Preliminary Exam 2011
Morning Exam (3 hours)

PART I. Solve 4 of the following 5 problems.

1. Let the sequence $\{x_n\}_0^\infty$ of real numbers be defined by

$$x_n = \frac{3}{4}x_{n-1} + \frac{1}{4}, \qquad n = 1, 2, \ldots$$

a. For what values of x_0 does the sequence converge?
b. What is the limit of this sequence?
c. What is the rate of convergence to the limit?

2. Consider the system of equations

$$u = x + y^2 + z^3, \qquad v = x^3 + y + z^2, \qquad w = x^2 + y^3 + z,$$

and consider (x, y, z) as a function of (u, v, w). Show that there is a unique solution near the origin as long as u, v, and w are sufficiently small.

3. Evaluate the following contour integral:

$$I = \int_C \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy,$$

where C is the triangle with vertices at $(5, 5), (-5, 5)$, and $(0, -5)$ traversed counterclockwise. Be careful about the hypotheses of any theorem you use.

4. Let $t \in \mathbb{R}$, and let $y : \mathbb{R} \to \mathbb{R}$.
a. Prove that every solution $y(t)$ of the ordinary differential equation

$$\frac{dy}{dt} = \cos(y)$$

is bounded above and below for all t.
b. Prove that every solution $y(t)$ of the ordinary differential equation

$$\frac{dy}{dt} = 1 + \cos(y)$$

is bounded above and below for all t.
c. Is the solution $y(t)$ of the ordinary differential equation

$$\frac{dy}{dt} = t^3 \cos(y)$$

bounded above and below for all t?

5. Determine whether or not the following limit exists, and find its value if it exists:

$$\lim_{n\to\infty} \left[n - \frac{n}{e}\left(1 + \frac{1}{n}\right)^n \right]$$

PART II. Solve 3 of the following 6 problems.

6. Let $f(x)$ be a continuous, monotonically decreasing function on the semi-infinite interval $[1, \infty)$. Furthermore, let $\lim_{x\to\infty} f(x) = 0$. Establish the Integral Test from Calculus, namely prove that the series $\Sigma_{n=1}^{\infty} f(n)$ converges if and only if the integral $\int_1^\infty f(x)dx$ converges.

7. Consider the functions $u = f(x, y) = x^2 - y^2$ and $v = g(x, y) = 2xy$.
a. What does the Inverse Function Theorem tell you about defining $x = F(u, v)$ and $y = G(u, v)$?
b. Find an expression for

$$\frac{\partial G}{\partial u}.$$

8. Consider the interval $0 \leq x \leq 1$. Let

$$f(x) = \begin{cases} 0 & x \ \text{ irrational} \\ \frac{1}{q^2} & x = \frac{p}{q} \end{cases}$$

where p/q is the fraction in lowest terms. Show that $f(x)$ is continuous only if x is irrational.

9. Let $0 < a < \frac{1}{10}$.
a. Determine the number of roots that the function $f(x) = \sin(x) - ax$ has in the interval $\frac{\pi}{2} < x < \pi$.
b. Does this root/do these roots depend continuously on a?
c. For the root near π, show that $\frac{\pi}{1+a}$ is a better approximation to the root than π.

10. Let

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi \leq x < 0. \end{cases}$$

Find the Fourier sine series of $f(x)$ on $(-\pi, \pi)$.

11. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

PART III. Solve 1 of the following 3 problems.

12. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Answer the following four questions: Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

13. Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \quad \text{or} \quad x > \frac{1}{n} \\ \sin^2\left(\frac{\pi}{x}\right) & \frac{1}{n+1} \leq x \leq \frac{1}{n}. \end{cases}$$

Show that $\{f_n(x)\}$ converges to a continuous function, but not uniformly. Also, use the series $\Sigma_{n=1}^{\infty} f_n(x)$ to show that absolute convergence, even for all x, does not imply uniform convergence.

14. Suppose

$$f(x) = \Sigma_{n=-\infty}^{\infty} a_n e^{2\pi i n x}.$$

What condition on the coefficients a_n will guarantee that f is k times differentiable? (The more general condition you give, the better.)