for some integer $n \geq 1$. Show that for all $y \in R$, $1 + xy$ is a unit of $R$.

**Problem 15.** Let $R$ be a commutative ring, let $I$ be an ideal of $R$, and let $\sqrt{I}$ be the set of all $x \in R$ such that $x^m \in I$ for some positive integer $m$.

a) Show that $\sqrt{I}$ is an ideal of $R$.

b) If $I$ and $J$ are two ideals of $R$, prove that $\sqrt{I} + \sqrt{J} \subset \sqrt{I + J}$.

c) If $R = \mathbb{Z}$ and $I$ is the ideal generated by a positive integer $b$, then what is a generator of $\sqrt{I}$?