Preliminary Exam 2006
Morning Exam (3 hours)

PART I. Solve 4 of the following 5 problems.

1. **a.** Let C denote the straight line segment connecting the two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. Show that

$$\int_C x dy - y dx = x_1 y_2 - x_2 y_1.$$

b. Let the points $(x_1, y_1), \ldots, (x_n, y_n)$ denote the vertices of a regular n-gon in the plane, taken in counterclockwise order. Show that the area of the n-gon is

$A = (1/2)[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$.

2. Given a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_0, y_0) = 0$. Suppose that

$$\frac{\partial f}{\partial y}|_{(x_0,y_0)} \neq 0.$$

The Implicit Function Theorem states that the level set $\{(x, y) : f(x, y) = 0\}$ is the graph of a smooth function $y = \phi(x)$ near $(x, y) = (x_0, y_0)$.
a. Compute

$$\frac{d\phi}{dx}|_{x_0}.$$

b. Compute

$$\frac{d^2\phi}{dx^2}|_{x_0}.$$

3. Let $I : \mathbb{R}^2 \to \mathbb{R}$ and $U : \mathbb{R}^2 \to \mathbb{R}$ both be smooth functions. Suppose that $I(x, y) \to \infty$ as $\|(x, y)\| \to \infty$. Also, suppose that $U(x, y) > 0$ for all (x, y) and that $U(x, y) \to 0$ as $\|(x, y)\| \to \infty$. Show that for each value of c such that the set $\{(x, y) : I(x, y) = c\} \neq \emptyset$ there exists a point $(x_1, y_1) \in \{(x, y) : I(x, y) = c\}$ such that $\nabla U|_{(x_1,y_1)}$ and $\nabla I|_{(x_1,y_1)}$ are multiples of each other.

4. Find the solution $u(x)$ of the equation

$$\frac{d^2u}{dx^2} + u = 3\sin(x)$$

that passes through the initial condition $u(0) = 1$ and $(du/dx)(0) = 1$. Here, x and u are real-valued.

5. Determine whether or not the series

$$\frac{\sin(t)}{1} + \frac{\cos(2t)}{4} + \frac{\sin(3t)}{9} + \frac{\cos(4t)}{16} + \frac{\sin(5t)}{25} + \frac{\cos(6t)}{36} + \cdots$$

is uniformly convergent on $[-\pi, \pi]$. Also, determine whether or not this series defines a continuous function on $[-\pi, \pi]$.

PART II. Solve 3 of the following 7 problems.

6. Let ℓ denote the line in $\mathbb{R}^3$ defined by the equations

$$\begin{aligned} x + y - z &= 1 \\ 2x - y + 2z &= 2. \end{aligned} \qquad (1)$$

Find the point P on the line ℓ that is closest to the point $Q = (0, 0, 1)$.

7. Let $f(x)$ be a real-valued, three times differentiable function on $[-1,1]$ such that

$$f(-1)=0, \qquad f(0)=0, \qquad f(1)=1, \qquad f'(0)=0.$$

Prove that there exists at least one value of $x \in (-1,1)$ at which $(d^3 f/dx^3)(x) \geq 3$. (Note that equality holds for $(1/2)(x^3+x^2)$.)

8. Every rational number x may be written in the form $x = p/q$, where $q > 0$, and p and q are integers without any common divisors. When $x = 0$, we take $q = 1$. Consider the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q}. \end{cases}$$

Prove that f is continuous at every irrational point and that it has a simple discontinuity at every rational point.

9. Find the Fourier series (in terms of sine and/or cosine functions) of the function

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0) \\ +1 & x \in [0, \pi] \end{cases}$$

10. A model of population growth claims that the population $y(t)$ grows according to the law

$$\frac{dy}{dt} = \kappa y^{1+\epsilon},$$

where $\kappa, \epsilon > 0$ and ϵ is small.
a. How long does it take an initial population of y_0 to become infinite?
b. Answer the same question for the case $\epsilon = 0$, and compare this case to that in which ϵ is very small.

11. Let $U : \{(x,y) \neq (0,0)\} \to \mathbb{R}$ be a C^2 function which satisfies

$$U(\lambda x, \lambda y) = \frac{1}{\lambda^2} U(x,y)$$

a. Prove or disprove

$$\left.\frac{\partial U}{\partial x}\right|_{(\lambda x, \lambda y)} = \frac{1}{\lambda^3} \left.\frac{\partial U}{\partial x}\right|_{(x,y)}.$$

b. If (r,θ) represent polar coordinates, give a simple formula for

$$\frac{\partial U}{\partial r}.$$

12. Prove or give a counter example: If f is a continuous function on a compact subset Y of a metric space X, then f is uniformly continuous on Y.

PART III. Solve 1 of the remaining 4 problems.

13. The Alternating Series Test (usually attributed to Leibniz) states: Given an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

with $a_n \geq 0$ for all n, if **(i)** $a_{n+1} \leq a_n$, and **(ii)** $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. In this problem, you are to consider the same general class of alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $a_n \geq 0$ for all n. Prove or find a counterexample to the statement: If $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. In other words, you are to determine whether or not the first hypothesis, **(i)**, of the Alternating Series Test is necessary.

14. Let $p(x)$ be a polynomial of degree n, and let α be any real number. Show that between any two successive roots of $p(x)$ the function

$$\frac{dp}{dx}(x) + \alpha p(x)$$

has a root.

15. **a.** Does there exist a continuous map $f : (0,1) \to \mathbb{R}$ which is onto (surjective)? (If so, give an example; but, if not, state why not.)
b. Does there exist a continuous map $f : [0,1) \to \mathbb{R}$ which is onto (surjective)? (If so, give an example; but, if not, state why not.)
c. Does there exist a continuous map $f : [0,1) \to \mathbb{R}$ which is 1-1 (injective) and onto (surjective)? (If so, give an example; but, if not, state why not.)

16. Define the following two functions:

$$f(x) = x^3 - \sin^2(x)\tan(x), \qquad g(x) = 2x^2 - \sin^2(x) - x\tan(x).$$

For each of them separately, determine if it is positive or negative for $x \in (0, \pi/2)$, or whether it changes sign in this interval. Prove your answers.