Part I. Solve four of the following five problems.

1. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous. Prove that
\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.
\]
(You may assume that continuous functions are Riemann integrable.)

2. (a) Solve the initial-value problem
\[
\frac{d^2y}{dt^2} + 16y = \cos 6t, \quad y(0) = 0, \quad y'(0) = 0.
\]
(b) Give a rough sketch of the graph of the solution \( y(t) \). Make sure that your sketch includes a scale on both axes. Hint:
\[
\cos \theta_2 - \cos \theta_1 = 2 \left( \sin \left( \frac{\theta_1 + \theta_2}{2} \right) \right) \left( \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \right)
\]

3. Consider the space of all \( n \times n \) matrices with real entries with the standard metric, i.e., view the matrix as an element of \( \mathbb{R}^{n^2} \) and use the usual Euclidean metric on \( \mathbb{R}^{n^2} \). Prove that the subset of all invertible matrices is open.

4. Rewrite the integral
\[
\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx
\]
as an equivalent iterated integral in the order \( dy \, dx \, dz \).

5. Which is bigger, \( e^\pi \) or \( \pi^e \)? Hint: Compare \( \left( \frac{1}{e} \right)^{1/e} \) to \( \left( \frac{1}{\pi} \right)^{1/\pi} \) by considering the function \( f(x) = x^x \).
Part II. Solve three of the following six problems.

6. Consider two real-valued functions \( f \) and \( g \) defined on a punctured neighborhood of the number \( a \) in \( \mathbb{R} \). Give a precise statement and a rigorous proof of the fact that the limit of the product \( fg \) at \( a \) is the product of the limits of \( f \) and \( g \) at \( a \).

7. Show that
   \[
   \int_0^1 (\ln x)^n \, dx = (-1)^n n!
   \]
   for all positive integers \( n \). Make sure that your calculation includes a rigorous justification of the convergence of these integrals.

8. Given data points of the form \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), the line \( y = mx + b \) is the best least squares fit to the data if it minimizes the sum of the squares of the vertical deviations \( d_i = y_i - (mx_i + b) \) of the points to the line. In other words, it must minimize
   \[
   \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - (mx_i + b))^2.
   \]
   Find \( m \) and \( b \) in terms of the sums
   \[
   S_x = \sum_{i=1}^{n} x_i, \quad S_y = \sum_{i=1}^{n} y_i, \quad S_{x^2} = \sum_{i=1}^{n} x_i^2, \quad \text{and} \quad S_{xy} = \sum_{i=1}^{n} x_i y_i.
   \]
   (Other term(s) may also appear in your answer.) Make sure that you provide a complete justification for your answer.

9. For what values of \( r \) is the function
   \[
   f(x, y, z) = \begin{cases} 
   \frac{(x + y + z)^r}{x^2 + y^2 + z^2} & \text{if } (x, y, z) \neq (0, 0, 0); \\
   0 & \text{if } (x, y, z) = (0, 0, 0).
   \end{cases}
   \]
   continuous on \( \mathbb{R}^3 \)?

10. Let \( b : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \(-1 < b(t) < 2\) for all \( t \in \mathbb{R} \). Describe the long-term behavior of the solutions to the differential equation
   \[
   \frac{dy}{dt} + 2y = b(t)
   \]
   as precisely as possible and justify your answer.
11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with compact support and let $X$ be a smooth vector field on $\mathbb{R}^3$. Show that

$$\iiint_{\mathbb{R}^3} (\nabla f) \cdot (\text{curl} X) \, dV = 0.$$ 

**Part III.** Solve one of the remaining three problems.

12. Consider a power series

$$\sum_{k=0}^{\infty} a_k x^k$$

whose interval of convergence is $[-1, 1]$. Abel’s Theorem says that this series converges uniformly on $[-1, 1]$.

(a) Prove Abel’s Theorem. Hint: Let $R_n = a_n + a_{n+1} + a_{n+2} + \ldots$ and

$$R_n(x) = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots.$$ 

Then $R_n(x) = (R_n - R_{n+1}) x^n + (R_{n+1} - R_{n+2}) x^{n+1} + (R_{n+2} - R_{n+3}) x^{n+2} + \ldots$.

(b) State the power series expansion of the inverse tangent function centered around 0, and explain the significance of Abel’s Theorem for this series.

13. (a) Let $\{v_i\}$ be an infinite sequence in $\mathbb{R}^n$ that tends to the origin $\mathbf{0}$ as $i \to \infty$ in terms of the usual Euclidean metric $d$, i.e., $d(v_i, \mathbf{0}) \to 0$ as $i \to \infty$. Show that $||v_i|| \to 0$ as $i \to \infty$ for any norm $||\cdot||$ on $\mathbb{R}^n$.

(b) Let $V = C^\infty[0, 1]$ be the vector space of smooth real-valued functions defined on the interval $[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$ and the associated metric

$$d(f, g) = \sqrt{\int_0^1 (f(x) - g(x))^2 \, dx}.$$ 

Let $||\cdot||$ be the norm on $V$ given by

$$||f|| = \sqrt{\langle f, f \rangle^2 + \langle f', f' \rangle^2}.$$ 

Find an infinite sequence $\{f_i\}$ in $V$ that tends to the zero function $\mathbf{0}$ as $i \to \infty$ in terms of the metric $d$, i.e., $d(f_i, \mathbf{0}) \to 0$ as $i \to \infty$, for which $||f_i|| \not\to 0$ as $i \to \infty$. 

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14. For a smooth function $f: \mathbb{R}^+ \to \mathbb{R}$, write $f(t) \sim \sum_{k=r}^{\infty} a_k t^k$ for some $r \in \mathbb{R}$ if

$$
\lim_{t \to 0^+} \frac{f(t) - \sum_{k=r}^{r+n} a_k t^k}{t^{r+n}} = 0
$$

for all $n = 0, 1, 2, \ldots$. Note that $r$ can be negative and need not be an integer. The sum

$$
\sum_{k=r}^{r+n} a_k t^k = a_r t^r + a_{r+1} t^{r+1} + \ldots + a_{r+n} t^{r+n},
$$

as usual.

(a) Prove that $e^{-1/t} + t^{-3} \cos t \sim \sum_{k=-3}^{\infty} \frac{(-1)^{k+3}}{(2k+6)!} t^{2k+3}$.

(b) Suppose $f(t) \sim \sum_{k=-1/2}^{\infty} a_k t^k$. Prove that

$$
\int_0^t f(s) \, ds \sim \sum_{k=-1/2}^{\infty} \frac{a_k}{k+1} t^{k+1}
$$

or give a counterexample.

(c) Under the same assumptions as in part (b), prove that

$$
f'(t) \sim \sum_{k=-1/2}^{\infty} k a_k t^{k-1}
$$

or give a counterexample.