Preliminary Exam 2007
Morning Session (3 hours)

Part I. Solve four of the following five problems.

1. Consider the initial-value problem
$$\frac{dy}{dt} = -2ty^2, \quad y(-1) = y_0.$$
Find all values of y_0 such that the solution is defined for all real t.

2. Suppose a_n and b_n are two sequences that tend to infinity as $n \to \infty$. We say that b_n tends to infinity faster than a_n if
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0,$$
and we write $a_n << b_n$. (We could use little-o notation, but we do not.) Arrange the six sequences
(a) $\sqrt{n}$ (b) e^n (c) $n!$ (d) n^2 (e) $\ln n$ (f) n^n
in terms of the ordering $<<$. Justify each limit involved.

3. A C^2 function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies Laplace's equation if
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
Derive Laplace's equation in terms of polar coordinates. Hint: Consider the expression
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Let $\epsilon > 0$. Show that there exists a continuous, piecewise linear function $g : [a, b] \to \mathbb{R}$ such that
$$|f(x) - g(x)| < \epsilon$$
for all $x \in [a, b]$.

5. Let $f(x) = \sin(x^3)$. Calculate $f^{(15)}(0)$. Hint: You do not have to calculate $f^{(15)}(x)$ to do this problem.

Part II. Solve three of the following six problems.

6. Find the positively oriented, piecewise-smooth, simple, closed curve C for which the value of the line integral

$$\int_C (y^3 - y)\,dx - 2x^3\,dy$$

is a maximum. Provide a brief justification for your answer.

7. (a) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} n^3 x^n$.

   (b) Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

   $$f_n(x) = n^3 x^n (1-x).$$

   Show that this sequence f_n converges pointwise on $[0,1]$.

   (c) Calculate $\int_0^1 f_n(x)\,dx$.

   (d) Does the sequence f_n converge uniformly? Provide a brief justification for your answer.

8. The second-order differential equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

can be used to model a harmonic oscillator, and an undamped or underdamped harmonic oscillator can be used to make a clock. If we arrange for the clock to tick whenever the mass passes the rest position, then the time between ticks is equal to one-half of the natural period of the oscillator.

   (a) Suppose dirt increases the damping coefficient slightly, will the clock run fast or slow?

   (b) Suppose the spring provides slightly less force for a given compression or extension as it ages. Will the clock run fast or slow?

   (c) If grime collects on the harmonic oscillator and slightly increases the mass, will the clock run fast or slow?

9. Given a solid sphere of radius R, remove a "cylinder" whose central axis goes through the center of the sphere. Let h denote the height of the remaining solid. Calculate the volume of the remaining solid. (Hint: Your answer should be independent of R.)

10. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Divide $[a, b]$ into n subintervals of equal length, and let M_n, T_n, and S_n denote the result of the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule applied to f respectively.

   (a) Let $x_0 < x_1 < x_2$ be three real numbers such that $x_1 = (x_0 + x_2)/2$. Given three positive numbers y_0, y_1, and y_2 such that the points (x_0, y_0), (x_1, y_1), and (x_2, y_2) lie on a parabolic arc of the form $y = Ax^2 + Bx + C$, show that the area above the x-axis and below the arc between x_0 and x_2 is

   $$\frac{x_1 - x_0}{3}(y_0 + 4y_1 + y_2).$$

   (b) Derive Simpson's Rule S_n. (Recall that Simpson's Rule is only valid for even n.)

   (c) Show that $S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$.

11. Suppose that f is a real-valued function defined on the closed interval $[a, b]$.

   (a) Suppose f is zero except on a finite subset of $[a, b]$. Show that f is Riemann integrable and that the value of the integral is 0.

   (b) Is it possible for the result in part (a) to hold for some f that is zero except on a countable set? Provide a proof or a counterexample.

Part III. Solve one of the remaining three problems.

12. Suppose that f is twice differentiable on the interval $[a, b]$ and that $f(a) < 0$ and $f(b) > 0$. Moreover, suppose that there exist a $\delta > 0$ and an $M > 0$ such that

$$f'(x) \geq \delta \quad \text{and} \quad 0 \leq f''(x) \leq M$$

for all $x \in [a, b]$. Finally, let x_* be the unique root of f in the interval $[a, b]$.

Choose $x_1 \in (x_*, b)$ and define the sequence $\{x_n\}$ using Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(a) Prove that $x_{n+1} < x_n$ and that $x_n \to x_*$ as $n \to \infty$.

(b) Show that

$$x_{n+1} - x_* = \frac{f''(t_n)}{2f'(x_n)}(x_n - x_*)^2$$

for some $t_n \in (x_*, x_n)$.

(c) If $A = M/(2\delta)$, conclude that

$$0 \leq x_{n+1} - x_* \leq \frac{1}{A}\left(A(x_1 - x_*)\right)^{2^n}.$$

(d) What does this result say about the convergence of Newton's method under these assumptions?

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and let $(t_0, y_0) \in \mathbb{R}^2$. Suppose $y(t)$ is a solution to the initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

in a neighborhood of the point t_0. Give a careful proof of the following statement:

If $y(t)$ cannot be extended to a solution to $dy/dt = f(t, y)$ on the interval $[t_0, \infty)$, then there exists a $t_1 > t_0$ such that $|y(t)| \to \infty$ as $t \to t_1$ from below.

14. Recall that an infinite series

$$\sum_{k=1}^{\infty} a_k$$

of real numbers is absolutely convergent if the series

$$\sum_{k=1}^{\infty} |a_k|$$

converges.

(a) Prove that an absolutely convergent series converges.

(b) Show that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \dots$$

converges to a number between $\frac{5}{6}$ and $\frac{7}{12}$. (It actually converges to $\ln 2$, but that's another story.)

(c) Show that the alternating harmonic series is not absolutely convergent.

(d) Let s be a real number. Show how the terms of the alternating harmonic series can be rearranged so that the "new" series converges to s.