Preliminary Exam 2004
Afternoon exam (3 hours)

Part I. Solve 4 of the following 5 problems.

1. Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,0,-1)$, $(2,1,1,0)$, $(1,1,1,1)$, $(1,2,3,4)$, and $(0,1,2,3)$. Find a subset of these vectors which is a basis for V.

2. Find an orthonormal basis for the subspace $V = \{(x,y,z) \in \mathbb{R}^3 : 3x+y+2z = 0\}$ of $\mathbb{R}^3$. Here "orthonormal" means "orthonormal relative to the usual dot product on $\mathbb{R}^3$."

3. A certain group G contains elements g and h satisfying $ghg^{-1} = h^2$ and $g^3 = h^7 = e$, where e denotes the identity element of G. Show that $(hg)^3 = e$.

4. Find the sum of the squares of the roots of the polynomial

$$x^3 + 5x^2 - x - 1.$$

5. Find the greatest common divisor of 1122211 and 1234321.

Part II. Solve 3 of the following 6 problems.

6. Find $\lim_{n\to\infty} \dfrac{1}{2^n} A^n$, where

$$A = \begin{pmatrix} 11 & 18 \\ -6 & -10 \end{pmatrix}.$$

7. Let n be a positive integer and f a polynomial of degree n with coefficients in $\mathbb{C}$. Write $f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ with complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not necessarily distinct, and put $f_j(x) = f(x)/(x-\alpha_j)$ $(1 \leqslant j \leqslant n)$. State and prove a necessary and sufficient condition for the polynomials $f_1, f_2, \ldots, f_n$ to constitute a basis for the vector space V of polynomials of degree $\leqslant n-1$ over $\mathbb{C}$.

8. Prove Fermat's Little Theorem: if x is any integer x and p is any prime, then $x^p \equiv x \bmod p$.

9. Let A be a 2×2 matrix with even integer entries and determinant 84, and let M be the subgroup of $\mathbb{Z}^2$ generated by the columns of A. Express the quotient group $\mathbb{Z}^2/M$ up to isomorphism as a direct sum of cyclic groups of prime power order.

10. Let $G = GL(2,\mathbb{R})$ be the group of 2×2 matrices with real entries and nonzero determinant. Let H be the subgroup of G generated by r and s, where

$$r = \begin{pmatrix} \cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) \\ \sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) \end{pmatrix}$$

and

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Is H a finite group?
(b) Is H a commutative group?

11. Let G be a group, H a normal subgroup of G, and K an arbitrary subgroup of G. Let HK be the subset of G consisting of all products of the form hk with $h \in H$ and $k \in K$. Prove that HK is a subgroup of G.

Part III. Solve 1 of the remaining 4 problems.

12. Let $\mathbb{R}(t)$ denote the field of rational functions over $\mathbb{R}$, *i. e.,* the field consisting of quotients $f(t)/g(t)$ where f and g are polynomials with real coefficients and g is not the zero polynomial. Prove that the exponential function e^t is not algebraic over $\mathbb{R}(t)$.

13. Let S_{10} denote the group of permutations of the set $\{1, 2, \ldots, 10\}$. For each integer n in the range $11 \leqslant n \leqslant 20$ determine whether S_{10} contains an element of order n.

14. View the polynomial $P(x) = x^6 + 1$ as a polynomial over $\mathbb{F}_2$, the field with 2 elements. Factor P into irreducibles over $\mathbb{F}_2$.

15. Let $K = \mathbb{R}(\tan t)$ be the field generated over $\mathbb{R}$ by the tangent function, and let $L = \mathbb{R}(\cos t, \sin t)$ be the extension field of K generated over $\mathbb{R}$ by the cosine and sine functions.

(a) Determine the degree $[L : K]$.

(b) Verify that L is Galois over K and determine the structure of $\mathrm{Gal}(L/K)$.

(c) Show that if $\sigma \in \mathrm{Gal}(L/K)$ then there is a constant $c \in \mathbb{R}$ such that $\sigma(f)(t) = f(t + c)$ for $r \in \mathbb{R}(\cos t, \sin t)$ and $t \in \mathbb{R}$.