Part I. Solve four of the following five problems.

1. Find a basis for the span of the columns of the matrix

\[
\begin{pmatrix}
1 & 2 & 0 & 2 & 0 \\
4 & 12 & 2 & 10 & 1 \\
3 & 8 & 1 & 7 & 1 \\
4 & 10 & 1 & 9 & 0
\end{pmatrix}
\]

2. Are the polynomials

\[x^2 + 3x + 1, \quad 2x^2 - 2x - 1, \quad \text{and} \quad 18x^2 - 2x - 3\]

linearly independent over \(\mathbb{R}\)?

3. Let \(P_n\) denote the vector space of polynomials in \(\mathbb{R}[x]\) with degree less than or equal to \(n\). Compute the trace of the linear operator \(\frac{d}{dx}\) on \(P_n\).

4. Let \(A\) be a \(2 \times 2\) matrix with characteristic polynomial \(x^2 + x + \frac{1}{2}\). Compute

\[
\lim_{n \to \infty} \left( A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).
\]

5. Let \(V\) be a vector space and let \(T_1\) and \(T_2\) be linear transformations that map \(V\) to itself.

   (a) Assume \(T_1\) and \(T_2\) commute, that is, \(T_1(T_2(v)) = T_2(T_1(v))\) for all \(v \in V\). If \(v\) is an eigenvector for \(T_1\) with eigenvalue \(\lambda\) and \(T_2(v) \neq 0\), prove that \(T_2(v)\) is also an eigenvector for \(T_1\).

   (b) Give an example where part (a) fails if \(T_1\) and \(T_2\) do not commute.
Part II. Solve three of the following six problems.

6. Let $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ denote the field of 2 elements.
   
   (a) Is $x^4 + x^2 + 1$ irreducible in $\mathbb{F}_2[x]$? Find a complete factorization.
   
   (b) How many irreducible polynomials of degree 4 are there in $\mathbb{F}_2[x]$?

7. Let $A$ be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and let $I_c$ denote the set of functions that vanish at some fixed $c \in \mathbb{R}$.
   
   (a) Prove that $I_c$ is a prime ideal.
   
   (b) Is $I_c$ a maximal ideal? Justify your answer.
   
   (c) Give an example of a proper non-zero ideal of $A$ that is not of the form $I_c$ for some $c \in \mathbb{R}$.

8. Let $p$ be a prime number, and let $\mathbb{F}_p$ denote the finite field with $p$ elements. Find the order of the group $\text{SL}_3(\mathbb{F}_p)$ of invertible $3 \times 3$ matrices over $\mathbb{F}_p$ with determinant 1.

9. Let $A$ be the $n \times n$ matrix which has 0’s on the main diagonal and 1’s everywhere else. Find the eigenvalues of $A$, determine the eigenspaces of $A$, and compute the determinant of $A$.

10. Prove that the group $\mathbb{Q}/\mathbb{Z}$ does not contain any finite index subgroups.

11. Let $K$ be the smallest subfield of $\mathbb{C}$ that contains the roots of $x^3 - 2$.
   
   (a) Prove that $K$ contains some quadratic extension of $\mathbb{Q}$.
   
   (b) Prove that $K$ does not contain $\sqrt{2}$. 

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Part III. Solve one of the remaining three problems.

12. For each of the following statements, either provide the requested example or prove that no such example exists.
   (a) A group $G$ whose list of sizes of conjugacy classes is $1, 1, 2, 3, 5$.
   (b) A non-abelian group $G$ such that every subgroup of $G$ is normal.
   (c) A group $G$ with a chain of subgroups $H \subseteq N \subseteq G$ such that $H$ is normal in $N$ and $N$ is normal in $G$, but $H$ is not normal in $G$.

13. The following three rings all have 125 elements:
   (a) $\mathbb{Z}_5[x]/\langle x^3 - x^2 + x - 1 \rangle$
   (b) $\mathbb{Z}_5[x]/\langle x^3 + 4 \rangle$
   (c) $\mathbb{Z}_5[x]/\langle x^3 + 4x^2 + 1 \rangle$

Determine which of the these rings are isomorphic and which are not. Justify your assertions by either providing the appropriate isomorphism or by proving that no such isomorphism exists.

14. (a) Give an example of a polynomial in $\mathbb{Q}[x]$ whose splitting field has degree 8 over $\mathbb{Q}$. Justify your answer.
   (b) Can the answer to part (a) be a cubic polynomial? Justify your assertion.