Morning Exam 2003

PART I. Solve 4 out of the next 5 problems.

1. Let S be the plane $x+y+2=0$ in the three-dimensional $x-y-z$ space. Consider the function $f(x,y,z)=x^2+2xy+z^3$. For an arbitrary point (x,y,z) on S, evaluate

$$\frac{\partial}{\partial \mathbf{n}} f(x,y,z),$$

which represents the directional derivative of f in the direction of the normal vector $\mathbf{n}$ to the plane, and you may choose either of the two possible perpendicular directions.
Also, give the value of this directional derivative at the point $(-1,-1,3)$ on S.

2. Evaluate the path integral

$$\int_C x^2y\,dx - xy^2\,dy,$$

where C is the circle of radius two centered at $(0,0)$ traveled in the counterclockwise direction.

3. By computing either a double or a triple integral, show that the volume of a sphere of radius R is

$$\frac{4\pi}{3}R^3.$$

4. Consider the ordinary differential equation

$$\frac{dy}{dt} = 2y\left(1-\frac{y}{5}\right)$$

for the real-valued function $y(t)$, with t real. For any arbitrary initial condition $y(0)>0$, find $\lim_{t\to\infty} y(t)$.

5. Given $a_n = \sqrt{n^2+n}-n$ for $n=1,2,\ldots$, compute $\lim_{n\to\infty} a_n$ and determine whether the infinite series $\sum_{n=1}^{\infty} a_n$ converges or diverges.

PART II. Solve 3 out of the next 6 problems.

6. Determine whether or not the vector field

$$\mathbf{F}(x,y,z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$$

can be written as the curl of another vector field; that is, whether or not there exists a $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$.

7. A parallelogram in the plane is called a rectangle if all four interior angles are right angles (and here a square is treated as a special case of a rectangle). By measuring the distances between no more than two pairs of vertices, show how one can determine whether or not the parallelogram is a rectangle.

8. Consider the system of ordinary differential equations

$$\dot{x} = -2x + ay, \qquad \dot{y} = -2x,$$

for the real-valued functions $x(t)$ and $y(t)$ where a is a real parameter. Classify the type of the fixed point $(0,0)$ for each value of a, and make a qualitative sketch of how the solutions approach the fixed point in each of the different cases (your sketch does not have to be quantitatively accurate).

9. Give a careful qualitative description of the behavior of the solutions of the third-order differential equation

$$\frac{d^3y}{dt^3} = ky,$$

where $y(t)$ is real-valued and k is a real constant. Be sure you consider the different cases (and note that you do **not** need to find the general solution).

10. Consider $x \in [0, 1]$. Define the sequence of real-valued functions

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, \qquad n = 1, 2, \ldots$$

First, show that the sequence $\{f_n(x)\}$ is uniformly bounded on $[0, 1]$. Second, find $\lim_{n\to\infty} f_n(x)$ for all $x \in [0, 1]$. Finally, find a sequence $\{x_n\}$ that converges to zero so that $\{f_n\}$ has no uniformly convergent subsequence.

11. Give an example of an open cover of the interval $(0, 1)$ that has no finite subcover.

PART III. Solve 1 out of the next 3 problems.

12. A real-valued function f defined on (a, b) is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b, a < y < b$, and $0 < \lambda < 1$. Prove that every convex function is continuous, and prove that every increasing convex function of a convex function is convex.

13. Suppose a solution $y_1(t)$ of an ODE

$$\frac{dy}{dt} = f(y)$$

satisfies

$$\lim_{t\to\infty} y_1(t) = y_0.$$

First, prove that if f is continuously differentiable then $f(y_0) = 0$. Second, state whether or not you can weaken the hypothesis on f and still obtain the same conclusion.

14. Suppose $f(x)f(y) = f(x + y)$ for all real x and y. Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant. Can you also prove this same result assuming only that f is continuous?