INVARIANT VALUATION AND ECONOMIC DEPRECIATION: A CONSTRUCTIVE PROOF OF THE SAMUELSON THEOREM

Boston University School of Law Working Paper No. 11-06
(February 14, 2011)

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ABSTRACT: I elaborate on Samuelson’s (1964) result that a tax on capital income will leave asset values unaffected by the tax rates of their holders only if "economic" depreciation is allowed as a deduction in computing taxable income, extended by Lyon (1990) to the case of time-varying marginal rates. Neither offers any insight into why, with economic depreciation, economic agents in a taxable environment should behave as though (as Lyon puts it) they were discounting all pre-tax "cash receipts . . . at the pre-tax interest rate.” Using discrete time, I formulate a constructive proof of Samuelson’s result, which shows that economic depreciation induces pure accrual taxation, with the result that the impact of income taxation on the accrual of value and on discounting exactly offset one another. That is why taxpayers behave as though they were discounting pre-tax cash flows.

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Over the past 45 years a handful of papers has established that, in the presence of a per-
sonal tax that extends to capital income, asset values are independent of the marginal tax rates of their holders if and only if "economic depreciation" is allowed as a deduction against gross income from capital in determining taxable income. This "neutrality" result is originally due to Samuelson (1964), with an elaboration by Lyon (1990). In this paper I address a shortcoming in both papers: neither provides any insight into why, in the presence of a capital income tax, all taxpayers should (as Lyon’ puts it) value assets as "the present value of all [pre-tax] receipts discounted at the pre-tax interest rate." Approaching the problem differently than both Samuelson and Lyon, I provide a new proof of the original proposition in discrete time, one that lays bare why taxpayers should behave in that way. Economic depreciation induces pure accrual taxa-
tion, with the consequence that the impact of the income tax on the accrual of value is exactly offset by its impact on the discounting process, effectively neutralizing (in a different sense) the effects of the tax.

Samuelson and Lyon proceeded in fundamentally similar ways. In continuous time, Sam-
uelson obtained the time derivatives of the value function without taxes, and with both taxes and depreciation. He then equated the two to find that the equation is identically satisfied (after eliminating the tax factors) only when the depreciation term equals the negative of the time derivative of the value function in the absence of taxes. To obtain that result, Samuelson had to factor the tax rate out of the valuation integral, which required him to assume that the tax rate was constant over time. He also suggested in passing that the same result could be obtained in discrete time, and that is how Lyon proceeds. After writing down the valuation function, Lyon substitutes the bare change in that function ($V_t - V_{t+1}$) for the depreciation term and again shows that the tax factors may be eliminated, leaving all economic agents to behave as though they inhabited a world without taxes. Since they do so in every period, moreover, Lyon infers that they will behave that way even if tax rates vary over time.

Neither solution, however, provides any intuition into why agents in a world with taxes should behave as though there were none. I remedy that using the discrete time framework. But, in contrast with Lyon, I obtain an explicit representation of the change in the value function (it has the same form as Samuelson’s time derivative without taxes). When that is substituted into
the value function, what emerges is that the asset holder is taxed as value accrues to the asset with the passage of time. Proceeding by induction I then show constructively that the value function appears to all investors as independent of taxes because the effect of taxation on the accrual of value is exactly offset by its impact on the process of discounting -- taxes reduce asset values and the discount rate in ways that mirror one another -- and they do so at every marginal rate.¹ What is more, this approach confirms, consistent with Lyon’s finding, that the result holds in the presence of time varying tax and interest rates.

I. ECONOMIC DEPRECIATION AND ACCRUAL TAXATION

Consider an asset consisting of an arbitrary sequence of cash flows \{C_1, C_2, \ldots, C_N\}, discounted at an appropriate pre-tax discount rate \( r > -1 \). In discrete time the value of the asset at the time of acquisition is the sum of the present values of its cash flows, or

\[
V_0(r) = \sum_{j=1}^{N} \frac{C_j}{(1+r)^j}.
\]

(If \( V_0(r) \) is the purchase price, then \( r \) is the asset’s internal rate of return.) Thereafter, as of the end of any intervening period \( k \), the value of the remaining \( N-k \) cash flows, discounted to the end of period \( k \) (equivalently, the beginning of period \( k+1 \), is

(1)

\[
V_k(r) = \sum_{j=k+1}^{N} \frac{C_j}{(1+r)^{j-k}},
\]

with \( V_N = 0 \).

Suppose we tax this asset by including in gross income the cash flow in each period, allowing a deduction in computing taxable income for "economic" depreciation, defined as the actual change in the value of the asset during that period. To determine the change in value during period \( k+1 \), note that the value at the end of that period is

\[
V_{k+1} = \sum_{j=k+2}^{N} \frac{C_j}{(1+r)^{j-k-1}} = (1+r) \sum_{j=k+2}^{N} \frac{C_j}{(1+r)^{j-k}}
\]

¹ This approach thus formally corroborates an intuition about the problem by Strnad (1994).
\[
V_k - V_{k+1} = V_k - [(1+r)V_k - C_{k+1}]
\]
\[
= -rV_k + C_{k+1} = D_{k+1}.
\]

That is, between the ends of periods \( k \) and \( k+1 \), each cash flow comes one period closer to being realized and so is discounted for one fewer periods, so that \( V_k \) increases to \((1+r)V_k\); then the \( k+1 \)st cash flow is realized and so is subtracted from the remaining (augmented) sum. The difference between the values in periods \( k \) and \( k+1 \) is therefore

\[
D_{k+1}, \text{ as given by (2), is "economic" depreciation of the asset in period } k+1; \text{ it corresponds to the continuous depreciation function in Samuelson. The form of the depreciation allowance is itself of note. While depreciation is usually conceived of in terms of the physical characteristics of an asset, or of the myriad effects on the asset -- obsolescence, decay, and the like -- of time, it here is a financial phenomenon, consisting of the increase in the present value of the entire expected remaining income stream from the asset at the beginning of period } k+1, \text{ offset by the disappearance of the } k+1 \text{st cash flow. } "\text{Depreciation" is just the difference between those two.}

It follows that if in each period we include \( C_{k+1} \) in income, but allow a deduction for depreciation in the amount \( D_{k+1} \) as given by (2), we tax

\[
C_{k+1} - D_{k+1} = C_{k+1} - (-rV_k + C_{k+1}) = rV_k.
\]

But (3) is just the accrual of yield (at rate \( r \)) on the asset’s value at the beginning of period \( k+1 \). Thus, economic depreciation induces pure accrual taxation; the two are formally equivalent.\(^2\)

\(^2\) In effect, (3) formalizes a relationship articulated and illustrated in Chapter 14 of Fisher (1906) 237-238. What are here referred to as cash flow and accrual were denoted by Fisher “realized” and “earned” income, respectively. Fisher’s "general principle connecting realized and earned income is that they differ by the appreciation or depreciation of capital. It is thus possible to describe earned income as realized income less depreciation of capital, or else as realized income plus appreciation of capital.” It may also be obtained by substituting the depreciation function in Samuelson back into the valuation function. See Sims (1994).

The matter of which is more "properly" to be regarded as "income" for tax purposes, cash flow (Fisher’s "realized" income), or accrued value (Fisher’s "earnings"), occasions recurring heated disputes of both terminology and substance. E.g., Simons (1938) 77-102, 226-231. In substance the question dates at least back to Mill (1848).
II. A CONSTRUCTIVE PROOF OF THE SAMUELSON THEOREM

Since economic depreciation and pure accrual taxation are equivalent, both satisfy the conditions of the Samuelson theorem. Conceived of either way the asset’s value will be independent of the tax rate. And with the aid of expression (3) we can establish Samuelson’s result in a constructive fashion that differs from both Samuelson’s and Lyon’s.

To do so, note first that the value of the asset in the presence of an income tax at rate $t$ is the sum of the cash-flows after tax, discounted using an after-tax discount rate. In discrete time that is usually written

$$V_0(r, t) = \sum_{j=1}^{N} \frac{C_j - t(C_j - D_j)}{[1 + r(1-t)]^j}.$$  

Using (3), however, each after tax cash flow $j$ can be rewritten as

$$C_j - t(C_j - D_j) = C_j - trV_{j-1}.$$  

and the initial value of the asset as

$$(1') \quad V_0(r, t) = \sum_{j=1}^{N} \frac{C_j - trV_{j-1}}{[1 + r(1-t)]^j}.$$  

The first term in the sum is

$$(T1) \quad \frac{C_1 - trV_0}{1 + r(1-t)},$$  

using the fact that

$$V_0 - \sum_{j=2}^{N} \frac{C_j}{(1+r)^j} - \frac{C_1}{1-r} + \sum_{j=2}^{N} \frac{C_j}{(1+r)^j},$$  

so that

$$C_1 = (1+r)V_0 - (1+r)\sum_{j=2}^{N} \frac{C_j}{(1+r)^j} = (1+r)V_0 - \sum_{j=2}^{N} \frac{C_j}{(1+r)^j-1}$$  

$$= (1+r)V_0 - V_1,$$

we can rewrite the numerator of the first term as
\[ C_1 - trV_0 = (1-r)V_0 - V_1 - trV_0 = \left[ 1+r(1-t) \right] V_0 - V_1; \]

discounted for one period it is

\[ (T1') \quad \frac{\left[ 1+r(1-t) \right] V_0 - V_1}{\left[ 1+r(1-t) \right]} = V_0(r) - \frac{V_1}{\left[ 1+r(1-t) \right]}. \]

The basic intuition underlying Samuelson’s result is captured by (T1’). Because economic depreciation is equivalent to accrual taxation, value accrues to the initial value of the asset at rate \( r^*(1-t); \) at the same time discounting at the after-tax discount rate exactly offsets that accrual, and with the cancellation of the two the original value is preserved, taking into account the (after-tax) present value of the (after-tax) gain accrued in the first period, though not (necessarily) with respect to the asset’s remaining value (\( V_1 \)). What remains to be determined is whether this pattern recurs. To see that it does, note first that by the same argument the second period cash flow, discounted for two periods, can be written as

\[ (T2') \quad \frac{C_2 - trV_1}{\left[ 1+r(1-t) \right]^2} = \frac{V_1}{\left[ 1+r(1-t) \right]} - \frac{V_2}{\left[ 1+r(1-t) \right]^2}, \]

and more generally, discounting the \( k \)th term for \( k \) periods produces

\[ (Tk') \quad \frac{C_k - trV_{k-1}}{\left[ 1+r(1-t) \right]^k} = \frac{V_{k-1}}{\left[ 1+r(1-t) \right]^{k-1}} - \frac{V_k}{\left[ 1+r(1-t) \right]^k}. \]

Returning to (T1’ and T2’ and summing, the terms in \( V_1 \) cancel, leaving

\[ (S_2) \quad V_0(r) - \frac{V_2}{\left[ 1+r(1-t) \right]^2}. \]

We should therefore expect that extending the process to \( k \) terms will through repeated cancellation yield the \( k \)th partial sum

\[ (S_k) \quad \sum_{j=1}^{k} \frac{C_j - trV_{j-1}}{\left[ 1+r(1-t) \right]^j} = V_0(r) - \frac{V_k}{\left[ 1+r(1-t) \right]^k}, \]

from which by induction it follows that
verifying that each partial sum takes the form $S_k$. But since $V_N = 0$, it follows, independently of $t$, that

$$V_0(r) = \frac{V_k}{[1 + r(1-t)]^k} + \frac{V_{k-1}}{[1 + r(1-t)]^{k-1}}$$

The same result can be obtained starting from the end of any period $k$, establishing that the values $V_k$ will likewise be independent of $t$.

Observe that given the result $(S_N)$ that

$$V_0(r, t) = \sum_{j=1}^{N} \frac{C_j - trV_{j-1}}{[1 + r(1-t)]^j} = V_0(r) - \frac{V_N}{[1 + r(1-t)]^N} = V_0(r)$$

it follows that for the pre-tax investment and subsequent after-tax cash flows

$$\{-V_0(r), C_1-trV_0, ..., C_N-trV_{N-1}\},$$

$\textit{r}^*(1-t)$ is the internal rate of return that solves $S_N$. That is, for each investment and every investor or the pre-tax rate of return is reduced by the tax rate.

**III. Time-Varying $r$ and $t$**

Now suppose that there are vectors $r = \{r_1, ..., r_N\}$ and $\tau = \{\tau_1, ..., \tau_N\}$ of interest rates and tax rates, respectively, prevailing during the $N$-period life of the asset. Then (1) becomes

$$(1t) \quad V_k(r) = \sum_{j=k+1}^{N} \frac{C_j}{\prod_{i=k+1}^j (1 + r_i)},$$

and

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\[ V_{k-1}(r) = \sum_{j=k}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} = (1 + r_{k-1}) \sum_{j=k}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} \]

\[ = (1 + r_{k-1}) \sum_{j=k+1}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 - r_i)} - C_{k-1} \]

so depreciation in period \( k+1 \) is now

\[ (2t) \quad - r_{k-1} V_k(r) + C_{k-1} \]

and taxable income in \( k+1 \) is

\[ (3t) \quad r_{k-1} V_k(r) \]

Hence the after tax value becomes

\[ V_0(r, \tau) = \sum_{j=1}^{N} \frac{C_j - \gamma_j r_j V_{j-1}(r)}{\prod_{i=1}^{j}[1 + r_i(1 - \gamma_i)]} \]

Since

\[ V_{k-1}(r) = \sum_{j=k}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} = \frac{C_k}{1 + r_k} + \sum_{j=k+1}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} \]

so that

\[ C_k = (1 + r_k)V_{k-1}(r) - (1 + r_k) \sum_{j=k+1}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} \]

\[ = (1 + r_k)V_{k-1}(r) - \sum_{j=k+1}^{N} \frac{C_j}{\prod_{i=k}^{j-1}(1 + r_i)} \]

\[ = (1 + r_k)V_{k-1}(r) - V_k(r) \]

we may again rewrite the \( k \)th term as
\[
\frac{C_k - \tau_k r_k V_{k-1}(r)}{\Pi_{t=1}^{k-1} [1 + r_t (1 - \tau_t)]} = \frac{[1 + r_k (1 - \tau_k)] V_{k-1}(r) - V_k(r)}{\Pi_{t=1}^{k-1} [(1 + r_t (1 - \tau_t)]}
\]

and the \(k+1\)st as

\[
\frac{V_{k-1}(r)}{\Pi_{t=1}^{k-1} [1 + r_t (1 - \tau_t)]} - \frac{V_k(r)}{\Pi_{t=1}^{k-1} [(1 + r_t (1 - \tau_t)]}
\]

so that the terms in \(V_k(r)\) are discounted by the same factor and therefore cancel when added, and in the summation \((S_N)\) all but the first and final terms again cancel, preserving the result obtained in \(\Pi\).

The formulation of the depreciation allowance in (2), as Samuelson suggests, does not require that depreciation in every (or even any) period be positive, \(\text{viz.}\) that the asset value declines. When negative it captures appreciation. At one extreme, for example, when the cash flow is zero in each but the final period, the depreciation allowance is \(-r^* V_k\), and (as always) taxable income is \(r^* V_k\). So the asset looks, for example, like a pure discount bond, which we tax on a pure accrual basis under § 1272(a) of the Internal Revenue Code. That is, even though we do not conventionally think of discount (or other) debt as subject to "depreciation," the depreciation function in (2) is sufficiently general to account for periodic accrual of discount in terms of (and in effect as entailing) economic depreciation. This highlights the fact that, while Samuleson’s paper was nominally about depreciation, it -- as well as the elaborations offered by Lyon and here -- is actually a solution to the more general question of what amount should be allowed as a deduction for tax purposes across the entire history of an asset if what is desired is a capital income tax that does not distort asset values. As such it provides insight into the treatment of the costs of asset acquisitions themselves, that is, into the "capitalization" requirement of I.R.C. § 263; as well as into the treatment of extraordinary costs incurred on retirement, as analyzed by (among others) Kiefer (1985) and Sims (1994), addressed by I.R.C. §§ 468-468A.
REFERENCES