QUALIFYING EXAM IN SYSTEMS ENGINEERING

Written Exam: MAY 22, 2018, 9:00AM to 1:00PM, EMB 105

Oral Exam: May 24 or 25, 2018
Time/Location TBA (-1 hour per student)

CLOSED BOOK, NO CHEAT SHEETS
BASIC SCIENTIFIC CALCULATOR PERMITTED
ALL EXAM MATERIALS STAY IN THE EXAM ROOM

GENERAL INSTRUCTIONS:

1) Please write on every sheet:
   a. Your Exam Number
   b. The page numbers (example: Page 1 of 4)

2) Only write on 1 side.
   Exams may be scanned and emailed to the faculty for grading.
   If using pencil, make sure it is dark.

COMPLETE THE REQUIRED SECTIONS AS BELOW:

The exam consists of three topical sections. Select three of the following five sections:

A. Dynamic Systems Theory (SE/EC/ME 501)
B. Continuous Stochastic Processes (EC505) OR Discrete Stochastic Processes (EK500 and SE/ME 714)
C. Optimization (SE/EC 524)
D. Dynamic Programming and Stochastic Control (SE/EC/ME 710)
E. Nonlinear Systems and Control (SE/ME 762)
1. For which of the following systems is the origin asymptotically stable?

(i) $\ddot{x} + a\dot{x} + bx = 0, \quad a > 0, b > 0,$

(ii) $\ddot{x} + a\dot{x} + bx = 0, \quad a < 0, b > 0,$

(iii) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a < 0,$

(iv) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -a & a \\ a & -1 & 0 \\ -a & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$

You should, of course, state your reasons.

2. For each of the following polynomials determine how many roots are in the right half-plane:

(i) $\lambda^2 - 2\lambda + 1,$

(ii) $\lambda^3 + 5\lambda^2 + 7\lambda + 3,$

(iii) $\lambda^5 + \lambda^4 - 4\lambda^3 + 5\lambda^2 - 9\lambda - 18.$

3. Find a polynomial having the least possible degree such that all coefficients are positive, but the polynomial has at least one root in the right half plane.

4. In the second-order transfer function $g(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$ $\zeta$ is called the damping ratio, and $\omega_0$ is called the undamped natural frequency. The dynamics of a dc-motor are given by $J\ddot{\theta} + c\dot{\theta} = u(t),$ where $J$ is the motor inertia, $c$ is a coefficient representing forces opposing the motion (back EMF, viscous friction, etc.), and $u(t)$ is a control input proportional to the applied current. Find the feedback gains in a control law of the form $u = -k_v\dot{\theta} - k_p\theta$ such that the natural undamped frequency of the system ($\omega$) is 1, and the damping ratio, $\zeta$ is also 1. ($k_p$ and $k_v$ are to be expressed in terms of $J$ and $c.$)
Problem #1:
Say whether each of the following statements is true or false and give a brief justification. One word answers will receive no credit.

(a) If $X(t)$ is zero mean WSS process with $R_X(\tau) = \max(1-|\tau|, 0)$, it is mean-squared continuous.
(b) A mean-square differentiable zero mean WSS Gaussian process, $X(t)$, and its derivative, $X'(t)$
are uncorrelated (and hence independent).
(c) The discrete time process, $W_k = SU_k + (1-S)V_k$, $k \in \mathbb{Z}$ is mean-squared ergodic if $U_k$ and $V_k$ are IID Bernoulli sequences with $U_k \sim \text{Ber}(1/4)$ and $V_k \sim \text{Ber}(3/4)$ and $S \sim \text{Ber}(1/2)$ is
a scalar Bernoulli random variable.
(d) If $X_n$ is a discrete-time IIP process then so is $X_n^2$.
(e) If $X_n$ is strict-sense stationary then so is $X_n^2$.

Problem #2:
Let $(X_k : k \geq 0)$ be a time-homogeneous discrete-time Markov process with the one-step transition probability diagram shown below.

(a) Write down the one step transition probability matrix $P$.
(b) Find the equilibrium probability distribution $\pi$.
(c) Find the mean hitting times for states 1 and 2 for hitting state 3.
Problem 1. Consider a non-homogeneous Poisson process $N = \{N(t); t \geq 0\}$ with the rate function

$$\lambda(t) = \frac{c}{t + 1}, \quad t \geq 0, \quad c > 0.$$ 

Let $\{S_n; n \geq 1\}$ be the sequence of arrivals of the process, $S_0 = 0$, and let $\{X_n; X_n = S_n - S_{n-1}, n \geq 1\}$ be the sequence of inter-arrivals of the process.

(a) Find the distribution (or density) of $X_1$.

(b) Are $X_1$ and $X_2$ independent? Explain your answer.

Problem 2. Let $\{S_n; n \geq 1\}$ be the sequence of arrivals of a renewal process, $S_0 = 0$, and let $\{X_n; X_n = S_n - S_{n-1}, n \geq 1\}$ be the sequence of inter-arrivals of the process. Let

$$Y(t) = S_{N(t)+1} - t; \quad t \geq 0$$

be the residual lifetime process of the renewal process. Find a condition on the renewal process such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t E[Y(u)]du = \infty.$$ 

Explain your answer.

Problem 3. Argue (prove) that in a Markov chain (finite or infinite state) open communication classes are transient.

(Recall that a communication class $C$ is open if there exists $i \in C$ and $j \notin C$ such that $P_{ij} > 0$; in other words, there is a positive probability of moving outside of $C$.)
There are three problems in this exam for a total of 100 points. Please justify your answers and provide detailed derivations. Answers without full and valid justifications will not get full credit. Good luck!

Problem 1 (K1 pts)

For each one of the statements below, state whether it is true or false. Include a 1-2 sentence supporting argument, or a counterexample, but not a formal proof.

Grading:
- Correct answer, with explanation: 0.3K1 points
- Correct answer, bad or no explanation: 0.2K1 points
- Incorrect answer, with explanation: 0.1K1 points
- No answer or incorrect answer with no explanation: 0 points.

1. Finding the optimal solution to a linear programming problem is in the worst case an NP-Hard problem.

2. Suppose we have two polyhedra defined by \( P = \{ x | Ax \leq b \} \) and by \( Q = \{ x | Bx \leq d \} \). We want to determine whether there is a point in \( P \cap Q \). This can be solved using linear programming.

3. A specific linear program can have multiple primal optimal solutions and multiple dual optimal solutions.

4. Every optimal solution of a linear program is a basic feasible solution.

5. Suppose we have a linear program written as a minimization problem:

   \[
   \min c^T x \quad \text{such that} \quad Ax = b, x \geq 0
   \]

   Let \( p \) be a feasible solution to the dual problem. Then, \( c^T x \geq b^T p \).

6. Let \( G = (\mathcal{N}, \mathcal{E}) \) denote a network with nodes \( \mathcal{N} \), directed edges \( \mathcal{E} \), where each edge \( e = (i, j) \in \mathcal{E} \) has a distance \( d_e \). The problem of finding the shortest path from node 1 to node \( n \in \mathcal{N} \) can be solved as a linear program.

7. Let \( f(x) \) be a strictly convex differentiable function. Define \( ||x|| \) to be the Euclidean norm of \( x \). The problem

   \[
   \min_{1 \leq ||x|| \leq 2} f(x)
   \]

   has a unique optimal solution \( x^* \).

8. The dual simplex algorithm starts with a basic feasible solution, and pivots to other basic feasible solutions until the optimal basic feasible solution is found, or one detects there that the problem is unbounded.

9. Consider a min-cost flow network optimization problem, where each of the arcs has a capacity constraint that restricts the flow on that arc to be less than or equal to 2. Then, if there exists an optimal solution, there exists an optimal solution where the basic arcs form a spanning tree on the graph.
10. Consider the integer knapsack problem

$$\text{maximize } \sum_{i=1}^{n} c_i x_i \text{ subject to } \sum_{i=1}^{n} w_i x_i \leq W, \ x_i \in \{0, 1\}$$

This problem can be solved exactly in time that is polynomial in $W$ and $n$.

**Problem 2 ($K_2$ pts)**
Consider the linear programming problem

$$\text{min } c^T x \text{ subject to } Ax \leq b$$

1. Form the dual of this problem.

2. Write the complementary slackness conditions.

3. Show that the set of vectors $c$ such that the primal has a finite optimal solution is a convex set.

**Problem 3 ($K_3$ pts)**
You have lost your keys, and you know you lost them in one of four places: the messy bedroom (1), the living room (2), the even messier office (3) or the rest room (4). You need to minimize the probability that you don’t find your keys in the next 10 minutes. The probability model used in this problem is standard in search theory, that indicates that the probability that you don’t find your keys that were left in room $j$ after searching for time $x_j$ is $e^{-c_j x_j}$. Assuming equal probability of having lost the keys in each of the rooms, the probability of not finding the keys with a search plan of allocating $x_j$ to room $j$ is

$$P(x) = \sum_{j=1}^{4} 0.25 e^{-c_j x_j}$$

Let $c_1 = 0.5, c_2 = 1.0, c_3 = 0.5, c_4 = 2.0$ when $x_j$ is expressed in units of minutes. If you have 10 total minutes to search among the four rooms (no time required to switch rooms), find the optimal allocation of non-negative search time $x$. Note you can’t allocate negative time to any room...

**Problem 4 (5 pts)**
Let $a_1, a_2, a_3$ be the fraction of problems 1, 2, 3 that you believe you got right. You choose these numbers. Write down the explicit choice of these numbers, and solve the following linear program explicitly:

$$\max_{K_1, K_2, K_3} \sum_{j=1}^{3} a_j K_j \text{ subject to } \sum_{j=1}^{3} K_j \leq 95, \ 20 \leq K_j \leq 45, j = 1, 2, 3.$$
Question 1

1.1 Consider first the stochastic DP problem with finite horizon $N$ and positive scalar state and control variables, $x_k, u_k, k=0,1,2,\ldots,N$. The DP Problem has:

dynamics, $x_{k+1} = x_k + 2u_k w_k$, with $w_k$ a uniformly distributed random variable over $[0, x_k]$,

allowable control space $U=\{u_k : u_k \geq 0\}$, period cost $g_k(x_k, u_k) = x_k + \frac{1}{u_k}$, and terminal cost $g_N(x_N) = x_N$.

- Find the optimal control $u^*_{N-1} = \mu_{N-1}(x_{N-1})$ and cost to go function $J_{N-1}(x_{N-1})$.

1.2 Consider next the imperfect state information problem ($x_k$ is not observed perfectly) with the same dynamics and costs. The observation equation is

$z_{k+1} = x_{k+1} + u_k v_{k+1}$ with $v_{k+1}$ a normally distributed observation noise, $\mathcal{N}(0, \sigma_v)$.

The initial probability distribution of the state, $P_0(x_0)$ after $z_0$ is observed, can be considered to be given (i.e. known at $k=0$).

- Comment on the sufficient statistic of this imperfect state information problem. In particular, is the sufficient statistic the full probability distribution of the state conditional upon the information vector, or is it simply the first moment of the distribution? Justify your answer.

- Develop an expression for the estimator $P_{x_{k+1} | z_k} = \Phi(P_{x_k | z_k}, u_k, z_{k+1})$, making sure to derive as explicitly as you can the prior probability distribution of $x_{k+1}$, namely, $P(x_{k+1} | u_k, P_{x_k | z_k})$.

Question 2

A cloud computing facility with $n$ processors accepts jobs at a price $u$ that it broadcasts and may modify dynamically. While the broadcasted price is $u$, jobs arrive at exponential inter arrival times with mean $\tau_\alpha = 1/(1-u)$. The allowed prices that may be broadcasted must lie in the interval $0 \leq u \leq 1$. When a job arrives, it pays the cloud computing facility $u$ dollars and is assigned to a processor that it employs for an exponential time with mean $\tau_s$, and then departs. If a job arrives while there are $n$ jobs being processed, it is turned away and the cloud computing facility is penalized by $c$ dollars. The objective of the cloud computing facility is to maximize its average income over time derived at job arrival events.

- Formulate the Hamilton-Jacoby-Bellman equation and use it to characterize the optimal price that the cloud computing facility ought to broadcast when the number of jobs being processed equals $x$, $u = \mu(x)$ for $x \in \{x : 0 \leq x \leq n\}$. Make sure to describe the dynamics of $x$. 
Problem 1. Give a precise mathematical definition of the uniformly asymptotic
stability of an equilibrium point \( x_0 \) of \( \dot{x}(t) = f(t, x(t)) \).

Problem 2. A phase-locked loop in communication networks can be described by
\[
\ddot{y} + [a + b \cos y] \dot{y} + c \sin y = 0
\]
1. Transform the system into state space form by choosing \( x_1 = y \) and \( x_2 = \dot{y} \).
2. Suppose \( c > 0 \). Using the Lyapunov function candidate
\[
V(x_1, x_2) = c(1 - \cos x_1) + (1/2)x_2^2
\]
show that \((0, 0)\) is a stable equilibrium point if \( a \geq b \geq 0 \), and that \((0, 0)\) is an
asymptotically stable equilibrium point if \( a > b \geq 0 \).

Problem 3. Consider the time-invariant differential equation \( \dot{x} = f(x) \) with equilib-
rium point \( 0 \). Suppose there exists \( V: \mathbb{R}^n \to \mathbb{R} \) such that
\begin{enumerate}
\item \( V \) is continuously differentiable
\item \( V(0) = 0 \)
\item \( V(x) > 0, \ \forall x \in \mathbb{R}^n - \{0\} \)
\item \( \dot{V}(x) < 0, \ \forall x \in \mathbb{R}^n - \{0\} \)
\end{enumerate}
Give a detailed proof from first principles that every bounded solution converges to
0 as \( t \to \infty \).