Afternoon Exam 2003

PART I. Solve 4 out of the next 5 problems.

1. Find a polynomial of degree 4 whose graph goes through the points $(1,1)$, $(2,-1)$, $(3,-59)$, $(-1,5)$, $(-2,-29)$.

2. What are the eigenvalues of the matrix A which represents the rotation of $\mathbb{R}^3$ by θ around an axis v?

3. Compute the inverse of the following matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

4. Find an orthonormal basis for the vector space V of the polynomials over $\mathbb{R}$ of degree less than or equal to 2, with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

5. Find the greatest common divisor of 2003^4+1 and 2003^3+1.

PART II. Solve 3 out of the next 6 problems.

6. Let A be a 3×3 orthogonal matrix whose determinant is -1. Prove that -1 is an eigenvalue of A.

7. Classify up to similarity all 3×3 complex matrices A such that $A^3 = I$.

8. Let $f(x)$ be a polynomial of degree n that takes integer values at all integer points. Prove that f can be written as a linear combination with integer coefficients of the polynomials $P_k = \frac{x(x-1)\cdots(x-k+1)}{k!}$, $0 \le k \le n$ (where $P_0 = 1$).

9. Assume that every nontrivial element g of a group G has order 2. Prove that G is commutative.

10. Let $f(x) = x^n - nx + 1$ and let A be an $n \times n$ matrix with characteristic polynomial f.
(a) Prove that if $n > 2$ then A is diagonalizable over the complex numbers. (Hint: Prove that f has no common zeros with f'.)
(b) Is the assertion in (a) true if $n = 2$? Either prove it or give a counterexamble.

11. Let $A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$. Find $\lim_{n\to\infty} A^n$.

PART III. Solve 1 out of the next 3 problems.

12. Let G be the dihedral group defined as the set of all formal symbols $x^i y^j$, with $i = 0,1$ and $j = 0, 1, \dots, n-1$, and where $x^2 = e$, $y^n = e$ for $n > 2$, and $xy = y^{-1}x$.
(a) Prove that the subgroup $N = \{e, y, y^2, \dots, y^{n-1}\}$ is normal in G.
(b) Prove that $G/N \approx W$, where $W = \{1, -1\}$ is the group under the multiplication of the real numbers.

13. For which values of n does the number of conjugacy classes in S_n (the group of permutations of n letters) equal n?

14. Let $f(x)$ and $g(x)$ be a pair of polynomials in one variable. Prove that there exists a nonzero polynomial $F(x,y)$ such that $F(f(x), g(x)) \equiv 0$. [Hint: consider the linear transformation $F \mapsto F(f,g)$ from the space of polynomials in two variables of degree $\le n$ to the space of polynomials in one variable.]