PART I. Solve 4 out of the next 5 problems.

1. Find a polynomial of degree 4 whose graph goes through the points (1, 1), (2, −1), (3, −59), (−1, 5), (−2, −29).

2. What are the eigenvalues of the matrix $A$ which represents the rotation of $\mathbb{R}^3$ by $\theta$ around an axis $v$?

3. Compute the inverse of the following matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}.
$$

4. Find an orthonormal basis for the vector space $V$ of the polynomials over $\mathbb{R}$ of degree less than or equal to 2, with the inner product

$$
(f, g) = \int_0^1 f(x)g(x)dx.
$$

5. Find the greatest common divisor of $2003^4 + 1$ and $2003^3 + 1$.

PART II. Solve 3 out of the next 6 problems.

6. Let $A$ be a $3 \times 3$ orthogonal matrix whose determinant is $−1$. Prove that $−1$ is an eigenvalue of $A$.

7. Classify up to similarity all $3 \times 3$ complex matrices $A$ such that $A^3 = I$.

8. Let $f(x)$ be a polynomial of degree $n$ that takes integer values at all integer points. Prove that $f$ can be written as a linear combination with integer coefficients of the polynomials $P_k = \frac{x(x-1)\ldots(x-k+1)}{k!}$, $0 \leq k \leq n$ (where $P_0 = 1$).

9. Assume that every nontrivial element $g$ of a group $G$ has order 2. Prove that $G$ is commutative.

10. Let $f(x) = x^n − nx + 1$ and let $A$ be an $n \times n$ matrix with characteristic polynomial $f$.
   (a) Prove that if $n > 2$ then $A$ is diagonalizable over the complex numbers. (Hint: Prove that $f$ has no common zeros with $f'$.)
   (b) Is the assertion in (a) true if $n = 2$? Either prove it or give a counterexample.

11. Let $A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$. Find $\lim_{n \to \infty} A^n$.

PART III. Solve 1 out of the next 3 problems.

12. Let $G$ be the dihedral group defined as the set of all formal symbols $x^iy^j$, with $i = 0, 1$ and $j = 0, 1, \ldots, n-1$, and where $x^2 = e$, $y^n = e$ for $n > 2$, and $xy = y^{-1}x$.
   (a) Prove that the subgroup $N = \{e, y, y^2, \ldots, y^{n-1}\}$ is normal in $G$.
   (b) Prove that $G/N \cong W$, where $W = \{1, -1\}$ is the group under the multiplication of the real numbers.

13. For which values of $n$ does the number of conjugacy classes in $S_n$ (the group of permutations of $n$ letters) equal $n$?

14. Let $f(x)$ and $g(x)$ be a pair of polynomials in one variable. Prove that there exists a nonzero polynomial $F(x, y)$ such that $F(f(x), g(x)) \equiv 0$. [Hint: consider the linear transformation $F \mapsto F(f, g)$ from the space of polynomials in two variables of degree $\leq n$ to the space of polynomials in one variable.]