Preliminary Exam 2004
Afternoon exam (3 hours)

Part I. Solve 4 of the following 5 problems.

1. Let $V$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 0, -1)$, $(2, 1, 1, 0)$, $(1, 1, 1, 1)$, $(1, 2, 3, 4)$, and $(0, 1, 2, 3)$. Find a subset of these vectors which is a basis for $V$.

2. Find an orthonormal basis for the subspace $V = \{(x, y, z) \in \mathbb{R}^3 : 3x + y + 2z = 0\}$ of $\mathbb{R}^3$. Here “orthonormal” means “orthonormal relative to the usual dot product on $\mathbb{R}^3$.”

3. A certain group $G$ contains elements $g$ and $h$ satisfying $ghg^{-1} = h^2$ and $g^3 = h^7 = e$, where $e$ denotes the identity element of $G$. Show that $(hg)^3 = e$.

4. Find the sum of the squares of the roots of the polynomial $x^3 + 5x^2 - x - 1$.

5. Find the greatest common divisor of 1122211 and 1234321.

Part II. Solve 3 of the following 6 problems.

6. Find $\lim_{n \to \infty} \frac{1}{2^n} A^n$, where

$$A = \begin{pmatrix} 11 & 18 \\ -6 & -10 \end{pmatrix}.$$

7. Let $n$ be a positive integer and $f$ a polynomial of degree $n$ with coefficients in $\mathbb{C}$. Write $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ with complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not necessarily distinct, and put $f_j(x) = f(x)/(x - \alpha_j)$ ($1 \leq j \leq n$). State and prove a necessary and sufficient condition for the polynomials $f_1, f_2, \ldots, f_n$ to constitute a basis for the vector space $V$ of polynomials of degree $\leq n - 1$ over $\mathbb{C}$.

8. Prove Fermat’s Little Theorem: if $x$ is any integer and $p$ is any prime, then $x^p \equiv x \mod p$.

9. Let $A$ be a $2 \times 2$ matrix with even integer entries and determinant 84, and let $M$ be the subgroup of $\mathbb{Z}^2$ generated by the columns of $A$. Express the quotient group $\mathbb{Z}^2/M$ up to isomorphism as a direct sum of cyclic groups of prime power order.

10. Let $G = GL(2, \mathbb{R})$ be the group of $2 \times 2$ matrices with real entries and nonzero determinant. Let $H$ be the subgroup of $G$ generated by $r$ and $s$, where

$$r = \begin{pmatrix} \cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) \\ \sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) \end{pmatrix}$$

and

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Is $H$ a finite group?
(b) Is $H$ a commutative group?
11. Let $G$ be a group, $H$ a normal subgroup of $G$, and $K$ an arbitrary subgroup of $G$. Let $HK$ be the subset of $G$ consisting of all products of the form $hk$ with $h \in H$ and $k \in K$. Prove that $HK$ is a subgroup of $G$.

Part III. Solve 1 of the remaining 4 problems.

12. Let $\mathbb{R}(t)$ denote the field of rational functions over $\mathbb{R}$, i.e., the field consisting of quotients $f(t)/g(t)$ where $f$ and $g$ are polynomials with real coefficients and $g$ is not the zero polynomial. Prove that the exponential function $e^t$ is not algebraic over $\mathbb{R}(t)$.

13. Let $S_{10}$ denote the group of permutations of the set $\{1, 2, \ldots, 10\}$. For each integer $n$ in the range $11 \leq n \leq 20$ determine whether $S_{10}$ contains an element of order $n$.

14. View the polynomial $P(x) = x^6 + 1$ as a polynomial over $\mathbb{F}_2$, the field with 2 elements. Factor $P$ into irreducibles over $\mathbb{F}_2$.

15. Let $K = \mathbb{R}(\tan t)$ be the field generated over $\mathbb{R}$ by the tangent function, and let $L = \mathbb{R}(\cos t, \sin t)$ be the extension field of $K$ generated over $\mathbb{R}$ by the cosine and sine functions.

   (a) Determine the degree $[L : K]$.

   (b) Verify that $L$ is Galois over $K$ and determine the structure of $\text{Gal}(L/K)$.

   (c) Show that if $\sigma \in \text{Gal}(L/K)$ then there is a constant $c \in \mathbb{R}$ such that $\sigma(f)(t) = f(t + c)$ for $r \in \mathbb{R}(\cos t, \sin t)$ and $t \in \mathbb{R}$. 