PART I. Solve 4 of the following 5 problems.

1. Give an $\varepsilon-\delta$ proof of the continuity of the function $f(x) = \sqrt{x}$ at $x = 0$.

2. Define the function
   
   $$f(x, y) = \begin{cases} 
   \frac{x^2y}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\
   0 & \text{for } (x, y) = (0, 0).
   \end{cases}$$

   Prove or disprove that $f(x, y)$ is continuous at $(0, 0)$.

3. Evaluate the path integral
   
   $$I = \int_C \frac{-y \, dx + x \, dy}{x^2 + y^2},$$

   where $C$ is any simple, closed curve that encircles the origin and that is traversed in the counterclockwise direction. (Hint: think carefully about the hypotheses of Green’s Theorem before you apply it.)

4. Let $y \in \mathbb{R}$, $t \in \mathbb{R}$, and $y = y(t)$. Consider the differential equation
   
   $$\frac{dy}{dt} = y^2.$$

   (a) Let $\alpha$ be a nonzero constant. Consider a new dependent variable $u$ defined by the transformation $u = y^\alpha$. Find the differential equation satisfied by $u(t)$.

   (b) Show that, no matter how one chooses $\alpha$, one cannot put the new equation into the form
   
   $$\frac{du}{dt} = ku,$$

   where $k$ is another constant (which may depend on $\alpha$).

   (c) Give a qualitative explanation of why you cannot transform the original equation into the type of equation in (b) for $u$.

5. Consider the sequence
   
   $$\sqrt{3}, \sqrt[3]{\sqrt{3}}, \sqrt[3]{\sqrt[3]{\sqrt{3}}}, \ldots.$$

   Prove that this sequence has a limit, and find the limit. (Hint: It may be useful to first show that if $0 < a < 3$, then $a < \sqrt[5]{a} < 3$.)

PART II. Solve 3 of the following 6 problems.

6. Let
   
   $$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

   Prove that, for $n$ even, $P_n(x) > 0$ for all real numbers $x$; whereas, for $n$ odd, $P_n(x)$ has exactly one real root. (Hint: differentiate.)

7. Maximize the function $f(x, y, z) = \cos(\frac{7}{2}(x + y + z))$ subject to the constraints $x^2 + y^2 + z^2 = 1, x \geq 0, y \geq 0, z \geq 0$.

8. Let $x \in \mathbb{R}^3$ and let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Define
   
   $$F(x) = \nabla f \big|_x \times \nabla g \big|_x$$

   and let $r(t)$ satisfy the differential equation
   
   $$\frac{dr}{dt} = F(r).$$
(a) Suppose that we know $13. Suppose outline the reason why the condition is sufficient.

(b) Describe all the equilibrium points of the differential equation for $r(t)$.

(c) Relate your answer in part (b) to a topic in vector calculus.

9. The following system of three nonlinear algebraic equations is to be solved for $x, y, z$ as functions of the variables $u, v, w$:

$$
\begin{align*}
  u &= x + y^2 + z^3 \\
  v &= x^3 + y + z^2 \\
  w &= x^2 + y^3 + z.
\end{align*}
$$

Prove or find a counterexample to the statement that there is a unique solution near $(x, y, z) = (0, 0, 0)$ if $u, v, w$ are small.

10. Consider the sequence

$$
1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, \ldots
$$

For which numbers $\alpha$ is there a subsequence converging to $\alpha$?

11. Let $x, y \in \mathbb{R}$. Define

$$
\begin{align*}
  d_1(x, y) &= (x - y)^2 \\
  d_2(x, y) &= \sqrt{|x - y|} \\
  d_3(x, y) &= |x^2 - y^2| \\
  d_4(x, y) &= |x - 2y| \\
  d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}.
\end{align*}
$$

For each of these, determine whether it is a metric or not, being careful to state your reasons.

**PART III.** Solve 1 of the remaining 3 problems.

12. Let $x \in \mathbb{R}$. Suppose you are given a Fourier series

$$
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).
$$

State a general condition on the real-valued coefficients $(a_0, a_1, \ldots, b_1, \ldots)$ that suffices to guarantee that $f(x)$ is three times continuously differentiable and outline the reason why the condition is sufficient.

13. Suppose $f : \mathbb{R} \to \mathbb{R}$ and suppose $f$ is three times continuously differentiable.

(a) Suppose that we know $|f''(x)| < 10$ for all $x \in [-1, 1]$. What are the values of $n$ for which the above hypotheses suffice to guarantee that $f(x) \neq 0$ for all $x \in [-1, 1]$ if we also know that $1 \leq f(x) \leq 2$ for the specific numbers $x = -1, -1 + \frac{1}{n}, -1 + \frac{2}{n}, \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \ldots, 1 - \frac{2}{n}, 1 - \frac{1}{n}, 1$?

(b) Suppose instead that, while we do not know any bound on $|f''(x)|$, we know $|f''(x)| < 10$ for all $x \in [-1, 1]$. Also, suppose, as above, that we know $1 \leq f(x) \leq 2$ for the specific numbers $x = -1, -1 + \frac{1}{n}, -1 + \frac{2}{n}, \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \ldots, 1 - \frac{2}{n}, 1 - \frac{1}{n}, 1$. What is the set of values of $n$ for which this information suffices to guarantee that $f(x) \neq 0$ for all $x \in [-1, 1]$?

14. (a) Let $a_{ij} \in \mathbb{R}$ for $i = 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$. Prove or give a counterexample to the statement that

$$
\lim_{i \to \infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left( \lim_{i \to \infty} a_{ij} \right).
$$
(b) Let $f_n : [0, 1] \to \mathbb{R}$, $n = 1, 2, \ldots$, be continuous. Suppose that there exists a function $f_0(x) : [0, 1] \to \mathbb{R}$ such that $f_n(x) \to f_0(x)$ as $n \to \infty$ for all $x \in [0, 1]$. Prove or give a counter example to the statement that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f_0(x) \, dx.$$ 

(c) Consider the same hypotheses as in (b) but now also require that $f_n(x) \to f_0(x)$ uniformly. Prove or give a counter example to the statement that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f_0(x) \, dx.$$