Preliminary Exam 2005
Morning Exam (3 hours)

PART I. Solve 4 of the following 5 problems.

1. a. Determine whether the series

$$\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$$

converges or diverges, showing which test(s) you used.
b. Determine whether the series

$$\Sigma_{n=1}^{\infty} \frac{3^n n!}{n^n}$$

converges or diverges, showing which test(s) you used.

2. You are given the following two algebraic equations:

$$x^2 + zw + zx = 0, \qquad \text{and} \qquad y^3 + zy + z^2 = 0.$$

a. Find $\frac{\partial x}{\partial z}$, assuming that x and y are functions of z and w.
b. Evaluate the Jacobian matrix of the transformation from the (x, y) plane to the (z, w) plane that is implied by the equations for z and w as functions of x and y.

3. A rectangular box without a lid is to be made from twelve (12) square meters of cardboard. Find the maximum volume of such a box.

4. Let x and y be real numbers. Establish that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

5. Consider the second-order, linear, ordinary differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0$$

for the real-valued function $y = y(t)$. Fix the initial value $y(0) = 4$. How should $\frac{dy}{dt}(0)$ be chosen so that the solution $y(t)$ through the initial condition $(y(0), \frac{dy}{dt}(0)) = (4, \frac{dy}{dt}(0))$ vanishes the fastest?

PART II. Solve 3 of the following 6 problems.

6. Prove that

$$\lim_{t\to\infty} \int_1^2 \frac{\sin(tx)}{x^2\sqrt{x-1}} dx = 0.$$

7. The following system of three nonlinear algebraic equations is to be solved for x, y, z as functions of the variables u, v, w:

$$\begin{aligned} u &= x + y + z \\ v &= x^2 + y^2 + z^2 \\ w &= x^3 + y^3 + z^3. \end{aligned} \qquad (1)$$

a. Prove or find a counter example: for each (u, v, w) near $(0, 2, 0)$, there is a unique solution (x, y, z) near $(-1, 0, 1)$.
b. Is the Implicit Function Theorem applicable for (u, v, w) near $(2, 4, 8)$ and (x, y, z) near $(0, 0, 2)$?

8. Show that the curvature of the circular helix defined by the vector-valued function

$$\mathbf{r}(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$$

is constant.

9. Let $\mathbf{Q}$ denote the set of all rational numbers. Introduce the distance function $d(p, q) = |p - q|$ for any pair $p, q \in \mathbf{Q}$. It is known that, with this distance function as the metric, $\mathbf{Q}$ is a metric space. Now, let

$$E = \{p \in \mathbf{Q} | 2 < p^2 < 3\}.$$

a. Show that E is closed and bounded in $\mathbf{Q}$.
b. Determine whether the set E is compact or not.

10. Prove the following statement or find a counter-example: Every convergent sequence that is uniformly bounded on a compact set of real numbers contains a uniformly convergent subsequence.

11. Consider any function $f : \mathbf{R} \to \mathbf{R}$ which satisfies the following properties:
(i) f is continuous for $x \geq 0$;
(ii) $f'(x)$ exists for $x > 0$;
(iii) $f(0) = 0$; and,
(iv) f' is monotonically increasing.
For $x > 0$, define

$$g(x) = \frac{f(x)}{x}.$$

Prove that g is monotonically increasing.

PART III. Solve 1 of the remaining 3 problems.

12. Assume that f is a continuous, real-valued function defined in (a, b) such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex. (Recall that a function $f(x)$ on (a, b) is convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ whenever $x \in (a, b)$, $y \in (a, b)$, and $\lambda \in (0, 1)$.)

13. Let X denote a metric space. Consider space $C(X)$, the set of all complex-valued, continuous, bounded functions with domain X. For $f \in C(X)$, let

$$\|f\| = \sup_{x \in X} |f(x)|$$

denote the supremum norm of f.
a. Show that the distance function

$$d(f, g) = \|f - g\|$$

for any two functions $f, g \in C(X)$ is a metric on $C(X)$.
b. Show that, with this metric, $C(X)$ is a complete metric space.

14. Let $f(x)$ be a real-valued function on $[0, 1]$. Let all the derivatives of f be continuous. Assume that $|f(1)| \geq |f(0)|$. Show that either there is an $x \in (0, 1)$ such that $f(x)$ and $f'(x)$ have the same sign or that $f(x)$ is a constant function.