Preliminary Exam 2006
Afternoon exam (3 hours)

Part I. Solve 4 of the following 5 problems.

1. Let $t \in \mathbb{R}$ and let
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$
Express e^{tA} as a 2×2 matrix whose entries are functions from $\mathbb{R}$ into $\mathbb{R}$.

2. Find a polynomial of degree three whose graph goes through the points $(-2,-5)$, $(-1,1)$, $(1,1)$, and $(3,25)$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $T(x,y,z,w) = (a,b,c)$ where
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ -1 & 2 & 1 & 2 \\ 1 & 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
Find the dimension of the kernel (null space) of T and of the image (range) of T.

4. Let V be the real, inner product space of continuous functions on the closed interval $[0,\pi]$ with inner product
$$(f,g) = f \cdot g = \int_0^\pi f(x)g(x)dx.$$
Let $W \subset V$ be the subspace of V spanned by the functions 1, $\sin(x)$, and $\cos(x)$. Find an orthonormal basis of W.

5. How many elements are there in the group of invertible 2×2 matrices over the field of seven elements?

Part II. Solve 3 of the following 6 problems.

6. Let U and V be two subspaces of a finite-dimensional vector space. Show that
$$\dim(U+V) + \dim(U \cap V) = \dim(U) + \dim(V).$$

7. Consider the 3×3 matrix
$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$
a. Show $X \cdot AX = 0$ for all $X \in \mathbb{R}^3$, where $X \cdot Y$ is the usual dot product.
b. Find a non-zero vector Y so that $AY = 0$.
c. For Y as in part (b), show that $AX \cdot Y = 0$ for all $X \in \mathbb{R}^3$.
d. For Y as in part (b), show that there is a real number λ so that if X is any vector orthogonal to Y (i.e., $X \cdot Y = 0$) then $A^2X = \lambda X$. Determine λ.

8. **a.** Let $G = \mathrm{GL}(n,\mathbb{R})$ and $H = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det A > 0\}$ where $n > 1$. Is H a subgroup of G? If so, is it a normal subgroup?
b. Answer the same questions with H replaced by $\{A \in \mathrm{GL}(n,\mathbb{R}) : AA^t = I\}$, where A^t denotes the transpose of A.

9. **a.** Let G be any group. Show that a normal subgroup of order 2 must be contained in the center of G.
b. Consider the permutation group S_n of n objects. Find the center of S_n.

10. Is there a non-abelian group of order $n = 49$? Either find one or explain why none exists. Do the same for $n = 50$ and $n = 51$.

11. Suppose A is a real, symmetric, $n \times n$ matrix with eigenvalues $1, 2, \cdots, n-1, n$. Compute $\|A\|$, the norm of A, where

$$\|A\| = \sup\{\|A\vec{x}\| \text{for all vectors } x \in \mathbb{R}^n \text{ with norm} \|\vec{x}\| = 1\} ,$$

where $\|\vec{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $\vec{x} = (x_1, \cdots, x_n)$. Justify your conclusion.

Part III. Solve 1 of the remaining 4 problems.

12. Which of the following rings is an integral domain? Which is a field? Justify your assertions.
a. $\mathbf{Z}[x]/(x^2 + 7)$
b. $\mathbb{R}[x]/(x^4 + 3x^2 + 2)$
c. $\mathbf{Q}[x]/(x^3 - 2)$

13. What are all of the possible degrees for irreducible polynomials over the following fields, F?
a. $F = \mathbf{C}$, the field of complex numbers.
b. $F = \mathbf{Z}_p (= \mathbf{Z}/p\mathbf{Z})$, where p is any prime.
c. $F = \mathbb{R}$.

14. The three matrices A, B, and C satisfy

$$A^2 = B^2 = C^2 = Id, \qquad \text{and} \qquad BC - CB = iA.$$

a. What are $AB + BA$ and $AC + CA$?
b. Derive a set of explicit forms of A, B, and C in the case of 2×2 matrices.

15. **a.** Suppose $p, n \in \mathbf{Z}$, where p is prime and p does not divide n. Must there exist integers a and b such that $ap + bn = 1$?
b. Suppose that $f, g \in \mathbf{Q}[x]$, where f is irreducible and f does not divide g. Must there exist $h, k \in \mathbf{Q}[x]$ such that $hf + kg = 1$?
c. Repeat part (b) with $\mathbf{Q}[x]$ replaced by $\mathbf{Q}[x, y]$. Justify your assertions.