Part I. Solve four of the following five problems.

1. Let \( P_3 \) denote the subspace of \( \mathbb{R}[x] \) of polynomials of degree at most 3. Find a basis for the subspace of \( P_3 \) of polynomials \( f(x) \) such that
   \[ f(0) = f(1) \quad \text{and} \quad f'(1) = f''(2). \]

2. Let \( A \) be a 2\( \times \)2 matrix over \( \mathbb{R} \) such that \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is an eigenvector for \( A \) with eigenvalue 1, and \( \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) is an eigenvector with eigenvalue 1/2.
   (a) Compute \( A^3 \begin{pmatrix} 3 \\ 4 \end{pmatrix} \).
   (b) Compute \( \lim_{n \to \infty} A^n \begin{pmatrix} 3 \\ 4 \end{pmatrix} \).

3. Let \( P \) be the subspace of \( \mathbb{R}^3 \) spanned by \( \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix} \), and let \( Q \) be the span of the vectors \( \begin{pmatrix} 2 \\ 0 \\ 13 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \). Find a basis for \( P \cap Q \).

4. Let \( A \) be a 3\( \times \)5 matrix over \( \mathbb{R} \) and let \( T_A \) be the associated linear transformation. If the dimension of \( \ker(T_A) \) is two, does the equation
   \[ Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \]
   have infinitely many solutions \( x \) in \( \mathbb{R}^5 \)? Justify your answer.

5. Consider the space \( M_{2\times2}(\mathbb{C}) \) of 2\( \times \)2 matrices with complex entries. If \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) in \( M_{2\times2}(\mathbb{C}) \), let \( \overline{A} \) denote \( \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \overline{\gamma} & \overline{\delta} \end{pmatrix} \), where \( \overline{z} \) is the complex conjugate of \( z \in \mathbb{C} \), and let
   \[ V = \{ A \in M_{2\times2}(\mathbb{C}) \mid \text{tr}(A) = 0 \quad \text{and} \quad A^T = -\overline{A} \} \]
   Note that \( V \) is a vector space over \( \mathbb{R} \), the real numbers, with an inner product given by
   \[ \langle A, B \rangle = -\text{tr}(AB). \]
   Find an orthonormal basis for \( V \) over \( \mathbb{R} \) with respect to this inner product.
Part II. Solve three of the following six problems.

6. Let $G = S_8$ be the group of permutations of the set \{1, 2, 3, 4, 5, 6, 7, 8\}. In each part, indicate whether the statement is true or false and justify your answer:
   
   (a) $G$ has a cyclic subgroup of order 15.
   
   (b) $G$ has a cyclic subgroup of order 14.
   
   (c) If $H$ is any abelian group of order 8, then $H$ is isomorphic to a subgroup of $G$.

7. Let $R$ and $S$ be commutative rings with identity, and let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(1_R) = 1_S$. In each part, indicate whether the statement is true or false and justify your answer:
   
   (a) If $P$ is a prime ideal of $S$, then $\varphi^{-1}(P)$ is a prime ideal of $R$.
   
   (b) If $P$ is a maximal ideal of $S$, then $\varphi^{-1}(P)$ is a maximal ideal of $R$.
   
   (c) If $P$ is a principal ideal of $S$, then $\varphi^{-1}(P)$ is a principal ideal of $R$.

8. Suppose that $G$ is a finite group with exactly two conjugacy classes. Show that $|G| = 2$.

9. Let $(2, x^4 + x + 1)$ denote the ideal in $\mathbb{Z}[x]$ generated by the elements 2 and $x^4 + x + 1$. Is the quotient ring $\mathbb{Z}[x]/(2, x^4 + x + 1)$ a field? Why or why not?

10. Prove that the trace of a $2 \times 2$ matrix over $\mathbb{R}$ is 0 if and only if it is a linear combination of matrices of the form $XY - YX$, where $X$ and $Y$ denote arbitrary $2 \times 2$ matrices over $\mathbb{R}$.

11. Let $GL(2, \mathbb{C})$ act on itself by conjugation. Classify the orbits of this action.
Part III. Solve one of the remaining three problems.

12. (a) Let $F = \mathbb{Q}(\sqrt{2})$, and let $\beta$ be an element of $F$ that is not in $\mathbb{Q}$. Show that $\mathbb{Q}(\beta) = F$.

(b) Is the question in part (a) true if $F$ is replaced with $\mathbb{Q}(e^{\frac{2\pi i}{5}})$?

(c) Is the question in part (a) true if $F$ is replaced with $\mathbb{Q}(\sin(\frac{2\pi}{11}))$?

13. Let $\mathbb{Z}[x]$ be the ring of polynomials in one variable over the integers, and let $M$ be a maximal ideal of $\mathbb{Z}[x]$.

   (a) Show that $M$ is not a principal ideal.

   (b) Show that $M$ can be generated by two elements of $\mathbb{Z}[x]$.

14. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, and let $T : V \to V$ be a linear transformation. If $W = \ker(T)$, let

$$\overline{T} : V/W \to V/W$$

denote the natural map given by

$$\overline{T}(v + W) = T(v) + W.$$

Prove that $\overline{T}$ is injective if $x^2$ does not divide $f(x)$, where $f(x)$ denotes the minimal polynomial of $T$. 