Preliminary Exam 2011
Afternoon Session (3 hours)

Part I. Solve four of the following five problems.

1. Let V be the span of the vectors $v_1 = (1, 2, 2)$ and $v_2 = (3, -1, 1)$ in $\mathbb{R}^3$, and suppose that $\{u_1, u_2\}$ is an orthonormal basis for V. If u_1 is a scalar multiple of v_1 then what are the possibilities for u_2? The "possibilities" should be expressed as explicit vectors in $\mathbb{R}^3$.

2. Put

$$A = \begin{pmatrix} 2 & -3/2 \\ 1 & -1/2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} A^n = B$.

3. An $n \times n$ matrix A over $\mathbb{C}$ is said to be hermitian if $A = \overline{A}^{\mathrm{t}}$, where $\overline{A}$ is the complex conjugate of A. Let H_n be the set of all $n \times n$ hermitian matrices.
(a) Is H_n a subspace of the complex vector space of all $n \times n$ matrices over $\mathbb{C}$? Why or why not?
(b) What is the dimension of H_n as a real vector space?

4. Let V be a vector space and $T : V \to V$ a linear transformation with the property that $T(W) \subseteq W$ for every subspace W of V. Prove that T is a scalar multiplication. In other words, prove that there is an element λ in the field of scalars such that $T(v) = \lambda v$ for all $v \in V$.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map

$$T(w, x, y, z) = (3w + x - 7y - 2z, w + 3x - 5y + 2z, w + x - 3y).$$

Find bases for the kernel of T and the image of T.

Part II. Solve three of the following six problems.

6. Let n be a nonnegative integer. Show that the functions $\sin(2^j x)$ $(0 \leqslant j \leqslant n)$ are linearly independent as real-valued functions on $\mathbb{R}$.

7. Let A be a 7×7 matrix such that $(A-I)^3 = 0$ and $(A-I)^2$ has rank 2. Find the Jordan normal form of A.

8. Let A be an $n \times n$ matrix over $\mathbb{R}$. Prove that the rank of A equals the rank of AA^{t}.

9. Let G be a group and H a normal subgroup of order 2.
(a) Prove that if G/H is cyclic then G is abelian.
(b) Give an example to show that if G/H is merely abelian then G need not be abelian.

10. Exhibit a subgroup H of order 8 in S_4 (the group of permutations of 4 objects) by listing the elements of H and showing that they form a subgroup.

11. Let R be a finite commutative ring with the property that if $x, y \in R$ and $xy = 0$ then $x = 0$ or $y = 0$. Show that R is a field. (You may assume that $1 \neq 0$ in R.) Hint: For a given nonzero $y \in R$, consider the map $R \to R$ defined by $x \mapsto xy$.

Part III. Solve one of the remaining three problems.

12. Suppose that $G = C_{25} \times C_{45} \times C_{48} \times C_{150}$, where C_n denotes a cyclic group of order n.

(a) How many elements of order 5 does G have?

(b) How many subgroups of order 5 does G have?

(c) Write $G \cong C_{d_1} \times C_{d_2} \times \cdots \times C_{d_k}$ with positive integers $d_1, d_2, \ldots, d_k$ such that d_i divides d_{i+1} for $1 \leqslant i \leqslant k-1$.

13. In each case define a surjective ring homomorphism $\mathbb{Z}[x] \to R$ or explain why none exists:

(a) $R = \mathbb{F}_2$, the field with 2 elements.

(b) $R = \mathbb{F}_4$, the field with 4 elements.

(c) $R = \mathbb{Q}$.

14. Consider the quotient ring $R = \mathbb{R}[x]/(f(x))$, where $f(x) \in \mathbb{R}[x]$. Let n be the degree of f.

(a) Explain why R is a field if $n = 1$.

(b) Give examples showing that if $n = 2$ then R may or may not be a field.

(c) Prove that if $n = 3$ then R is not a field.