Preliminary Exam 2005
Afternoon exam (3 hours)

Part I. Solve 4 of the following 5 problems.

1. Find all solutions over $\mathbb{R}$ to the system of equations

$$\begin{cases} 3x - y + 8z = 0 \\ 2x + 2y + 5z = 0. \end{cases}$$

2. Find the inverse of the matrix $\begin{pmatrix} 0 & 2 & 0 \\ 5 & 1 & 7 \\ 0 & 9 & 1 \end{pmatrix}$.

3. Give an explicit example of a prime number $p > 100$ such that the integers $2^{100} - 1$, $3^{100} - 1$, and $5^{100} - 1$ are divisible by p. Justify your answer.

4. Determine whether or not the vectors

$$\begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}$$

span $\mathbb{R}^4$.

5. Let A be a 4×4 matrix with complex coefficients such that $(A - 3I)^2 = 0$, where I denotes the 4×4 identity matrix. List the possibilities for the Jordan normal form of A.

Part II. Solve 3 of the following 6 problems.

6. Prove that a continuous group homomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is actually $\mathbb{R}$-linear, i. e. satisfies $\varphi(rv) = r\varphi(v)$ for $r \in \mathbb{R}$ and $v \in \mathbb{R}^n$. (Hint: first consider integer values of r.)

7. Show that an abelian group of order < 1024 has a set of generators of cardinality < 10.

8. Let A, B, and M be $n \times n$ matrices with real-valued entries such that

$$B = M^{-1}AM,$$

i.e., such that A and B are similar matrices. Show that A and B have the same eigenvalues with the same multiplicities.

9. Let A be an $n \times n$ matrix and U an invertible $n \times n$ matrix, both with coefficients in $\mathbb{R}$, and suppose that $UAU^{-1} = cA$ for some $c \in \mathbb{R}$, $c \neq 0, \pm 1$. Prove that $A^n = 0$.

10. Let V be the real vector space of polynomials of degree equal to or less than three,

$$V = \{ax^3 + bx^2 + cx + d \; | a, b, c, d \in \mathbb{R}\}.$$

Define an inner product on V by the formula

$$< P, Q >= \int_{-\infty}^{\infty} e^{-x^2} P(x)Q(x)dx.$$

Find an orthonormal basis for V.

11. Show that the ring $\mathbb{F}_2[x]/(x^3+x+1)$ is a field but that the ring $\mathbb{F}_3[x]/(x^3+x+1)$ is not a field.

Part III. Solve 1 of the remaining 4 problems.

12. Prove that a subgroup of index 2 in a group is normal.

13. Compute the Galois group of the polynomial $f(x) = x^3 - 5x + 5$ over $\mathbb{Q}$. (Hint: the discriminant of the cubic polynomial $x^3 + bx + c$ is $-4b^3 - 27c^2$.)

14. Let V and W be vector spaces over a field $\mathbb{F}$. Consider the set of all vector space homomorphisms of V into W, denoted $Hom(V, W)$. Assume that $S, T \in Hom(V, W)$ and $v_i S = v_i T$ for all elements v_i of a basis of V. Prove that $S = T$.

15. Let R and S be commutative rings and let I and J be ideals of R and S respectively. Viewing the cartesian product $I \times J$ as an ideal of the product ring $R \times S$, prove that $I \times J$ is a prime ideal of $R \times S$ if and only either $I = R$ and J is a prime ideal of S or $J = S$ and I is a prime ideal of R. You may quote general facts about prime ideals without proof.