ENG ME 500 Engineering Mathematics

Instructor:

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Prerequisites:

Multivariate Calculus; Ordinary Differential Equations; or instructor permission. It is expected that you can already:

- solve simple first and second order linear ordinary differential equations
- differentiate and integrate elementary functions, including trigonometric, exponential and hyperbolic functions
- integrate by parts; evaluate simple surface and volume integrals
- use the binomial theorem and the series expansions of elementary functions (sine, cosine, exponential, logarithmic and hyperbolic functions)

It is your responsibility to test yourself by

- taking the **prerequisites self test** at the end of these notes.

Textbook:

Lectures are based on *Mathematical Methods for Mechanical Sciences* (M. S. Howe; 6th edition). It can be downloaded in pdf form from the ME 500 *BlackBoard* web site. You are expected to 'read around' the subject, and are recommended to consult other textbooks such as those listed on page 5.

Course grading:

- 4 take-home examinations (12.5% each)
- final **closed book** examination (50%)

Homework:

Four ungraded homework assignments provide practice in applying techniques taught in class – model answers will be posted on *BlackBoard*. In addition there are four take home exams each consisting of a short **essay** and 5 problems.

Each take-home examination problem is graded out of 10 as follows: up to 7 points for obtaining the correct answer and showing all your working; up to 3 points for setting-out the answer in a neat and readable manner. The essay is graded out of 25.

You have at least **ONE WEEK** for each take-home examination; exam scripts are due at the end of class on the date specified.

The following example illustrates how solutions should be set-out, and how you should include written explanations of your steps:

Question: Use the divergence theorem to evaluate $\oint_{\mathrm{S}} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \frac{1}{3}(x^3, y^3, z^3)$ and S is the surface of the sphere $|\mathbf{x}| \leq R$.

Solution: To evaluate

$$I = \oint_{\mathrm{S}} \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F} = \frac{1}{3}(x^3, y^3, z^3)$ and S is the surface $|\mathbf{x}| = R$.

By the divergence theorem

$$I = \int_{\mathrm{V}} \mathrm{div}\, \mathbf{F}\, dV,$$

where the integration is over the volume V of the sphere, and

$$\mathrm{div}\, \mathbf{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = \frac{1}{3}(3x^2 + 3y^2 + 3z^2) = r^2, \ r = \sqrt{x^2 + y^2 + z^2}.$$

Using spherical polar coordinates, and taking $dV = 4\pi r^2 dr$ (because the integrand is radially symmetric):

$$I = \int_{r<R} r^2 dV = \int_0^R 4\pi r^4 dr = \left[\frac{4\pi}{5} r^5 \right]_0^R$$

$$\therefore \quad I = \frac{4\pi}{5} R^5$$

SAMPLE ESSAY

Essay: Write a short essay (not more than two sides of paper) in which you give the definition of a vector field, define the *curl* of a vector field $\mathbf{F}(\mathbf{x})$, and state and prove Stokes' theorem.

A single valued vector function defined over a region of space is called a vector field. For example, the velocity vector in a fluid, and vector gravitational and electrical force distributions are vector fields.

The 'curl' of a vector field $\mathbf{F}(\mathbf{x})$ is a new vector field $\mathbf{curl\,F}$ whose value at $\mathbf{x}$ is determined by the following limiting procedure. For any fixed unit vector $\hat{\mathbf{a}}$, the component of $\mathbf{curl\,F}$ in the direction of $\hat{\mathbf{a}}$ is

$$\hat{\mathbf{a}} \cdot \mathbf{curl\,F} = \lim_{\mathrm{A}\to 0} \frac{1}{\mathrm{A}} \oint_{\mathrm{C}} \mathbf{F} \cdot d\mathbf{r} \qquad (1)$$

where C is a closed contour enclosing $\mathbf{x}$ in the plane through $\mathbf{x}$ with normal $\hat{\mathbf{a}}$. C encloses an area A of the plane, and is traversed in the positive sense with respect to the direction $\hat{\mathbf{a}}$.

For rectangular coordinates $\mathbf{x} = (x, y, z)$, with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively parallel to the x, y, and z directions, this definition yields

$$\mathbf{curl\,F} \equiv \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},$$

where $\mathbf{F} = (F_1, F_2, F_3)$.

The definition (1) also leads to a direct proof of Stokes' theorem

$$\oint_{\mathrm{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathrm{S}} \mathbf{curl\,F} \cdot d\mathbf{S} \equiv \int_{\mathrm{S}} \mathbf{n} \cdot \mathbf{curl\,F}\, dS,$$

where C is a closed contour, S an open, two-sided surface bounded by C, and the line integral is taken along C in the positive direction with respect to the unit normal $\mathbf{n}$ on S.

We can write $\oint_{\mathrm{C}} \mathbf{F} \cdot d\mathbf{r} = \lim_{k\to\infty} \sum_k \oint_{\mathrm{C}_k} \mathbf{F} \cdot d\mathbf{r}$, where the surface S is partitioned into infinitesimal elements of area $\delta \mathrm{S}_k$ with unit normal $\mathbf{n}_k$ and boundary C_k, because the integral along each section of C_k common to two adjacent elements enters twice, once for each element, but with opposite signs, and therefore makes no contribution to the sum. What remains is just the integral along those sections of the C_k that make up the contour C. The definition (1), with $\hat{\mathbf{a}} = \mathbf{n}_k$, then gives

$$\oint_{\mathrm{C}} \mathbf{F} \cdot d\mathbf{r} = \lim_{k\to\infty} \sum_k \oint_{\mathrm{C}_k} \mathbf{F} \cdot d\mathbf{r} = \lim_{k\to\infty} \sum_k \mathbf{n}_k \cdot \mathbf{curl\,F}\, \delta \mathrm{S}_k = \int_{\mathrm{S}} \mathbf{curl\,F} \cdot d\mathbf{S}. \quad \text{Q.E.D.}$$

Schedule

Sep R2		Vector Calculus
Sep T7		
Sep R9		
Sep T14		
Sep R16		
Sep T21	TH #1	
Sep R23		Complex Variable
Sep T28	TH #1 due	
Sep R30		
Oct T5		
Oct R7		
Oct R14		
Oct T19	TH #2	
Oct R21		PDE′s and Fourier transforms
Oct T26	TH #2 due	
Oct R28		
Nov T2		
Nov R4		
Nov T9		
Nov R11	TH #3	
Nov T16		
Nov R18	TH #3 due	
Nov T23		
Nov T30	TH #4	Review
Dec R2		
Dec T7	TH #4 due	
Dec R9		

Syllabus

During the course of the semester you are expected to read and understand the material in those sections of **MMMS**, 6th edition, indicated after each heading in the following syllabus:

1. Vector Calculus

Sections 2.1 - 2.5; Section 2.7.

Assignment 1: Take-home examination 1; Essay 1.

2. Complex Variable

Sections 3.1 - 3.10.

Assignment 2: Take-home examination 2; Essay 2.

3/4. Partial differential equations; Fourier series and transforms

Sections 1.9, 1.10, 1.11;

Sections 4.1 - 4.3, 4.6, 4.9, 4.10.

Assignment 3: Take-home examination 3; Essay 3.

Assignment 4: Take-home examination 4; Essay 4.

Alternative Texts: (*not* recommended for purchase):

Lopez, R. J. *Advanced Engineering Mathematics*. Addison Wesley.

Kreyszig, E. *Advanced Engineering Mathematics*. Wiley.

Greenberg, M. E. *Advanced Engineering Mathematics*. Prentice Hall.

Zill, D. G. and Cullen, M. R. *Advanced Engineering Mathematics*. Jones and Bertlett.

ASSIGNMENT 1

Problems 2A

1. Find the value of x given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + x\mathbf{j}$ are perpendicular. [6].

3. Solve for $\mathbf{x}$ the vector equation $\mathbf{x} + \mathbf{a}(\mathbf{b}\cdot\mathbf{x}) = \mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors. What happens when $\mathbf{a}\cdot\mathbf{b} = -1$? $[\mathbf{x} = \mathbf{c} - \mathbf{a}(\mathbf{b}\cdot\mathbf{c})/(1+\mathbf{a}\cdot\mathbf{b})]$.

5. Solve the simultaneous equations $\mathbf{x} + \mathbf{y}\times\mathbf{p} = \mathbf{a}$, $\mathbf{y} + \mathbf{x}\times\mathbf{p} = \mathbf{b}$.
$[\mathbf{x} = \{(\mathbf{p}\cdot\mathbf{a})\mathbf{p} + \mathbf{a} - \mathbf{b}\times\mathbf{p}\}/(1+p^2),\quad \mathbf{y} = \{(\mathbf{p}\cdot\mathbf{b})\mathbf{p} + \mathbf{b} - \mathbf{a}\times\mathbf{p}\}/(1+p^2)\ ]$.

7. Show that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})\times\mathbf{d} = (\mathbf{a}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})$.

Problems 2B

Calculate the gradients of

1. $\varphi = x$ $\quad [\mathbf{i}]$.

3. $\varphi = r^n$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\quad [nr^{n-2}\mathbf{r}]$.

5. $\varphi = \mathbf{r}\cdot\nabla(x+y+z)$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\quad [\mathbf{i}+\mathbf{j}+\mathbf{k}]$.

Find the directional derivatives in the direction of $\mathbf{a}$ of

7. $\varphi = \mathrm{e}^x \cos y$, $\mathbf{a} = (2,3,0)$, $\mathbf{x} = (2,\pi,0)$ $\quad [-2\mathrm{e}^2/\sqrt{13}]$.

Find the unit normal to the surfaces:

9. $z = \sqrt{x^2+y^2}$ at $\mathbf{x} = (3,4,5)$ $\quad [(3,4,-5)/5\sqrt{2}]$.

Problems 2C

Find the divergence of

1. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ $\quad [0]$.

3. $\mathbf{F} = (x, y^2, z^3)$ $\quad [1 + 2y + 3z^2]$.

5. $\mathbf{F} = xyz(\mathbf{i}+\mathbf{j}+\mathbf{k})$ $\quad [yz + xz + xy]$.

7. $\mathbf{F} = \mathbf{r}(\mathbf{r}\cdot\mathbf{a})$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} =$ constant. $\quad [4\mathbf{r}\cdot\mathbf{a}]$.

Prove that

9. $\nabla^2(r^n) = n(n+1)r^{n-2}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

11. $\mathrm{div}(f\nabla g) - \mathrm{div}(g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

13. $V = \frac{1}{6}\oint_S \mathbf{n}\cdot\nabla(r^2)dS$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and V is the volume enclosed by S.

Evaluate $\oint_S \mathbf{n}\cdot\mathbf{F}dS$ when

15. $\mathbf{F} = (x, x^2y, -x^2z)$ and S is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ $(0,0,1)$ $\quad [\frac{1}{6}]$.

17. $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$ where a, b, c are constants, and S is the unit sphere $|\mathbf{x}| = 1$ $\quad [\frac{4}{3}\pi(a+b+c)]$.

Problems 2E

Evaluate the line integrals $\int_C \mathbf{F}\cdot d\mathbf{r}$:

1. $\mathbf{F} = (3x^4, 3y^6, 0)$ where C is the curve: $x^2 + y^2 = 4,\ z = 0$ from $(2, 0, 0)$ to $(-2, 0, 0)$ $[\frac{-192}{5}]$.

3. $\mathbf{F} = (e^x, e^{4y/x}, e^{2z/y})$ where C is $\mathbf{r} = (t, t^2, t^3),\ 0 < t < 1$ $[\frac{3}{8}e^4 + \frac{3}{4}e^2 + e - \frac{13}{8}]$.

Show that the following integral is path-independent and find the corresponding potential function:

5. $\int_C \left[y \cos xy\, dx + x \cos xy\, dy - dz\right]$, $[\varphi = \sin xy - z]$.

7. Show that if $\mathbf{r} = \mathbf{r}(t)$ on C, the *length of arc* between $t = a$ and $t = b$ is given by $\ell = \int_a^b |\dot{\mathbf{r}}(t)| dt$.

9. $\mathbf{r} = (a \cos\theta,\ a \sin\theta,\ a\theta \tan\alpha)$ on a helix, where $a,\ \alpha$ are constants. Show that the length of arc measured from $\theta = 0$ is given by $\ell = a\theta \sec\alpha$.

Problems 2F

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$:

1. $\mathbf{F} = (2x, 2y, 0)$ where S is the surface: $z = 2x + 3y,\ 0 < x < 2,\ -1 < y < 1,$ $[-16]$.

3. $\mathbf{F} = (1, x^2, xyz)$ where S is $z = xy,\ 0 < x < y,\ 0 < y < 1,$ $[-\frac{59}{180}]$.

5. $\mathbf{F} = (2xy, x^2, 0)$ where S is $\mathbf{r} = (\cosh u, \sinh u, v),\ 0 < u < 2,\ -3 < v < 3,$ $[2\cosh^3 2 - 2]$.

7. $\mathbf{F} = (y^2, z^2, x^2 z)$ where S is the surface bounding the region $x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0,\ |z| \le 1,$ $[2\pi]$.

Use Stokes's theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$:

9. $\mathbf{F} = x^2 yz\mathbf{j}$ where C is the quadrilateral with vertices $(0, 1, 0),\ (1, 1, 0),\ (1, 0, 1),\ (0, 0, 1)$ traversed in this order. $[-\frac{1}{6}]$.

11. $\mathbf{F} = xyz\mathbf{j}$ where C is the triangle with vertices $(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),$ $[0]$.

13. $\mathbf{F} = (-3y, 3x, z)$ where C is $x^2 + y^2 = 4,\ z = 1,$ $[24\pi]$.

15. $\mathbf{F} = 2z\mathbf{i} + 4x\mathbf{j} + 5y\mathbf{k}$ where C is $x^2 + y^2 = 4,\ z - x = 4$, orientated anticlockwise when viewed from above. $[-4\pi]$.

ASSIGNMENT 2

Problems 3A

1. Express in the form $a+ib$:
   (i) $(2+i)^2+(2-i)^2$; (ii) $\frac{\alpha+\beta i}{\alpha-\beta i}-\frac{\alpha-\beta i}{\alpha+\beta i}$. [6, $4i\alpha\beta/(\alpha^2+\beta^2)$].

3. Find the modulus and principal value of the argument of: $z=-1,\ i,\ 3+4i,\ -i-\sqrt{3}$.
   [$1,\ \pi;\ 1,\ \frac{\pi}{2};\ 5,\ 0.927$ radians; $2,\ -\frac{5\pi}{6}$].

5. Express $\sqrt{5+12i},\ \sqrt{-5+12i},\ \sqrt{i}$ in the form $a+ib$. [$\pm(3+2i),\ \pm(2+3i),\ \pm\frac{1}{\sqrt{2}}(1+i)$].

7. Find two real numbers $a,\ b$ such that $(1+i)a+2(1-2i)b-3=0$. [$a=2,\ b=\frac{1}{2}$].

9. If $z_1,\ z_2,\ z_3$ are complex numbers, show that
   (i) $|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2$
   (ii) $|2z_1-z_2-z_3|^2+|2z_2-z_3-z_1|^2+|2z_3-z_1-z_2|^2=3\{|z_2-z_3|^2+|z_3-z_1|^2+|z_1-z_2|^2\}$

11. Show that

$$\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}=\cot\left(\frac{\theta}{2}\right)\mathrm{e}^{i(\theta-\frac{\pi}{2})}.$$

Evaluate the roots:

13. $\sqrt[8]{1}$ [$\pm1,\ \pm i,\ \pm(1\pm i)/\sqrt{2}$]

15. $z^2-(5+i)z+8+i=0$ [$z=3+2i,\ 2-i$]

Problems 3D Evaluate:

1. $\oint_{|z|=2}\frac{dz}{z+i}$ [$2\pi i$].

3. $\oint_{\mathrm{C}}\frac{dz}{z^2-\frac{1}{4}}$ where C is the square with corners $\pm(1\pm i)$ [0].

5. $\oint_{|z-\frac{\pi}{2}|=1}\frac{\sin z\,dz}{\left(z^2-\frac{\pi^2}{4}\right)}$ [$2i$]

7. $\oint_{|z|=\frac{3}{2}|a|}\frac{dz}{z-a}$. [$2\pi i$].

9. $\oint_{|z|=1}\frac{(2z-3)dz}{z^2-3z}$. [$2\pi i$].

Problems 3E

Evaluate the residues of:

1. $z\cosh(3/z)$. [$\frac{9}{2}$ at $z=0$].

3. $z^4/(z^2+1)$. [$-\frac{i}{2}$ at $z=i$; $\frac{i}{2}$ at $z=-i$].

5. $\frac{\sin z}{z^4}$. [$-\frac{1}{6}$ at $z=0$].

7. $z^2\mathrm{e}^{\frac{1}{z}}$. [$\frac{1}{6}$ at $z=0$].

9. $(\cosh 2z)/z^5$. [$\frac{2}{3}$ at $z=0$].

Evaluate by the residue theorem (contours are traversed in the anticlockwise sense):

11. $\oint_{|z|=1} \frac{z^6+7}{z^2-2z} dz \quad [-7\pi i].$

13. $\oint_{|z|=1} \frac{\sinh z}{4z^2+1} dz \quad [\pi i \sin\left(\frac{1}{2}\right)].$

15. $\oint_{|z-\frac{i}{2}|=1} \frac{z^4}{z^2+1} dz \quad [\pi].$

17. $\oint_{|z-\frac{i}{2}|=1} \frac{1}{z^3(z-1)^2} dz \quad [6\pi i].$

19. $\oint_{|z|=\frac{3}{2}} \mathrm{e}^{\frac{1}{z}}/(z-1)^2 \, dz \quad [0].$

Problems 3F

Evaluate by the residue theorem:

1. $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}, \quad |a| < 1 \quad \left[\frac{2\pi}{\sqrt{1-a^2}}\right].$

3. $\int_0^{2\pi} \frac{\cos 2\theta \; d\theta}{1-2p\cos\theta+p^2}, \quad |p| < 1 \quad [\frac{2\pi p^2}{1-p^2}].$

5. $\int_0^{\pi} \frac{\sin^4\theta \; d\theta}{a+\cos\theta}, \quad a > 1, \quad \left[\pi\left(\frac{3}{2}a - a^3 + (a^2-1)^{\frac{3}{2}}\right)\right].$

7. $\int_0^{\infty} \frac{dx}{(1+x^2)(4+x^2)} \quad \left[\frac{\pi}{12}\right].$

11. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} \quad \left[\frac{\pi}{4}\right].$

15. $\int_0^{\infty} \frac{x\sin 2x \; dx}{1+x^2} \quad \left[\frac{\pi}{2\mathrm{e}^2}\right].$

17. $⨍_{-\infty}^{\infty} \frac{\cos kx \; dx}{x-a}, \; (a, \; k \text{ real}, \; k > 0) \quad [-\pi\sin ka].$

19. $⨍_{-\infty}^{\infty} \frac{dx}{x(x-ai)}, \; (a > 0) \quad \left[\frac{\pi}{a}\right].$

ASSIGNMENT 3

Problems 1H

Find the eigenvalues and eigenfunctions of:

1. $y'' + \lambda y = 0,\ \ y(0) = 0,\ y'(1) = 0 \quad [\lambda_n = \left(\frac{(2n+1)\pi}{2}\right)^2,\ n = 0,\ 1,\ 2, \ldots,\ y_n = \sin(\sqrt{\lambda_n}x)]$.
2. $y'' + \lambda y = 0,\ \ y'(0) = 0,\ y'(\pi) = 0 \quad [\lambda_n = n^2,\ n = 0,\ 1,\ 2, \ldots,\ y_n = \cos(nx)]$.
3. Transform the equation $y''+2y'+(1-\lambda)y = 0,\ (y(0) = 0,\ y(1) = 0)$ into Sturm-Liouville form by multiplying by e^{2x}. Calculate the eigenvalues and eigenfunctions and show that $\int_0^1 e^{2x} y_n(x) y_m(x) dx = 0,\ m \neq n$. $[\lambda_n = -n^2\pi^2,\ n = 1,\ 2,\ 3, \ldots,\ y_n = e^{-x}\sin(n\pi x)]$.

Problems 4A

1. If

$$f(x) = \begin{cases} x, & 0 < x < \ell/2, \\ \ell - x, & \ell/2 < x < \ell, \end{cases}$$

in the boundary value problem for the diffusion equation (4.3.8), show that

$$u = \frac{4\ell}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{(2n+1)\pi x}{\ell}\right) \exp\left(\frac{-(2n+1)^2\pi^2\kappa t}{\ell^2}\right), \quad t > 0.$$

3. When $f(x) = x$ in problem 2, deduce that

$$u = \frac{\ell}{2} - \frac{4\ell}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{(2n+1)\pi x}{\ell}\right) \exp\left(\frac{-(2n+1)^2\pi^2\kappa t}{\ell^2}\right)}{(2n+1)^2}, \quad t > 0.$$

5. Show that the solution of the *Dirichlet* problem for u:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the rectangular domain $0 < x < a,\ 0 < y < b$, where $u = 0$ on each side of the rectangle *except* that along the x-axis, where $u = f(x)$ $(0 < x < a,\ y = 0)$, can be expressed in the form

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right), \quad \text{where} \quad A_n = \frac{2}{a}\int_0^a \frac{f(x)\sin(n\pi x/a)\,dx}{\sinh(n\pi b/a)}.$$

7. Obtain all solutions of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = u$$

of the form $u(x,y) = (A\cos\lambda x + B\sin\lambda x)f(y)$, where $A,\ B,\ \lambda$ are constants. Show that the particular solution that satisfies $u(0,y) = 0,\ u(\pi,y) = 0,\ u(x,1) = x$ $(0 < x < \pi)$ is given by

$$u = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{(1+n^2)(1-y)} \sin nx.$$

9. Show that

$$u(x,t) = \frac{1}{2}a - \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\{(2n+1)\pi x/a\}}{(2n+1)^2}\,\mathrm{e}^{-t[(2n+1)\pi/a]^2}$$

satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x}(\pm a, t) = 0, \qquad u(x,0) = |x|, \quad |x| < a.$$

11. Show that when the conditions of Problem 10 are replaced by (i) $u(0,t) = 1$ and $u(1,t) = 0$ for $t > 0$, (ii) $u(x,0) = \cos(\frac{\pi}{2}x)$ for $0 \le x \le 1$,

$$u(x,t) = 1 - x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n(4n^2-1)}\,\mathrm{e}^{-n^2\pi^2 t}.$$

13. Show that the solution $u(x,y)$ of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi,\ 0 < y < \pi,$$

where $\partial u/\partial x = 0$ at $x = 0$ and $x = \pi$, $u(x,\pi) = 0$, and $u(x,0) = f(x)$, is given by

$$u = A_0\frac{(\pi - y)}{2\pi} + \sum_{n=1}^{\infty}A_n\frac{\cos nx \sinh n(\pi - y)}{\sinh n\pi}, \quad A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

15. If $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < a,\ t > 0, \text{ subject to } u(0,t) = 0,\ \frac{\partial u}{\partial x}(a,t) = 0,\ u(x,0) = \sin^3\left(\frac{\pi x}{2a}\right),$$

show that

$$u(x,t) = \frac{3}{4}\sin\left(\frac{\pi x}{2a}\right)\mathrm{e}^{-\pi^2 t/4a^2} - \frac{1}{4}\sin\left(\frac{3\pi x}{2a}\right)\mathrm{e}^{-9\pi^2 t/4a^2}.$$

Problems 1J

Show that:

1. $\delta(x-y) = \delta(y-x)$.

3. If $xf(x) = 0$ for all values of x, then $f(x) = A\delta(x)$, where A is an arbitrary constant. $\left[\int_{-\infty}^{\infty} g(x)f(x)dx = \int_{-\infty}^{\infty} f(x)\left\{x\left(\frac{g(x)-g(0)}{x}\right) + g(0)\right\}dx = 0 + g(0)\int_{-\infty}^{\infty} f(x)dx = \text{constant} \times g(0),\ \therefore\ f(x) = \text{constant} \times \delta(x).\right]$

5. $F(x)\delta(x-a) = F(a)\delta(x-a)$.

7. $\frac{d}{dx}\mathrm{H}\big(f(x)\big) = \frac{df(x)}{dx}\delta\big(f(x)\big)$.

9. $\frac{d^2|x|}{dx^2} = 2\delta(x)$.

11. $\int_{-\infty}^{\infty}\delta^{(n)}(x-a)f(x)dx = (-1)^n f^{(n)}(a)$.

13. $\lim_{\epsilon\to 0}\frac{-2\epsilon x}{\pi(x^2+\epsilon^2)^2} = \delta'(x)$.

ASSIGNMENT 4

Problems 4F

Verify the following Fourier transform pairs:

1. $f(x) = e^{-|x|}, \quad \hat{f}(k) = \dfrac{\sqrt{2}}{\sqrt{\pi}(1+k^2)}.$

3. $f(x) = e^{-ax^2}, \quad \hat{f}(k) = \dfrac{1}{\sqrt{2a}} e^{-k^2/4a}, \ \ a > 0.$

5. $f(x) = \delta(x), \quad \hat{f}(k) = \dfrac{1}{\sqrt{2\pi}}.$

7. $f(x) = \begin{cases} x, & 0 < x < a, \\ 0, & x > a, \end{cases} \quad \hat{f}_c(k) = \sqrt{\dfrac{2}{\pi}} \dfrac{[ka \sin ka + \cos ka - 1]}{k^2}.$

9. $f(x) = e^{-x}, \quad \hat{f}_s(k) = \sqrt{\dfrac{2}{\pi}} \dfrac{k}{1+k^2}.$

11. $f(x) = x^n e^{-ax}$, $n =$ positive integer, $a > 0$, $\hat{f}_s(k) = \sqrt{\dfrac{2}{\pi}} \dfrac{n!}{(k^2+a^2)^{n+1}} \mathrm{Im}(a+ik)^{n+1}.$

13. $f(x) = e^{-ax} \ (a > 0), \quad \hat{f}_c(k) = \sqrt{\dfrac{2}{\pi}} \dfrac{a}{a^2+k^2}.$

15. $f(x) = \dfrac{e^{-ax}}{x} \ (a > 0), \quad \hat{f}_s(k) = \sqrt{\dfrac{2}{\pi}} \tan^{-1}\left(\dfrac{k}{a}\right).$

17. $f(x) = \cos(x^2/2), \quad \hat{f}_c(k) = \dfrac{1}{\sqrt{2}}\big(\cos(k^2/2) + \sin(k^2/2)\big).$

19. $f(x) = \dfrac{1}{x}, \quad \hat{f}_s(k) = \sqrt{\dfrac{\pi}{2}}.$

Problems 4G

1. The steady temperature distribution $u(x, y)$ in $x > 0, \ y > 0$ is governed by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{and} \quad \begin{cases} u = 0, \quad x = 0, \ 0 < y < \infty, \\ \partial u / \partial y = -\sigma \delta(x - a), \ (a > 0), \quad 0 < x < \infty, \ y = 0. \end{cases}$$

Use the sine transform to show that

$$u = \frac{\sigma}{2\pi} \ln\left(\frac{y^2 + (a+x)^2}{y^2 + (a-x)^2}\right), \quad x > 0, \ y > 0.$$

3. Show that the bounded solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{and} \quad \begin{cases} u = 0, \quad x = 0, \ 0 < y < \infty, \\ u = u_0 x/(1 + x^2), \quad 0 < x < \infty, \ y = 0. \end{cases}$$

is

$$u(x, y) = \frac{u_0 x}{(1+y)^2 + x^2} \equiv \mathrm{Re}\left(\frac{u_0}{z + i}\right), \quad z = x + iy.$$

6. Find the function $u(x,y)$ that is bounded in $y>0$ and satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad -\infty<x<\infty,\quad \text{and}\quad u=\delta(x-\xi)\quad \text{on}\quad y=0.\quad \left[\ u=\frac{y}{\pi[(x-\xi)^2+y^2]}\ \right]$$

7. If $u(x,y)$ is bounded in $y>0$ and satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad -\infty<x<\infty,\quad \text{and}\quad u=f(x)\quad \text{on}\quad y=0,$$

show that

$$u=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{[(x-\xi)^2+y^2]}.$$

8. If $\nabla u(x,y)$ is bounded in $y>0$ and satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad -\infty<x<\infty,\quad \text{and}\quad \frac{\partial u}{\partial y}=\delta(x-\xi)\quad \text{on}\quad y=0,$$

show that

$$\nabla u=\frac{(x-\xi,\ y)}{\pi[(x-\xi)^2+y^2]},\quad u=\frac{1}{2\pi}\ln[(x-\xi)^2+y^2]+\text{constant},\quad y>0.$$

9. If $\nabla u(x,y)$ is bounded in $y>0$ and satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad -\infty<x<\infty,\quad \text{and}\quad \frac{\partial u}{\partial y}=f(x)\quad \text{on}\quad y=0,$$

show that

$$u=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\xi)\ln[(x-\xi)^2+y^2]\,d\xi+\text{constant},\quad y>0.$$

10. Use the cosine transform to show that, if $u(x,y)$ is bounded in $y>0$ and satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad -\infty<x<\infty,\quad \text{and}\quad u=\mathrm{H}(a-|x|),\ a>0,\quad \text{on}\quad y=0,$$

then

$$u=\frac{1}{\pi}\left\{\tan^{-1}\left(\frac{a+x}{y}\right)+\tan^{-1}\left(\frac{a-x}{y}\right)\right\},\quad y>0.$$

11. Show that the solution $u(x,y)$ of

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} &= 0,\quad x>0,\quad 0<y<a,\\ u(x,0) &= f(x),\ 0<x<\infty,\\ u(x,a) &= 0,\ 0<x<\infty,\\ u(0,y) &= 0,\ 0<y<a,\end{aligned}$$

is given by

$$u=\frac{2}{\pi}\int_0^{\infty}f(\xi)d\xi\int_0^{\infty}\frac{\sinh[k(a-y)]}{\sinh ka}\sin kx\sin k\xi\,dk.$$

12. If $u(x,t)$ is bounded and satisfies

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, \quad x>0, \quad t>0,\\ u(x,0) &= x\mathrm{e}^{-x^2/4},\\ u(0,t) &= 0,\\ \text{show that} \qquad u &= \frac{x}{(1+t)^{\frac{3}{2}}}\mathrm{e}^{-x^2/4(1+t)}.\end{aligned}$$

REVIEW TOPICS

VECTOR CALCULUS

Vector algebra, applications to geometry, simple linear equations (MMMS: §2.1)

Scalar and vector fields, div, grad and curl (§§2.2 - 2.4)

Divergence theorem, Stokes theorem (§2.4)

Green's identities (§2.5)

How to evaluate line and surface integrals (§2.7)

COMPLEX VARIABLES

Algebra of complex numbers (§3.1)

Functions of a complex variable, definition of derivative (§3.2)

Cauchy-Riemann equations (§3.2)

Integration in the complex plane (§3.3)

Cauchy's theorem (§3.4)

Taylor and Laurent expansions (§§3.5 - 3.7)

Poles, residue theorem (§§3.8, 3.9)

Applications to evaluate simple trig and infinite integrals (§3.10)

Principal value integrals, Fourier integrals (§3.10)

The ML-theorem (§3.3)

PARTIAL DIFFERENTIAL EQUATIONS

Well posed problems (§4.2)

D'Alembert's solution of the wave equation (§4.1)

Method of separation of variables (§4.3)

Sturm-Liouville equation, eigenvalues and eigenfunctions (§1.9)

Eigenfunction expansions (generalized Fourier series) (§§1.9, 4.3)

Dirac delta function, Heaviside and sgn functions, and their derivatives, ϵ-sequences (§1.11)

Use of $e^{-\epsilon|x|}$ and $e^{-\epsilon x}$ (§§1.11, 4.9, 4.10)

Fourier, Sine and Cosine transforms, inversion formulae (§4.9)

Applications to partial differential equations (§4.10)

Time: 20 minutes

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \, \sin 2x \, dx$. [Ans. $\frac{1}{3}$]

2. Evaluate $\int_{\mathrm{V}} \{(x^2+y^2+z^2)^{\frac{1}{2}} + 3x^2\}\, dxdydz$ where V is the sphere $x^2+y^2+z^2=R^2$.
   [$\pi R^4 + \frac{4\pi}{5} R^5$]

3. Evaluate $\int_0^1 dy \int_{y-1}^{1-y} y dx$. [$\frac{1}{3}$]

4. Find the general solution of $d^2y/dx^2 + k^2 y = 2x$, where $k > 0$ is constant.
   [$y = A\cos kx + B\sin kx + 2x/k^2$]

5. Solve $dy/dx + x^3 y = 0$, given that $y = 1$ when $x = 1$. $\left[y = \mathrm{e}^{\frac{1}{4}(1-x^4)}\right]$

6. Evaluate $\dfrac{d}{dx}\left(\dfrac{\tan x}{1+x}\right)$. $\left[\dfrac{\sec^2 x}{1+x} - \dfrac{\tan x}{(1+x)^2}\right]$

7. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin^4 x}{(\mathrm{e}^x - 1 - x)^2}$. [4]