Preliminary Exam 2009
Afternoon session (3 hours)

Part I – answer 4 out of 5

1. Let U denote the subspace of $\mathbb{R}^4$ spanned by the vectors $(0, 1, 0, 1)$ and $(1, 0, 1, 0)$. Find an orthonormal basis for the orthogonal complement of U with respect to the standard Euclidean inner product on $\mathbb{R}^4$.

2. Let $M_{2\times 2}$ denote the vector space of 2×2 real matrices. Given

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

let $T : M_{2\times 2} \to M_{2\times 2}$ denote the linear transformation

$$T(X) = AX$$

Find a basis for $\ker(T)$ and a basis for $\text{image}(T)$.

3. Let $\mathbf{P}_{10}$ denote the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to 10. Let $T : \mathbf{P}_{10} \to \mathbf{P}_{10}$ denote the linear map given by differentiation, *i.e.*,

$$T(f(x)) = f'(x)$$

for $f(x) \in \mathbf{P}_{10}$. Compute the characteristic polynomial of T.

4. Suppose that A is a square complex matrix whose characteristic polynomial is $(x - 2)^2(x + 3)^2$.

   a) What are the trace and determinant of A ?

   b) Describe the possible Jordan canonical forms of A.

5. Find the greatest common divisor of 2111 and 4327.

Part II – answer 3 out of 6

6. Let G be a finite group, and let H be the subgroup generated by elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$.

   a) Prove that H is normal in G.

   b) Prove that G/H is abelian.

   c) For $G = S_3$, compute H and G/H.

7. Give examples of each the following or explain why no such example is possible.

   a) A non-abelian group with 8 elements.

   b) A non-cyclic abelian group with 15 elements.

   c) A group with exactly 5 conjugacy classes.

8. Suppose that R is a commutative ring and that x is an element of R such that $x^n = 0$ for some $n \geq 1$. Prove that x is contained in every prime ideal of R.

9. Let R be a commutative ring, and let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

   a) Prove that if $A \in M_n(R)$ such that all of its entries lie in some *proper* ideal of R, then A is not invertible in $M_n(R)$.

   b) Is the converse to the above part true? Prove it or find a counter-example.

10. Let $\alpha \in \mathbb{C}$ be an algebraic number whose minimum polynomial over $\mathbb{Q}$ is $x^3 + ax^2 + bx + c$.

    a) Find a basis of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

    b) Compute the matrix for multiplication by α in this basis.

    c) Determine the characteristic polynomial of this matrix.

11. Let A be an $n \times n$ matrix with entries in $\mathbb{R}$ such that $A^2 = -\mathrm{Id}$.

    a) Prove that A is diagonalizable over $\mathbb{C}$ and describe the corresponding diagonal matrices.

    b) What can you say about the parity of n ?

Part III – answer 1 out of 3

12. Let S, T be linear transformations acting on a complex vector space V such that $ST = TS$. Prove that if S has more than one eigenvalue, then there exist subspaces W and U of V such that

    (a) $V = W + U$

    (b) $W \cap U = 0$

    (c) W and U are invariant under both S and T.

13. Let $K = \mathbb{Q}(\sqrt[4]{2}, e^{2\pi i/3})$. Prove that there is no subfield $L \subseteq K$ such that the degree of L over $\mathbb{Q}$ equals 3.

14. Suppose that I is a non-zero ideal of $\mathbb{R}[x]$ such that $\mathbb{R}[x]/I$ is an integral domain. What are the possible dimensions of the (real) vector space $\mathbb{R}[x]/I$?