Preliminary Exam 2010
Afternoon Session (3 hours)

Part I. Solve four of the following five problems.

1. Find a basis for the span of the columns of the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 4 & 12 & 2 & 10 & 1 \\ 3 & 8 & 1 & 7 & 1 \\ 4 & 10 & 1 & 9 & 0 \end{pmatrix}.$$

2. Are the polynomials

$$x^2 + 3x + 1, \quad 2x^2 - 2x - 1, \quad \text{and} \quad 18x^2 - 2x - 3$$

linearly independent over $\mathbb{R}$?

3. Let P_n denote the vector space of polynomials in $\mathbb{R}[x]$ with degree less than or equal to n. Compute the trace of the linear operator $\frac{d}{dx}$ on P_n.

4. Let A be a 2×2 matrix with characteristic polynomial $x^2 + x + \frac{1}{2}$. Compute

$$\lim_{n\to\infty} \left(A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

5. Let V be a vector space and let T_1 and T_2 be linear transformations that map V to itself.

   (a) Assume T_1 and T_2 commute, that is, $T_1(T_2(v)) = T_2(T_1(v))$ for all $v \in V$. If v is an eigenvector for T_1 with eigenvalue λ and $T_2(v) \neq 0$, prove that $T_2(v)$ is also an eigenvector for T_1.

   (b) Give an example where part (a) fails if T_1 and T_2 do not commute.

Part II. Solve three of the following six problems.

6. Let $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ denote the field of 2 elements.

   (a) Is $x^4 + x^2 + 1$ irreducible in $\mathbb{F}_2[x]$? Find a complete factorization.

   (b) How many irreducible polynomials of degree 4 are there in $\mathbb{F}_2[x]$?

7. Let A be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and let I_c denote the set of functions that vanish at some fixed $c \in \mathbb{R}$.

   (a) Prove that I_c is a prime ideal.

   (b) Is I_c a maximal ideal? Justify your answer.

   (c) Give an example of a proper non-zero ideal of A that is not of the form I_c for some $c \in \mathbb{R}$.

8. Let p be a prime number, and let $\mathbb{F}_p$ denote the finite field with p elements. Find the order of the group $\mathrm{SL}_3(\mathbb{F}_p)$ of invertible 3×3 matrices over $\mathbb{F}_p$ with determinant 1.

9. Let A be the $n \times n$ matrix which has 0's on the main diagonal and 1's everywhere else. Find the eigenvalues of A, determine the eigenspaces of A, and compute the determinant of A.

10. Prove that the group $\mathbb{Q}/\mathbb{Z}$ does not contain any finite index subgroups.

11. Let K be the smallest subfield of $\mathbb{C}$ that contains the roots of $x^3 - 2$.

    (a) Prove that K contains some quadratic extension of $\mathbb{Q}$.

    (b) Prove that K does *not* contain $\sqrt{2}$.

Part III. Solve one of the remaining three problems.

12. For each of the following statements, either provide the requested example or prove that no such example exists.

    (a) A group G whose list of sizes of conjugacy classes is $1, 1, 2, 3, 5$.

    (b) A non-abelian group G such that every subgroup of G is normal.

    (c) A group G with a chain of subgroups $H \subseteq N \subseteq G$ such that H is normal in N and N is normal in G, but H is not normal in G.

13. The following three rings all have 125 elements:

    (a) $\mathbb{Z}_5[x]/\langle x^3 - x^2 + x - 1\rangle$

    (b) $\mathbb{Z}_5[x]/\langle x^3 + 4\rangle$

    (c) $\mathbb{Z}_5[x]/\langle x^3 + 4x^2 + 1\rangle$

    Determine which of the these rings are isomorphic and which are not. Justify your assertions by either providing the appropriate isomorphism or by proving that no such isomorphism exists.

14. (a) Give an example of a polynomial in $\mathbb{Q}[x]$ whose splitting field has degree 8 over $\mathbb{Q}$. Justify your answer.

    (b) Can the answer to part (a) be a cubic polynomial? Justify your assertion.