Preliminary Exam 2008
Morning Session (3 hours)

Part I. Solve four of the following five problems.

1. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Prove that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

(You may assume that continuous functions are Riemann integrable.)

2. (a) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 16y = \cos 6t, \quad y(0) = 0, \quad y'(0) = 0.$$

(b) Give a rough sketch of the graph of the solution $y(t)$. Make sure that your sketch includes a scale on both axes. Hint:

$$\cos\theta_2 - \cos\theta_1 = 2\left(\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\right)\left(\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right)$$

3. Consider the space of all $n \times n$ matrices with real entries with the standard metric, i.e., view the matrix as an element of $\mathbb{R}^{n^2}$ and use the usual Euclidean metric on $\mathbb{R}^{n^2}$. Prove that the subset of all invertible matrices is open.

4. Rewrite the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

as an equivalent iterated integral in the order $dy\, dx\, dz$.

5. Which is bigger, e^π or π^e? Hint: Compare $\left(\frac{1}{e}\right)^{1/e}$ to $\left(\frac{1}{\pi}\right)^{1/\pi}$ by considering the function $f(x) = x^x$.

Part II. Solve three of the following six problems.

6. Consider two real-valued functions f and g defined on a punctured neigborhood of the number a in $\mathbb{R}$. Give a precise statement and a rigorous proof of the fact that the limit of the product fg at a is the product of the limits of f and g at a.

7. Show that
$$\int_0^1 (\ln x)^n \, dx = (-1)^n n!$$
for all positive integers n. Make sure that your calculation includes a rigorous justification of the convergence of these integrals.

8. Given data points of the form (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), the line $y = mx + b$ is the best least squares fit to the data if it minimizes the sum of the squares of the vertical deviations $d_i = y_i - (mx_i + b)$ of the points to the line. In other words, it must minimize
$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - (mx_i + b))^2.$$
Find m and b in terms of the sums
$$S_x = \sum_{i=1}^{n} x_i, \quad S_y = \sum_{i=1}^{n} y_i, \quad S_{x^2} = \sum_{i=1}^{n} x_i^2, \quad \text{and} \quad S_{xy} = \sum_{i=1}^{n} x_i y_i.$$
(Other term(s) may also appear in your answer.) Make sure that you provide a complete justification for your answer.

9. For what values of r is the function
$$f(x,y,z) = \begin{cases} \dfrac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0); \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$$
continuous on $\mathbb{R}^3$?

10. Let $b : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $-1 < b(t) < 2$ for all $t \in \mathbb{R}$. Describe the long-term behavior of the solutions to the differential equation
$$\frac{dy}{dt} + 2y = b(t)$$
as precisely as possible and justify your answer.

11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with compact support and let $\mathbf{X}$ be a smooth vector field on $\mathbb{R}^3$. Show that

$$\iiint_{\mathbb{R}^3} (\nabla f) \cdot (\operatorname{curl} \mathbf{X}) \, dV = 0.$$

Part III. Solve one of the remaining three problems.

12. Consider a power series

$$\sum_{k=0}^{\infty} a_k x^k$$

whose interval of convergence is $[-1, 1]$. Abel's Theorem says that this series converges uniformly on $[-1, 1]$.

(a) Prove Abel's Theorem. Hint: Let $R_n = a_n + a_{n+1} + a_{n+2} + \ldots$ and

$$R_n(x) = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots .$$

Then $R_n(x) = (R_n - R_{n+1})x^n + (R_{n+1} - R_{n+2})x^{n+1} + (R_{n+2} - R_{n+3})x^{n+2} + \ldots .$

(b) State the power series expansion of the inverse tangent function centered around 0, and explain the significance of Abel's Theorem for this series.

13. (a) Let $\{\mathbf{v}_i\}$ be an infinite sequence in $\mathbb{R}^n$ that tends to the origin $\mathbf{0}$ as $i \to \infty$ in terms of the usual Euclidean metric d, i.e., $d(\mathbf{v}_i, \mathbf{0}) \to 0$ as $i \to \infty$. Show that $\|\mathbf{v}_i\| \to 0$ as $i \to \infty$ for any norm $\| \cdot \|$ on $\mathbb{R}^n$.

(b) Let $V = C^\infty[0, 1]$ be the vector space of smooth real-valued functions defined on the interval $[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ and the associated metric

$$d(f, g) = \sqrt{\int_0^1 (f(x) - g(x))^2 \, dx}.$$

Let $\| \cdot \|$ be the norm on V given by

$$\|f\| = \sqrt{\langle f, f \rangle^2 + \langle f', f' \rangle^2}.$$

Find an infinite sequence $\{f_i\}$ in V that tends to the zero function $\mathbf{0}$ as $i \to \infty$ in terms of the metric d, i.e., $d(f_i, \mathbf{0}) \to 0$ as $i \to \infty$, for which $\|f_i\| \not\to 0$ as $i \to \infty$.

14. For a smooth function $f : \mathbb{R}^+ \to \mathbb{R}$, write $f(t) \sim \sum_{k=r}^{\infty} a_k t^k$ for some $r \in \mathbb{R}$ if

$$\lim_{t \to 0^+} \frac{f(t) - \sum_{k=r}^{r+n} a_k t^k}{t^{r+n}} = 0$$

for all $n = 0,\ 1,\ 2, \ldots$. Note that r can be negative and need not be an integer. The sum

$$\sum_{k=r}^{r+n} a_k t^k = a_r t^r + a_{r+1} t^{r+1} + \ldots + a_{r+n} t^{r+n},$$

as usual.

(a) Prove that $e^{-1/t} + t^{-3} \cos t \sim \sum_{k=-3}^{\infty} \frac{(-1)^{k+3}}{(2k+6)!} t^{2k+3}$.

(b) Suppose $f(t) \sim \sum_{k=-1/2}^{\infty} a_k t^k$. Prove that

$$\int_0^t f(s)\, ds \sim \sum_{k=-1/2}^{\infty} \frac{a_k}{k+1} t^{k+1}$$

or give a counterexample.

(c) Under the same assumptions as in part (b), prove that

$$f'(t) \sim \sum_{k=-1/2}^{\infty} k\, a_k t^{k-1}$$

or give a counterexample.