Preliminary Exam 2004
Morning Exam (3 hours)

PART I. Solve 4 of the following 5 problems.

1. Give an $\epsilon - \delta$ proof of the continuity of the function $f(x) = \sqrt{x}$ at $x = 0$.

2. Define the function

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0). \end{cases}$$

Prove or disprove that $f(x,y)$ is continuous at $(0,0)$.

3. Evaluate the path integral

$$I = \int_C \frac{-ydx + xdy}{x^2+y^2},$$

where C is any simple, closed curve that encircles the origin and that is traversed in the counterclockwise direction. (Hint: think carefully about the hypotheses of Green's Theorem before you apply it.)

4. Let $y \in \mathbb{R}$, $t \in \mathbb{R}$, and $y = y(t)$. Consider the differential equation

$$\frac{dy}{dt} = y^2.$$

(a) Let α be a nonzero constant. Consider a new dependent variable u defined by the transformation $u = y^\alpha$. Find the differential equation satisfied by $u(t)$.
(b) Show that, no matter how one chooses α, one cannot put the new equation into the form

$$\frac{du}{dt} = ku,$$

where k is another constant (which may depend on α).
(c) Give a qualitative explanation of why you cannot transform the original equation into the type of equation in (b) for u.

5. Consider the sequence

$$\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots .$$

Prove that this sequence has a limit, and find the limit. (Hint: It may be useful to first show that if $0 < a < 3$, then $a < \sqrt{3a} < 3$.)

PART II. Solve 3 of the following 6 problems.

6. Let

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Prove that, for n even, $P_n(x) > 0$ for all real numbers x; whereas, for n odd, $P_n(x)$ has exactly one real root. (Hint: differentiate.)

7. Maximize the function $f(x,y,z) = \cos(\frac{\pi}{2}(x+y+z))$ subject to the constraints $x^2+y^2+z^2 = 1, x \geq 0, y \geq 0, z \geq 0$.

8. Let $\mathbf{x} \in \mathbb{R}^3$ and let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Define

$$F(\mathbf{x}) = \nabla f|_{\mathbf{x}} \times \nabla g|_{\mathbf{x}}$$

and let $\mathbf{r}(t)$ satisfy the differential equation

$$\frac{d\mathbf{r}}{dt} = F(\mathbf{r}).$$

(a) Show $f(\mathbf{r}(t))$ and $g(\mathbf{r}(t))$ are constant in time.
(b) Describe **all** the equilibrium points of the differential equation for $\mathbf{r}(t)$.
(c) Relate your answer in part (b) to a topic in vector calculus.

9. The following system of three nonlinear algebraic equations is to be solved for x, y, z as functions of the variables u, v, w:

$$\begin{aligned} u &= x + y^2 + z^3 \\ v &= x^3 + y + z^2 \\ w &= x^2 + y^3 + z. \end{aligned} \tag{1}$$

Prove or find a counter example to the statement that there is a unique solution near $(x, y, z) = (0, 0, 0)$ if u, v, and w are small.

10. Consider the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$$

For which numbers α is there a subsequence converging to α?

11. Let $x, y \in \mathbb{R}$. Define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2 - y^2| \\ d_4(x,y) &= |x - 2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

For each of these, determine whether it is a metric or not, being careful to state your reasons.

PART III. Solve 1 of the remaining 3 problems.

12. Let $x \in \mathbb{R}$. Suppose you are given a Fourier series

$$f(x) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

State a general condition on the real-valued coefficients ($a_0, a_1, \ldots$, and $b_1, \ldots$) that suffices to guarantee that $f(x)$ is three times continuously differentiable and outline the reason why the condition is sufficient.

13. Suppose $f : \mathbb{R} \to \mathbb{R}$ and suppose f is three times continuously differentiable.
(a) Suppose that we know $|\frac{df}{dx}(x)| < 10$ for all $x \in [-1, 1]$. What are the values of n for which the above hypotheses suffice to guarantee that $f(x) \neq 0$ for all $x \in [-1, 1]$ if we also know that $1 \leq f(x) \leq 2$ for the specific numbers $x = -1, -1 + \frac{1}{n}, -1 + \frac{2}{n}, \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \ldots, 1 - \frac{2}{n}, 1 - \frac{1}{n}, 1$?
(b) Suppose instead that, while we do not know any bound on $|\frac{df}{dx}(x)|$, we know $|\frac{d^3 f}{dx^3}(x)| < 10$ for all $x \in [-1, 1]$. Also, suppose, as above, that we know $1 \leq f(x) \leq 2$ for the specific numbers $x = -1, -1 + \frac{1}{n}, -1 + \frac{2}{n}, \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \ldots, 1 - \frac{2}{n}, 1 - \frac{1}{n}, 1$. What is the set of values of n for which this information suffices to guarantee that $f(x) \neq 0$ for all $x \in [-1, 1]$?

14. (a) Let $a_{ij} \in \mathbb{R}$ for $i = 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$. Prove or give a counter example to the statement that

$$\lim_{i \to \infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{j=1}^{\infty} \left(\lim_{i \to \infty} a_{ij} \right)$$

(b) Let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$, be continuous. Suppose that there exists a function $f_0(x) : [0,1] \to \mathbb{R}$ such that $f_n(x) \to f_0(x)$ as $n \to \infty$ for all $x \in [0,1]$. Prove or give a counter example to the statement that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f_0(x)dx.$$

(c) Consider the same hypotheses as in (b) but now also require that $f_n(x) \to f_0(x)$ uniformly. Prove or give a counter example to the statement that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f_0(x)dx.$$