Preliminary Exam 2008
Afternoon Session (3 hours)

Part I. Solve four of the following five problems.

1. Let P_3 denote the subspace of $\mathbb{R}[x]$ of polynomials of degree at most 3. Find a basis for the subspace of P_3 of polynomials $f(x)$ such that
$$f(0) = f(1) \quad \text{and} \quad f'(1) = f''(2).$$

2. Let A be a 2×2 matrix over $\mathbb{R}$ such that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A with eigenvalue 1, and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector with eigenvalue 1/2.

   (a) Compute $A^3 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

   (b) Compute $\lim_{n \to \infty} A^n \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

3. Let P be the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix}$, and let Q be the span of the vectors $\begin{pmatrix} 2 \\ 0 \\ 13 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$. Find a basis for $P \cap Q$.

4. Let A be a 3×5 matrix over $\mathbb{R}$ and let T_A be the associated linear transformation. If the dimension of $\ker(T_A)$ is two, does the equation
$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
have infinitely many solutions $\mathbf{x}$ in $\mathbb{R}^5$? Justify your answer.

5. Consider the space $M_{2\times2}(\mathbb{C})$ of 2×2 matrices with complex entries. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $M_{2\times2}(\mathbb{C})$, let $\overline{A}$ denote $\begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix}$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$, and let
$$V = \{ \, A \in M_{2\times2}(\mathbb{C}) \;\mid\; \mathrm{tr}(A) = 0 \;\text{ and }\; A^T = -\overline{A} \, \}$$
Note that V is a vector space over $\mathbb{R}$, the **real** numbers, with an inner product given by
$$\langle A, B \rangle = -\,\mathrm{tr}(AB).$$
Find an orthonormal basis for V over $\mathbb{R}$ with respect to this inner product.

Part II. Solve three of the following six problems.

6. Let $G = S_8$ be the group of permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. In each part, indicate whether the statement is true or false and justify your answer:
   (a) G has a cyclic subgroup of order 15.
   (b) G has a cyclic subgroup of order 14.
   (c) If H is any abelian group of order 8, then H is isomorphic to a subgroup of G.

7. Let R and S be commutative rings with identity, and let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(1_R) = 1_S$. In each part, indicate whether the statement is true or false and justify your answer:
   (a) If P is a prime ideal of S, then $\varphi^{-1}(P)$ is a prime ideal of R.
   (b) If P is a maximal ideal of S, then $\varphi^{-1}(P)$ is a maximal ideal of R.
   (c) If P is a principal ideal of S, then $\varphi^{-1}(P)$ is a principal ideal of R.

8. Suppose that G is a finite group with exactly two conjugacy classes. Show that $|G| = 2$.

9. Let $(2, x^4 + x + 1)$ denote the ideal in $\mathbb{Z}[x]$ generated by the elements 2 and $x^4 + x + 1$. Is the quotient ring $\mathbb{Z}[x]/(2, x^4 + x + 1)$ a field? Why or why not?

10. Prove that the trace of a 2×2 matrix over $\mathbb{R}$ is 0 if and only if it is a linear combination of matrices of the form $XY - YX$, where X and Y denote arbitrary 2×2 matrices over $\mathbb{R}$.

11. Let $\mathrm{GL}(2, \mathbb{C})$ act on itself by conjugation. Classify the orbits of this action.

Part III. Solve one of the remaining three problems.

12. (a) Let $F = \mathbb{Q}(\sqrt[7]{2})$, and let β be an element of F that is not in $\mathbb{Q}$. Show that $\mathbb{Q}(\beta) = F$.

    (b) Is the question in part (a) true if F is replaced with $\mathbb{Q}(e^{\frac{2\pi i}{5}})$?

    (c) Is the question in part (a) true if F is replaced with $\mathbb{Q}(\sin(\frac{2\pi}{11}))$?

13. Let $\mathbb{Z}[x]$ be the ring of polynomials in one variable over the integers, and let M be a maximal ideal of $\mathbb{Z}[x]$.

    (a) Show that M is not a principal ideal.

    (b) Show that M can be generated by two elements of $\mathbb{Z}[x]$.

14. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T : V \to V$ be a linear transformation. If $W = \ker(T)$, let

$$\overline{T} : V/W \to V/W$$

denote the natural map given by

$$\overline{T}(v + W) = T(v) + W.$$

Prove that $\overline{T}$ is injective if x^2 does not divide $f(x)$, where $f(x)$ denotes the minimal polynomial of T.