Preliminary Exam 2009
Morning Session (3 hours)

Part I. Solve four of the following five problems.

1. Let (a, b) be an open interval in $\mathbb{R}$ and suppose that f and g are two real-valued functions defined on (a, b). State and prove the Product Rule for the derivative of the product of the two functions f and g.

2. Let S be the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ that lies above the plane $z = 1/2$ in $\mathbb{R}^3$. Calculate the surface area of S.

3. Using estimates involving the function $f(x) = x^{3/2}$, show that it is uniformly continuous on the closed interval $[0, b]$ for each $b > 0$ but that it is not uniformly continuous on the interval $[0, \infty)$. (The phrase "Using estimates" means that you should not appeal to the theorem that says that a continuous function on a compact interval is uniformly continuous.)

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq 2^{-n}$. Prove that the sequence converges.

5. Is the vector-valued function $\mathbf{Y}(t) = (\cos 3t, 2\sin 3t)$ a solution to a linear system of the form $d\mathbf{Y}/dt = \mathbf{AY}$ where $\mathbf{A}$ is a 2×2 matrix? If so, determine $\mathbf{A}$. If not, why not?

Part II. Solve three of the following six problems.

6. Consider the autonomous system of differential equations

$$\frac{dx}{dt} = -x - y - 1$$
$$\frac{dy}{dt} = -2y.$$

   (a) Compute the general solution to this system.

   (b) Sketch its phase portrait.

7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

How smooth is f? Provide a rigorous justification for your answer.

8. Suppose that $f : [a, b] \to \mathbb{R}$ is twice differentiable and that $f''(x) > 0$ for all $x \in (a, b)$. Prove that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

for all x_1 and x_2 such that $a \leq x_1 < x_2 \leq b$.

9. Given two points F_1 and F_2 in the xy-plane and a positive constant k, an ellipse with foci F_1 and F_2 is the set of all points P such that the sum of the distances

$$d(P, F_1) + d(P, F_2)$$

is the constant k. The number k is called the length of the major axis of the ellipse.

(a) Derive an equation for an ellipse whose foci are located at the points $(-c, 0)$ and $(c, 0)$.

(b) Prove rigorously that the solution curves of the system

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y \\ \frac{dy}{dt} &= -5x - 3y \end{aligned}$$

lie on ellipses in the xy-plane.

10. (Parts (c) and (d) are on the next page.) Let $f : [1, \infty) \to (0, \infty)$ be a twice differentiable decreasing function such that $f''(x)$ is positive for $x \in (1, \infty)$. For each positive integer n, let a_n denote the area of the region bounded by the graph of f and the line segment joining the points $(n, f(n))$ and $(n + 1, f(n + 1))$.

(a) Show that $\sum_{n=1}^{\infty} a_n < \frac{1}{2}(f(1) - f(2))$.

(b) Show that the limit $\lim_{n\to\infty} \left[\sum_{k=1}^{n} f(k) - \frac{1}{2}(f(1) + f(n)) - \int_1^n f(x)\, dx \right]$ exists.

(c) Finally show that the limit $\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right]$ exists.

(d) Apply this result to the function $f(x) = 1/x$.

11. Let $P(x)$ and $Q(x)$ be polynomials of the same degree $d > 0$. Suppose that the coefficients of both polynomials are nonnegative real numbers. Prove that the series

$$\sum_{n=1}^{\infty} e^{-nP(n)/Q(n)}$$

converges.

Part III. Solve one of the remaining three problems.

12. Let C be the circle of radius r in the yz-plane centered at the point $(y, z) = (R, 0)$ where $R > r$, and let T be the surface that is obtained by rotating C about the z-axis in $\mathbb{R}^3$.

(a) Parametrize T and specify the ranges of the parameters necessary to sweep out T exactly once.

(b) Compute the surface area of T.

(c) Compute the volume enclosed by T.

13. Let C be the cylinder in $\mathbb{R}^3$ defined by $x^2 + y^2 = 2x$. Find the surface area of that part of the cylinder that lies inside the sphere $x^2 + y^2 + z^2 = 4$.

14. Suppose $0 < a < 1$. Does the series $\sum_{n=1}^{\infty}(1 - a^{1/n})$ converge? (Hint: Consider the function $f(x) = a^x$.)