Part I. Solve 4 of the following 5 problems.

1. Let $t \in \mathbb{R}$ and let
   
   $$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$ 

   Express $e^{tA}$ as a $2 \times 2$ matrix whose entries are functions from $\mathbb{R}$ into $\mathbb{R}$.

2. Find a polynomial of degree three whose graph goes through the points $(-2, -5)$, $(-1, 1)$, $(1, 1)$, and $(3, 25)$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by
   
   $$T(x, y, z, w) = (a, b, c)$$

   where
   
   $$\begin{bmatrix} 1 & -1 & 1 & -3 \\ -1 & 2 & 1 & 2 \\ 1 & 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$ 

   Find the dimension of the kernel (null space) of $T$ and of the image (range) of $T$.

4. Let $V$ be the real, inner product space of continuous functions on the closed interval $[0, \pi]$ with inner product
   
   $$(f, g) = f \cdot g = \int_0^\pi f(x)g(x)dx.$$ 

   Let $W \subseteq V$ be the subspace of $V$ spanned by the functions $1$, $\sin(x)$, and $\cos(x)$. Find an orthonormal basis of $W$.

5. How many elements are there in the group of invertible $2 \times 2$ matrices over the field of seven elements?

Part II. Solve 3 of the following 6 problems.

6. Let $U$ and $V$ be two subspaces of a finite-dimensional vector space. Show that
   
   $$\dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V).$$ 

7. Consider the $3 \times 3$ matrix
   
   $$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$ 

   a. Show $X \cdot AX = 0$ for all $X \in \mathbb{R}^3$, where $X \cdot Y$ is the usual dot product.
   b. Find a non-zero vector $Y$ so that $AY = 0$.
   c. For $Y$ as in part (b), show that $AX \cdot Y = 0$ for all $X \in \mathbb{R}^3$.
   d. For $Y$ as in part (b), show that there is a real number $\lambda$ so that if $X$ is any vector orthogonal to $Y$ (i.e., $X \cdot Y = 0$) then $A^2X = \lambda X$. Determine $\lambda$.

8. a. Let $G = \text{GL}(n, \mathbb{R})$ and $H = \{A \in \text{GL}(n, \mathbb{R}) : \det A > 0\}$ where $n > 1$. Is $H$ a subgroup of $G$? If so, is it a normal subgroup?
   b. Answer the same questions with $H$ replaced by $\{A \in \text{GL}(n, \mathbb{R}) : AA^t = I\}$, where $A^t$ denotes the transpose of $A$. 

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9. **a.** Let $G$ be any group. Show that a normal subgroup of order 2 must be contained in the center of $G$.

**b.** Consider the permutation group $S_n$ of $n$ objects. Find the center of $S_n$.

10. Is there a non-abelian group of order $n = 49$? Either find one or explain why none exists. Do the same for $n = 50$ and $n = 51$.

11. Suppose $A$ is a real, symmetric, $n \times n$ matrix with eigenvalues $1, 2, \ldots, n-1, n$. Compute $\|A\|$, the norm of $A$, where

$$\|A\| = \sup\{|Ax| \text{ for all vectors } x \in \mathbb{R}^n \text{ with norm } \|x\| = 1\},$$

where $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $x = (x_1, \ldots, x_n)$. Justify your conclusion.

**Part III.** Solve 1 of the remaining 4 problems.

12. Which of the following rings is an integral domain? Which is a field? Justify your assertions.

**a.** $\mathbb{Z}[x]/(x^2 + 7)$

**b.** $\mathbb{R}[x]/(x^4 + 3x^2 + 2)$

**c.** $\mathbb{Q}[x]/(x^3 - 2)$

13. What are all of the possible degrees for irreducible polynomials over the following fields, $F$?

**a.** $F = \mathbb{C}$, the field of complex numbers.

**b.** $F = \mathbb{Z}/p\mathbb{Z}$, where $p$ is any prime.

**c.** $F = \mathbb{R}$.

14. The three matrices $A$, $B$, and $C$ satisfy

$$A^2 = B^2 = C^2 = Id, \quad \text{and} \quad BC - CB = iA.$$

**a.** What are $AB + BA$ and $AC + CA$?

**b.** Derive a set of explicit forms of $A$, $B$, and $C$ in the case of $2 \times 2$ matrices.

15. **a.** Suppose $p, n \in \mathbb{Z}$, where $p$ is prime and $p$ does not divide $n$. Must there exist integers $a$ and $b$ such that $ap + bn = 1$?

**b.** Suppose that $f, g \in \mathbb{Q}[x]$, where $f$ is irreducible and $f$ does not divide $g$. Must there exist $h, k \in \mathbb{Q}[x]$ such that $hf + kg = 1$?

**c.** Repeat part (b) with $\mathbb{Q}[x]$ replaced by $\mathbb{Q}[x, y]$. Justify your assertions.