Abstract

Over the past decade, risk measurement has received a much needed amount of attention from the financial community. Risk measures based on fixed quantiles under the actual probability distribution, especially Value-at-Risk and its refinement the Conditional Tail Expectation, were instrumental in capturing the attention of financial decision-makers. However, these were developed in a way that is inconsistent with economic theory. Consequently, these instruments possess characteristics that make them invalid risk measures for the purposes they intend to serve, be it informing life-cycle investors or guaranteeing the firm's capital adequacy through regulation. In particular, in addition to failing to guarantee the intregity of financial firms when used for capital adequacy, these measures can eventually decrease with the investment horizon.

Risk-neutral fixed-quantile measures are valid for framing life-cycle decisions because of their economic content. When endowed with a dynamic replication technology, Q-measure fixed-quantile risk measures become least-cost insurance contracts that may be used for capital adequacy considerations.

However, no single quantile of the risk-neutral distribution can be used for the procurement of risk capital at *all horizons.* A risk-neutral *varying*-quantile instrument is needed. This unique instrument is a put option proposed by Merton-Perold (1993) and Bodie (1995). The Bodie-Merton-Perold Put is universally valid for both risk disclosure to investors and for the regulatory provision of risk capital at all horizons. It is a natural candidate for an industry standard in risk measurement.

On the Validity of Risk Measures over Time: Value-at-Risk, Conditional Tail Expectations and the Bodie-Merton-Perold Put

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1 Introduction

Since JP Morgan's Riskmetrics™ first publicized the methodology underlying Value-at-Risk amidst the Risk Management Revolution of the mid-1990's, Value-at-Risk and its refinement, the Conditional Tail Expectation, have become risk measures of choice within the financial sphere. Indeed, these tools were instrumental in capturing the attention of critical decision-makers. Quickly, senior management in financial firms adopted VaR as a transparent summary of risk exposures. This adoption of Value-at-Risk was further accelerated by the needs

*Many thanks are owed to Zvi Bodie for invaluable conversations. I am also grateful to Michael K. Johnston for comments. However, responsibility for all remaining errors is the sole property of the author.

associated with the growing volume of Over-the-Counter derivatives business. Nowadays, in-house risk managers routinely use Value-at-Risk measures to control the individual risk exposures taken by their firm's traders. In the context of the Basle Accords, the Bank for International Settlements promoted the use of multiples of 10-day 1%-VaR estimates for capital adequacy requirements in banking institutions. Lately, in an attempt to counter their inherent opacity, financial firms have begun to use Value-at-Risk or Conditional Tail Expectations to inform investors of their risk exposures. Recently, this trend toward a broader use of the VaR methodology in personal risk management was marked by significant events. Motivated by the fact that "individuals around the world are taking on more responsibility for their financial futures (Kim and Mina, Foreword, 2001)," the Riskmetrics™ Group introduced RiskGrades™ and X-Loss™ (Loss in Extreme Markets). RiskGrades™ are Value-at-Risk measures scaled for investor-friendliness and X-Loss™ are their Conditional Tail Expectation counterparts. Also, State Street Associates developed the investor-designed Risk Budget Tool™ to convert optimal portfolio allocations into Value-at-Risk statistics[1].

Academics as well as risk practitioners have often pointed to particular weaknesses of the Value-at-Risk methodology. Early, Beder (1995) noted that Value-at-Risk was "Seductive but Dangerous" because of the many pitfalls in the implementation of the methodology. Matz (2004) provides additional reflections on the "Use and Misuse of Value-at-Risk Analysis" by practitioners in the banking industry[2]. The development of Conditional Tail Expectations by

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[1]For more details on RiskMetrics'™ investor-oriented risk measures, see the RiskGrades™ Technical Document by Kim and Mina (2001). For State Street Associates' private-investments risk measures, see http://www.globallink.com/gl/ssa/overview.html

[2]A critical issue in practice is the selection the time series used to estimate VaR or CTE. Matz writes, "historical VaR is generally preferred for its ability to capture unlike events (but only to the extent that those unlikely events fall within the selected historical time period." Two critical problems are captured by this statement. The first relates to agency issues. Indeed, a favorable misrepresentation of a firm's risk exposures can be easily secured by a sufficient amount of data-mining. However, putting aside such inten-

Artzner, Delbaen, Eber and Heath (1999) itself resulted from concerns about the theoretical soundness of VaR[3].

This paper demonstrates that Value-at-Risk and Conditional Tail Expectations – both fixed-quantile instruments under the *actual probability measure* P – possess characteristics that invalidate them as risk measures for the purposes they intend to serve. In particular, in addition to being invalid for the provision of risk capital by financial firms, these statistics may eventually decrease with the investment horizon for commonplace values of expected equity returns, equity standard deviation and the risk-free rate. In fact, Value-at-Risk and Conditional Tail Expectations may even switch signs, seemingly indicating the absence of risk in the long-run. This contradicts the most profound results in the analysis of risk over time, particularly Bodie (1995) who proves that the risk associated with equity positions is increasing with the investment horizon[4]. However, when based on fixed *risk-neutral* quantiles, Value-at-Risk and Conditional Tail Expectations eventually increase with the time horizon

tional deception, virtually all time series that are used for the purpose of estimating empirical distributions suffer from survivorship bias. Simply put, U.S. markets have been fortunate to avoid (or bounce back from) financial catastrophes that other markets have failed to survive. In addition, strategies used are almost systematically strategies that have not suffered dramatic past accidents. Together, this produces time series that under-represent negative outcomes. This bias is most concentrated in the region of the probability distribution that is VaR's natural habitat, its left-tail. Survivorship bias was treated by Brown, Goetzman and Ross (1995). Jorion and Goetzmann (1999) provide vivid evidence of the importance of the bias in U.S. time series.

[3]CTE offered a resolution to the *incoherence* of Value-at-Risk. It was noted that Value-at-Risk may fail to satisfy sub-additivity, which amounts to it's being inconsistent with the risk-diversifying effects of the combination two portfolios that were previously separate. Acerbi and Tasche (2002) offered Expected Shortfall as an alternative coherent alternative to Value-at-Risk. Their contribution is computational in that the algorithm associated with their statistic guarantees coherence by handling the discreteness of distributions estimated from finite time series. Also, Rogachev (2002) proposes a Dynamic Value-at-Risk. His measure is amounts to a computationally-intensive refinement of the Monte Carlo approach to VaR calculations. Wirch and Hardy (2003) present a dynamic, iterated CTE, which they apply to the case of seggregated funds. Glasserman (2001) investigated "running risk," a probabilitic assessment of possible losses over a cycle of hedging using futures contracts.The above refinements in the implementation of Value-at-Risk – although well-needed within the VaR methodology itself – cannot protect themselves from the conceptual issues inherent to all VaR-like statistics that are discussed in this paper.

[4]Feinstein (1999) scrutinized Bodie's (1995) results under the particular conditions of mean-reversion in stock prices and found that Bodie's results remained valid for such particular price dynamics.

and do not feature the non-economic sign switches of their P-measure counterparts. Thus, by going to the risk-neutral measure, quantifiers of risk regain their validity as informative statistics when disclosed to life-cycle investors.

Indeed, the use of actual probabilities *alone* for the purpose of quantifying risk is meaningless within the economic theory of planning under uncertainty based on the work of Arrow (1953) and Debreu (1959). To provide a relevant risk statistic, one needs to consider both the statistical probability of risky events as well as their impact on the solution to the utility-maximization program of the investor[5]. Asking investors to base their decisions on a risk summary that informs them about only one of the two determinants of risk leads to non-optimal solutions. Graduate students might as well try to decide which of a lump-sum payment of $10,000,000 today or a perpetual annuity of $500,000 they prefer without knowing the prevailing rate of interest[6]. This definitional two-dimensionality of risk is discussed in Bodie and Merton (2000, p.256).

In contrast to the exclusive use of actual probabilities, option pricing theory – as it was developed since Samuelson and Merton (1969) – uses "risk-neutral probabilities[7]." As a con-

[5] For example, virtually all children face the likely prospect of scrapping their knees while playing. The very high probability of such an event would seem to indicate high risk. However, the child's knowledge that his parents will take care of him (and buy him candy as a utilitarian compensation) in the event that he skins his knee allows the child to play as if the "risky event" was of no consequence to him. A negative outcome is not a risk if the utilitarian cost associated with its realization is null. This is the case of the implicit insurance policy provided by parents to their children.

[6] A version of this puzzle was posed to the first graduating class of the Haas School's Master's in Financial Engineering Program by R.C. Merton in March 2002.

[7] Samuelson and Merton (1969) were the first to take the crucial methodological step towards risk-neutral pricing. Their Util-Prob Density Function (or Q-Density) is the combination of the actual probability density P and the marginal utility of wealth. The use of marginal utility of wealth as a weighing factor translated into what option theorists know today as the pricing kernel. The pricing kernel is strictly decreasing in the realization of the Brownian Motion, which corresponds to a lower marginal utility of wealth. Because the expected marginal use-value of wealth (the value of the Util-Prob density) must be proportional to the market price of a unit of consumption in any particular state, this value translates into an Arrow-Debreu price when discounted back to the present using the risk-free rate. The modern terminology originates in Cox and Ross (1976).

sequence, option pricing theory builds on the two-dimensionality of risk and is thereby deeply rooted in Arrow-Debreu theory. Let a state of the world be defined by a future date and a particular realization of the stock price at that date. A specific state's risk-neutral probability corresponds to this state's contribution to the value of a dollar's worth of consumption at this particular future date. Its present value is the current cost of securing one unit of consumption in this particular future state of the world and is known as an Arrow-Debreu price[8]. Probabilities under the actual probability measure have no such economic interpretation. As a consequence, the adoption of risk-neutral probabilities from option pricing theory for the purpose of defining fixed-quantile risk measures provides those measures with the solid bearings of economic theory. This makes them valid statistics for life-cycle planning by private investors[9].

This superiority of option pricing theory in the quantification of risk over time was used by Bodie (1995) in his demonstration that the risk of stocks increases with the investment

[8] The statement above is unambiguously rigorous in the discrete-state binomial model. Breeden and Litzenberger (1978) derive the price of Arrow-Debreu "elementary claims" from option prices. They provide an example for a discrete state-space (p.626). When the state-space is infinite, they show that the price of an elementary claim is proportional to the second partial derivative of the option valuation function with respect to the exercise price, evaluated by setting the exercise price equal to the value of the portfolio. They write, "With a continuous distribution for M, the probability of any given level of M is formally zero; however, M has a probability density function. The pricing function, $P(M,T)$, is analogous to a density function in that case" (p.627). They show that this price function is equal to the risk-neutral density function discounted at the risk-free rate for the Geometric Brownian Motion case (Footnote 12, p.630 and Equation 7, p.631).

[9] Students of risk quantification (e.g., Wang, Young and Panjer, 1997) have used the concept of probability distortions in their analysis of various risk measures. For any risk measure – e.g. VaR – the appropriate distortion function is defined as the one that allows for the representation of this risk measure as an expectation under the distorted measure. It was shown that the concavity of the distortion function is a sufficient condition for the coherence of this risk measure in the sense of Artzner et al. (1998). For instance, the distortion function associated with Value-at-Risk is a step function and therefore is not concave. This was presented as an alternative proof to the point originally made by Artzner et al. (1998) about the incoherence of VaR. Although technically related, generic probability distortions and risk-neutral probabilities differ significantly. For instance, when distortion functions are used to build new risk measures, "the determination of suitable parameters for these is essentially a political decision" (Wirch and Hardy, 1999, p. 347) based on the firm's attitude towards risk and a myriad of other internal considerations. Risk-neutral probabilities, because of their unique standing as bearers of equilibrium economic meaning, are not subject to such introspective fiddling. For a review of the literature on probability distortions, see Wirch and Hardy (1999).

horizon. Bodie's conclusions resonate in Baz and Chacko's (2004) "stock free-lunch paradox:"

> "What is the probability that the stock investment will outperform the bond investment by time T? ... The probability of outperformance goes to 100% as the holding period goes to infinity. However, if the investor wanted to buy insurance against the risk of the stock investment underperforming the bond investment, ... the investor would realize in dismay that the price of such insurance increases with the time horizon! It appears, paradoxically, that the insurance market is assigning a higher and higher price to an event with probability shrinking to zero (pp.18-19)."

Because they are fixed quantiles under the P-measure and therefore only capture one dimension of risk, Value-at-Risk and Conditional Tail Expectations fall victims of the fallacy that is exposed by Bodie and is implicit in Baz and Chacko's "paradox." This results from the fact that any α-percentile stock price under the actual probability measure may overtake the value of the investment compounded at the risk-free rate for long enough horizons, thus making investments that have a positive probability of severe losses look like a guaranteed winners to the ill-informed investor.

Option pricing theory relies on the sophisticated production capabilities of dynamic replication as its other theoretical pillar. Therefore, it might seem reasonable that actual probabilities and payoff-replication technology could be combined to define "well-behaved" risk measures. However, even when endowed with the replication technology necessary to produce "left-tail insurance," the ill-features described above remain. As an example, dynamic risk measures with upper bounds based on P-measure quantiles[10] can be decreasing. Inevitably, these contracts possess the unappealing feature of allowing for future cash payments by the contract holder when equity outperforms the risk-free asset within bounds. As the size of these potential payments increases with the horizon, the value of the contract is driven to zero and the risk measure indicates the near-absence of risk at long horizons. These possible future payments cannot occur for dynamic (i.e., using payoff-replicating) risk measures relying on fixed *Q-measure* quantiles. As least-cost downside insurance contracts, these Q-measure Rate Guarantee contracts seem to be candidates for the determination of financial firms' capital adequacy.

However, a re-examination of the definition of risk capital reveals that no one particular quantile under the risk-neutral measure can be used to guarantee the integrity of the firm for all horizons. This is caused by the log-normality of stock prices and the corresponding divergence between the risk free rate r and the drift of risk-neutral stock price dynamics $r - \frac{1}{2}\sigma^2$. Fortunately, a simple Q-measure *varying-quantile* risk measure – the already-existing Bodie-Merton-Perold Put – automatically provides the adjustment required to generate an instrument that is valid for risk capital provision at all horizons, in addition to being a valid statistic for life-cycle investors. Consequently, the Bodie-Merton-Perold Put is a universally-valid[11] risk measure and, as such, should be considered as a replacement to instruments currently used by the financial community.

The rest of the paper is organized as follows: Section 2 demonstrates that the P-measure

[10] For lack of better names or acronyms, these are labeled P-measure Rate Guarantees (PRG) since the contracts guarantee the return on the risk-free asset for all realizations of the stock price lower than the P-measure threshold stock price.

[11] A universally-valid risk measure is an instrument that is equally valid (1) as a statistic to private investors and (2) for the determination of risk capital at all possible horizons.

fixed-quantile risk measures Value-at-Risk and Conditional Tail Expectations fail to be valid tools for informing life-cycle investors of risk exposures. Section 3 shows the validity of Q-measure fixed-quantile measures in this context. Section 4 reviews the BMP Put in the context of risk capital provisions. Section 5 concludes.

2 Fixed-Quantile Risk Measures under The Actual Probability Distribution

Throughout the paper, the risky asset is a stock whose dynamics are a Geometric Brownian Motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

In addition to the risky asset, there is a risk-free asset with constant instantaneous rate of return r.

This Section investigates the properties of risk measures that are built on fixed quantiles of the actual probability distribution. The most popular instruments in this class are Value-at-Risk and Conditional Tail Expectation measures. Because Value-at-Risk and Conditional Tail Expectations are presented in the literature without a dynamic production technology[12], one is lead to believe that Value-at-Risk and Conditional Tail Expectations are amounts meant to be available in *all states of the world* at a future date T. Therefore, they must result from investments in the risk-free asset. Consequently, future values should be discounted back to the present at the risk-free rate r. The derivations in the rest of this Section

[12] This lack of a production technology is also true of the "Dynamic" refinements such as Rogachev's Dynamic Value-at-Risk (2002) and Wirch and Hardy's Dynamic Iterated CTE (2003).

demonstrate that these risk measures may decrease (and even become negative) over time. To the unsuspecting investor, this may represent a decrease in the riskiness of a stock investment as the holding horizon increases (or even a sure thing in the long-run). This, in turn, is a fallacy exposed by Bodie (1995). This Section also investigates the properties of a contingent-claim extension for fixed α's under the P-measure. The P-measure Return Guarantee (PRG) provides the risk-free rate of return in all states that correspond to a realization of the stock price below a particular stock-price threshold based on the fixed quantile α. However, precisely because Value-at-Risk may become negative for long horizons, the value of such a contingent claim may decrease with the horizon considered.

2.1 Determination of the Threshold

The future stock price that marks the α point – a fixed value between zero and one decided ahead of time – in the cumulative distribution of future stock prices is S_T^*[13] in

$$\begin{aligned} 1-\alpha &= 1-\Pr(S_T \leq S_T^* \mid S) \\ &= \Pr(S_T \geq S_T^* \mid S) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log S_T^*}^{\infty} \exp\left\{ \frac{-1}{2} \left(\frac{\log S_T - \log S - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)^2 \right\} d\log S_T. \end{aligned}$$

After executing the standard change of variables

$$Y_T = \frac{\log S_T - \log S - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

[13] The derivations below are closely related to a probabilistic derivation of the Black-Scholes formula. An accessible reference is Baz and Chacko (2004, pp. 57-61).

and the corresponding

$$dY_T = \frac{d\log S_T}{\sigma\sqrt{T}},$$

the above can be rewritten as

$$\begin{aligned}
1-\alpha &= \Pr(S_T \geq S_T^* \mid S) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{\log S_T^* - \log S - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}^{\infty} \exp\left\{\frac{-1}{2}\left(Y_T\right)^2\right\} dY_T \\
&= \Phi\left(\frac{\log S - \log S_T^* + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

where Φ is the cumulative standard normal.

Now, one can solve for the variable of interest by inversion of the cumulative distribution.

$$\begin{aligned}
\log S_T^* &= \log S + (\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha) \text{ or} \\
S_T^* &= S\exp\left\{(\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\right\}.
\end{aligned}$$

Note that as α goes to zero, the probability of the stock price's ending up above the threshold is driven to 1. Indeed, since $\Phi^{-1}(1) = +\infty$ and $\exp(-\infty) = 0$, S_T^* goes to zero, which is the lowest possible value for a stock price when it is modeled as a Geometric Brownian Motion. Alternatively, as α goes to one, one obtains $\Phi^{-1}(0) = -\infty$ and $\exp(+\infty) = +\infty$ and S_T^* goes to infinity.

2.2 Value-at-Risk

The risk of taking an equity position is the utility-impacting uncertainty inherent in the instantaneous dynamics of the stock price. The risk-free investment, not being subject to such instantaneous risk, is the sole benchmark for the purpose of defining a risk measure[14]. In all that follows, Value-at-Risk is defined as the negative of the α-quantile shortfall, where this shortfall is itself defined as the difference between S_T^* and the value of the investment at time T if it had been invested at the risk-free rate r. Value-at-Risk at maturity (i.e. not discounted back to the present) is

$$
\begin{aligned}
VaR_\alpha(T) &= -\left[S_T^* - \exp(rT)S\right] \\
&= S\left[\exp(rT) - \exp\left\{(\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\right\}\right].
\end{aligned}
$$

Value-at-Risk represents the dollar amount that – if made available at maturity *in all states of the world* – will be sufficient to cover potential shortfalls *with the exception* of the worst α% of all realizations. Figuratively speaking, this measure abandons the left tail of

[14]This is not to say that a time-varying risk-free rate would embody risk because instantaneously, the risk-free asset possesses no utility-impacting uncertainty. The choice of a constant r in this paper is solely for the purpose of simplicity. Another reason for chosing the risk-free investment as the anchor investment is that, as will be made clear below, the risk-free investment is the sole valid benchmark in the context of risk capital provision. Defining VaR as it is done here allows one not to disqualify Value-at-Risk on the grounds that it does not possess the correct benchmark in this particular context. Instead, the invalidity of VaR and other P-measure fixed-quantile instruments is shown to come from deep conceptual issues, not a simple slip in the anchor considered by the definition of the measure.

the distribution. The present value of Value-at-Risk[15] is

$$\begin{aligned} PV(VaR_\alpha(T)) &= \exp(-rT)\,[VaR_\alpha] \\ &= S\left[1-\exp\left\{(\mu - r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\right\}\right]. \end{aligned}$$

Therefore, as long as the equity risk premium is in excess of $\frac{1}{2}\sigma^2$(a consequence of the log-normality of the stock price), $(\mu - r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)$ will eventually increase over time and will become positive, yielding a *negative* Value-at-Risk for long horizons.

The graph below illustrates. The calibration values are (1) an initial stock price of \$100, (2) an annual drift of 10%, (3) a corresponding standard deviation of 20%, (4) an annual risk-free rate of 3%, and (5) a quantile α of 5%. Although these values are arbitrary, they fall within the commonplace spectrum of the parameter space. These calibrating values are used throughout, except if explicitly noted otherwise.

[15] VaR being a constant across states at a given future date, the discounting factor is, under the actual measure $E\xi_T = E\exp\left\{-rT - \frac{1}{2}\theta^2 T - \theta W_T\right\}$, where θ is the market price of risk. Because $E\exp(-\theta W_T)$ is the characteristic function of a normally-distributed random variable with mean zero and variance T, we get that $E\xi_T = \exp(-rT)$. Equivalently, the use of the pricing kernel above yields that the cost of assuring that a particular constant amount will be available in all states of the world at a particular date is the present value of this amount using the risk-free rate r.

Figure 1:

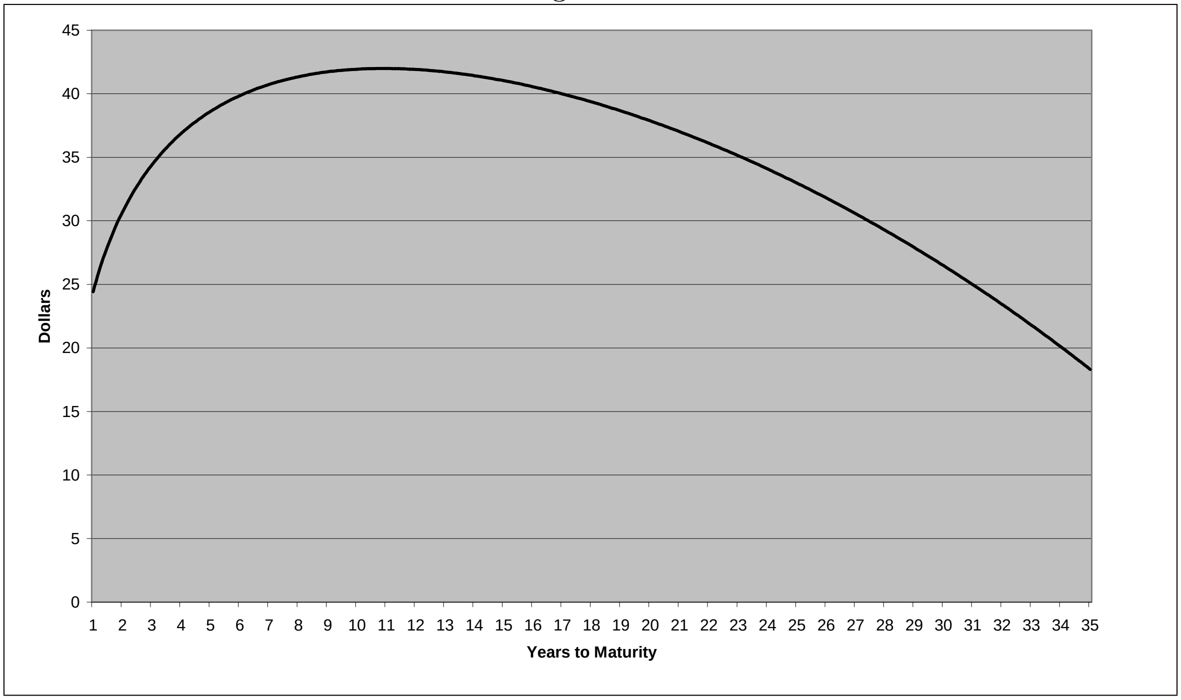

Figure 2: Present Value of 5% Value-at-Risk for Horizons up to 35 Years

2.3 Conditional Tail Expectations

Conditional Tail Expectations rely on the expected value of S_T – under the actual probability measure P – conditional on S_T being less than the threshold S_T^* derived above. Calculating a conditional expectation requires the use of a conditional density function, which is the origin of $\frac{1}{\alpha}$ (the inverse of the cumulative distribution evaluated at S_T^*) in the following derivation. Noteworthy features of this derivation are the nature of the upper bound (S_T^*) and the fact that the calculation is yields a future value, not a present value. Implicit again is the lack of a replication technology.

$$E(S_T \quad | \quad S_T \leq S_T^*) = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} \times$$
$$\int_{-\infty}^{\log S_T^*} \exp\left\{\log S_T - \frac{1}{2}\left(\frac{\log S_T - \log S - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T.$$

Consider the transformation

$$Z_T = \log S_T - \log S.$$

Then rewrite the above expression as

$$\begin{aligned}
E(S_T \quad | \quad S_T \leq S_T^*) &= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{Z_T - \frac{1}{2}\left(\frac{Z_T - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} dZ_T \\
&= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{\frac{2\sigma^2 T Z_T - Z_T^2 + 2Z_T(\mu - \frac{1}{2}\sigma^2)T - (\mu - \frac{1}{2}\sigma^2)^2 T^2}{2\sigma^2 T}\right\} dZ_T \\
&= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{\frac{-Z_T^2 + 2Z_T(\mu + \frac{1}{2}\sigma^2)T - (\mu - \frac{1}{2}\sigma^2)^2 T^2}{2\sigma^2 T}\right\} dZ_T.
\end{aligned}$$

Complete the square by adding and subtracting $2\sigma^2\mu T^2$ to the numerator inside the exponential. This yields

$$\begin{aligned}
E(S_T \quad &| \quad S_T \leq S_T^*) \\
&= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} \exp(\mu T) S \int_{-\infty}^{\log S^* - \log S} \exp\left\{-\frac{Z_T^2 - 2Z_T(\mu + \frac{1}{2}\sigma^2)T + (\mu + \frac{1}{2}\sigma^2)^2 T^2}{2\sigma^2 T}\right\} dZ_T \\
&= \frac{1}{\alpha}\frac{1}{\sqrt{2\pi\sigma^2 T}} \exp(\mu T) S \int_{-\infty}^{\log S^* - \log S} \exp\left\{-\frac{1}{2}\left[\frac{Z_T - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]^2\right\} dZ_T.
\end{aligned}$$

Now, let

$$W_T = \frac{Z_T - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Then

$$\begin{aligned}
E(S_T \quad | \quad S_T \leq S_T^*) &= \frac{1}{\alpha} S \exp(\mu T) \int_{-\infty}^{\frac{\log S_T^* - \log S - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} W_T^2\right\} dW_T \\
&= \frac{1}{\alpha} S \exp(\mu T) \Phi\left(\frac{\log S_T^* - \log S - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= \frac{1}{\alpha} S \exp(\mu T) \Phi\left(\frac{\log S + (\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha) \ - \log S_t - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= \frac{1}{\alpha} S \exp(\mu T) \Phi\left(\frac{-\sigma^2 T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)}{\sigma\sqrt{T}}\right) \\
&= \frac{1}{\alpha} S \exp(\mu T) \Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right).
\end{aligned}$$

where one goes from the third line to the fourth line by substituting in for S_T^*.

Therefore, the Conditional Tail Expectation is

$$CTE_\alpha(T) = S \exp(rT) - \frac{1}{\alpha} S \exp(\mu T) \Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right).$$

This expression is consistent with α being a truncation parameter of the expectation. In particular, as α goes to 1, the truncation is removed. As noted above, $\Phi^{-1}(0) = -\infty$. Then, $\Phi\left(-\sigma\sqrt{T} + \infty\right) = 1$ and the second term term in $CTE_1(T)$ becomes the unconditional mean $S\exp(\mu T)$. That is, the Conditional Tail Expectation is a negative value, indicating that – *on average* – one should not expect a loss but rather a gain compared to the risk-free investment. Furthermore, for the same reason as above – namely, the lack of a more dynamically-sophisticated production technology – this amount should be available in *all* *states of the world* at this future date. Thus, the present value of the Conditional Tail Expectation is

$$PV(CTE_\alpha(T)) = S\left[1 - \frac{1}{\alpha}\exp\left((\mu - r)T\right)\Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)\right].$$

It also may eventually become negative for long enough horizons in the presence of risk premium. This can be seen in the figure below.

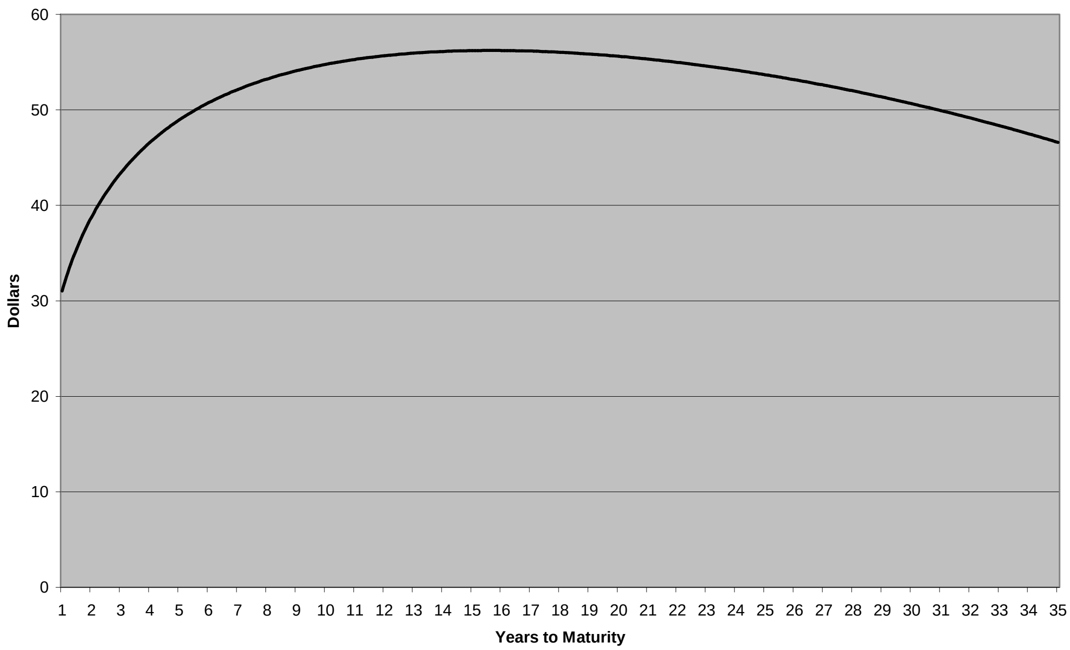

Figure 3: Present Value of 5% Conditional Tail Expectation for Horizons up to 35 Years.

2.4 Dynamic Risk Measures Based on Fixed Quantiles under the Actual Probability Distribution

One may think that the problem with the above risk measures is their lack of a replication technology. Certainly, a replication technology is required for a risk measure to be a least-cost insurance contract. This will be taken up in the context of risk capital provision by financial firms. The developments of this subsection, however, provide evidence that the failure of P-measure fixed-quantile instruments originates at a more basic level, namely, because fixed-quantile thresholds under the actual probability distribution can easily end up above the value of the risk-free investment. Therefore, dynamically-sophisticated risk measures that rely on such values for upper bounds may require future cash payments by the contract holder. As these potential payments increase over time, the value of the contract necessarily approaches zero.

Consider the following. Instead of defining a risk measure as a dollar amount that suffices to cover losses *except* in the worst cases (i.e., *abandoning* the left tail past a certain level as do VaR and CTE), define it as the cost of securing the risk-free rate of return whenever the stock falls below a certain threshold. This threshold is the stock price corresponding to a certain quantile in the actual stock-price probability distribution P, namely S_T^* above. The corresponding contingent claim's payoffs are $S\exp(rT) - S_T$ for all realizations of S_T below S_T^* and zero for all realizations that are above it. At first sight, this *embraces* the left-tail of the distribution in the same way as a put option. The cost of initiating the replication – this is already a present value – of this P-measure Return Guarantee (PRG) is priced below under the risk-neutral measure.

$$\begin{aligned}
PRG_\alpha(T) &= \exp(-rT)E^Q[1_{\{S_T \leq S_T^*\}}\{S\exp(rT) - S_T\}] \\
&= \frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \{S\exp(rT) - S_T\} \exp\left\{-\frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T \\
&= \frac{S}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \exp\left\{-\frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T \\
&\quad - \frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \exp\left\{\log S_T - \frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T.
\end{aligned}$$

Take the two terms in the above expression separately.

Start with the first term

$$\frac{S}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \exp\left\{-\frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T.$$

Consider the change of variables

$$Y_T = \frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Correspondingly,

$$dY_T = \frac{d\log S_T}{\sigma\sqrt{T}}.$$

Therefore,

$$\begin{aligned}
& \frac{S}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \exp\left\{-\frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T \\
= \; & \frac{S}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log S_T^* - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \exp\left\{-\frac{1}{2}\left(Y_T\right)^2\right\} dY_T \\
= \; & S\Phi\left(\frac{\log S_T^* - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \; & S\Phi\left(\frac{\log S + (\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha) - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \; & S\Phi\left((\frac{\mu - r}{\sigma})\sqrt{T} - \Phi^{-1}(1-\alpha)\right) \\
= \; & S\Phi\left(\theta\sqrt{T} - \Phi^{-1}(1-\alpha)\right),
\end{aligned}$$

where θ is the market price of risk (the Sharpe Ratio).

Consider now the second term

$$\frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^*} \exp\left\{\log(S_T) - \frac{1}{2}\left(\frac{\log S_T - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\} d\log S_T$$

An already-used change of variables yields

$$\begin{aligned}
&\frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{ Z_T - \frac{1}{2}\left(\frac{Z_T - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2 \right\} dZ_T \\
= \; & \frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{ \frac{2\sigma^2 T Z_T - Z_T^2 + 2Z_T(r - \frac{1}{2}\sigma^2)T - (r - \frac{1}{2}\sigma^2)^2 T^2}{2\sigma^2 T} \right\} dZ_T \\
= \; & \frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{ -\frac{Z_T^2 - 2Z_T(r + \frac{1}{2}\sigma^2)T + (r - \frac{1}{2}\sigma^2)^2 T^2}{2\sigma^2 T} \right\} dZ_T \\
= \; & \frac{S}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{ -\frac{1}{2}\left(\frac{Z_T - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2 \right\} dZ_T.
\end{aligned}$$

Executing the last change of variables

$$W_T = \frac{Z_T - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

the expression becomes

$$\begin{aligned}
&\frac{\exp(-rT)}{\sqrt{2\pi\sigma^2 T}} S \int_{-\infty}^{\log S_T^* - \log S} \exp\left\{ Z_T - \frac{1}{2}\left(\frac{Z_T - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2 \right\} dZ_T \\
= \; & S \int_{-\infty}^{\frac{\log S_T^* - \log S - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(W_T)^2 \right\} dW_T \\
= \; & S\Phi\left(\frac{\log S_T^* - \log S - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \; & S\Phi\left(\frac{\log S + (\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha) - \log S - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
= \; & S\Phi\left(\frac{(\mu - r - \sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)}{\sigma\sqrt{T}}\right) \\
= \; & S\Phi\left((\theta - \sigma)\sqrt{T} - \Phi^{-1}(1-\alpha)\right).
\end{aligned}$$

Putting the two terms together, one finds that the value of the PRG is

$$PRG_\alpha(T) = S\left\{\Phi\left(\theta\sqrt{T} - \Phi^{-1}(1-\alpha)\right) - \Phi\left((\theta-\sigma)\sqrt{T} - \Phi^{-1}(1-\alpha)\right)\right\}.$$

The value of this contract can be non-monotonic over time. Indeed, it allows for future cash outflows on the part of the contract holder whenever the Sharpe Ratio is greater than the volatility. These payments only occur when the stock outperforms the risk-free investment, but by no more than a certain amount $S_T^* - S_t \exp(rT)$. For short horizons, as the maturity grows, the contract provides insurance in a way that nears the coverage offered by a put option with strike price $S_t \exp(rT)$. However, over time, one must give up an ever-increasing amount of upside. These effects are reflected in the horse race into the right tail of the distribution between the two terms whenever $\theta - \sigma > 0$. The trend reversal appears to be captured by the switch from convexity to concavity of the cumulative normal distribution. This behavior may occur at only very long horizons for low values of α. For visualization purposes, both the value of PRG for horizons up to 35 years (the benchmark horizons) for $\alpha = 50\%$ and the PRG for a much longer horizon (200 years) for $\alpha = 5\%$ (our benchmark α) are provided.

Figure 4:

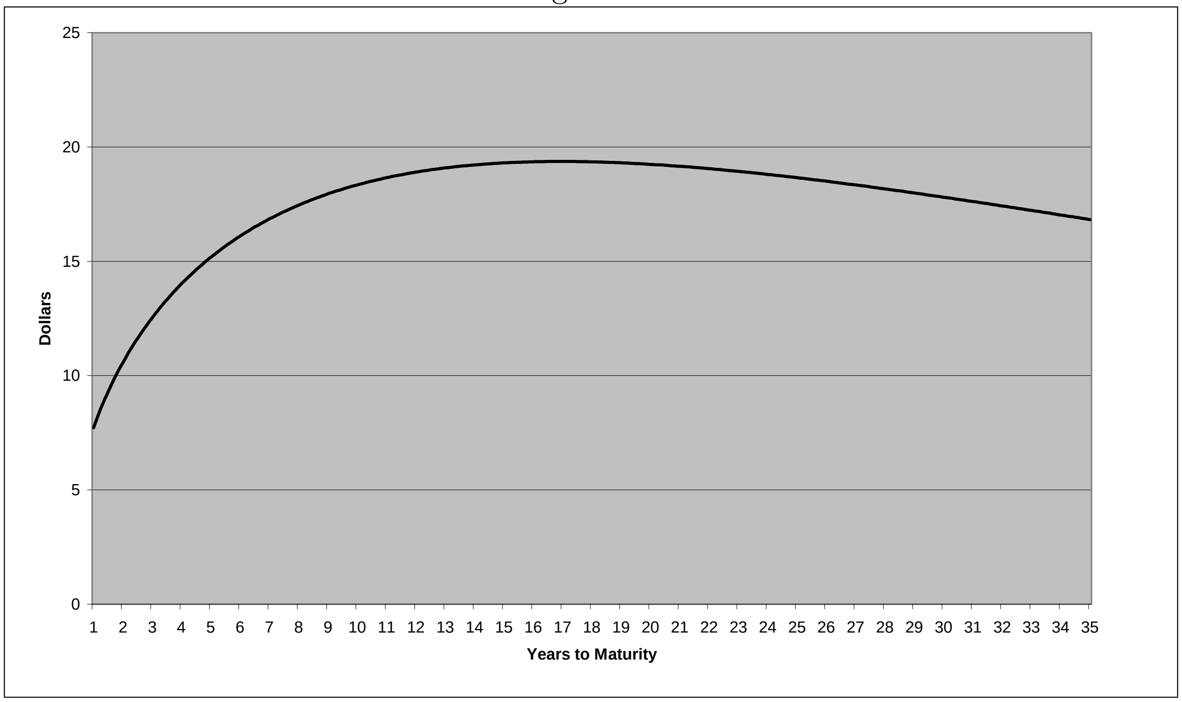

Figure 5: Value of 50%-P-Measure Return Guarantee for Horizons from 1 to 35 Years.

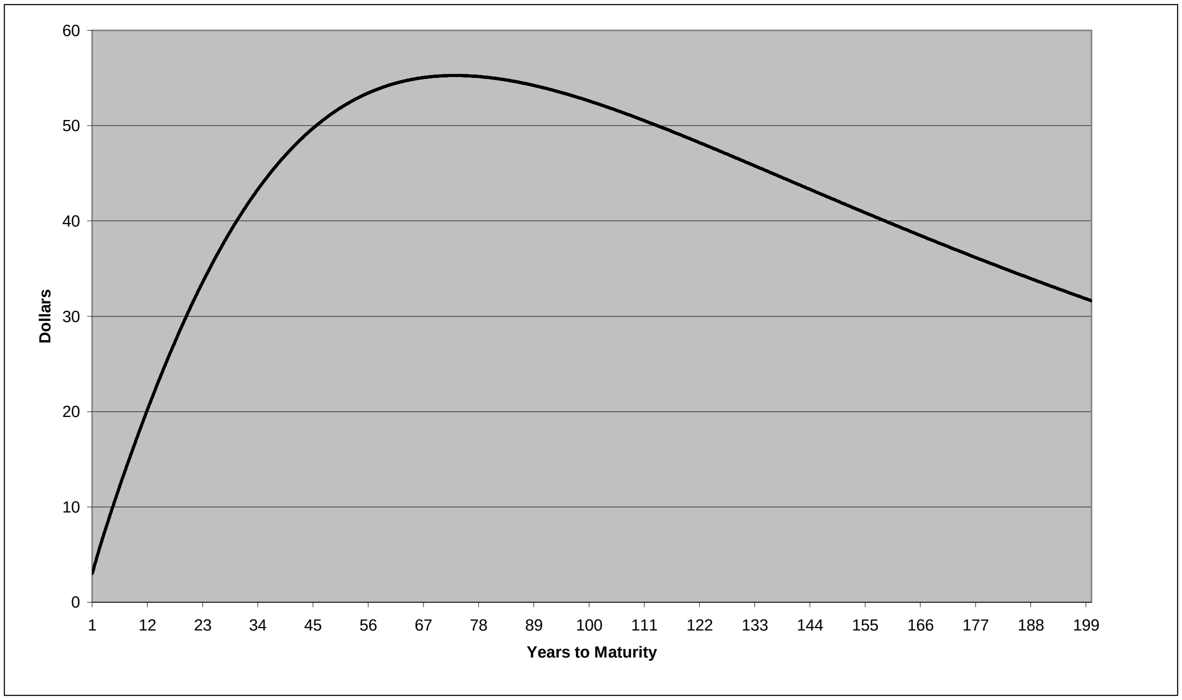

Figure 6: Value of 5%-P-Measure Return Guarantee for Horizons from 1 to 200 Years.

3 Risk-Neutral Fixed-Quantile Instruments

3.1 Quantiles under Risk Neutrality

There is an inherent asymmetry in the cost of purchasing contracts for the delivery of consumption units contingent on the relative performance of the equity portfolio. As a reflection of real production activity in the economy, when the equity does not perform well, the total stock of real wealth available for consumption is relatively low. This relative scarcity translates into higher prices for contracts that provide a unit of consumption in such "low-stocks" states. Breeden and Litzenberger (1978) write:

> "Note that the cost of a security paying $1.00 if the market increases by 10% or more is approximately 29¢, whereas a security paying $1.00 if the market is less

than 110% of its current value costs $B(1)$-29¢=65¢. For these securities, although they may have the same expected return (approximately), the security paying when the market is low has twice the price of a security paying when the market is high, reflecting the value of negative covariance (p.631)."

The relative scarcity implied in the "negative covariance" translates into a strictly decreasing and convex weighing function found in Samuelson-Merton (1969) in the form of the marginal utility schedule, or more recently, the pricing kernel. When this pricing kernel function is merged with the actual probability density, the resulting function is also a probability density function. This density function is known as the *risk-neutral density function.* Furthermore, the current cost of a unit of consumption in any particular state is equal to the height of *the risk-neutral* log-normal density function for the corresponding stock price discounted to the present at the risk-free rate[16]. Therefore, the cost of any flow of contingent payoffs becomes the expected value of the payoffs under this unique risk-neutral density function discounted back to the present using the risk-free rate.

[16]The formal expression can be derived from Equation 7 of Breeden and Litzenberger (1978). The price of a state-claim that pays a dollar if the stock price is in $[Y_1, Y_2]$ at time T is, in their notation,

$$\begin{aligned}\Delta(Y_1, Y_2, T) &= \exp(-rT) \\ &\times\{\Phi(\frac{\log S - \log Y_1 + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\ &-\Phi(\frac{\log S - \log Y_2 + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})\}.\end{aligned}$$

A trivial change of variables yields

$$\begin{aligned}\Delta(Y_1, Y_2, T) &= \exp(-rT) \\ &\times \int_{\frac{\log Y_1 - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}^{\frac{\log Y_2 - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \phi(x)dx.\end{aligned}$$

The informal statement above ought to be understood as a "limiting" statement when the interval over which the security pays a dollar becomes very small.

This Section demonstrates that instruments built around *fixed quantiles* under the Q-measure in a way that mirrors the P-measure instruments investigated above *cannot* eventually decrease with the horizon. This results from the "scaling-down" in the mean and the median of the risk-neutral lognormal compared to their P-measure counterparts at any horizon. This, in turn, is what prevents risk-neutral risk measures from displaying the fallacious properties of P-measure instruments. Because all derivations below are identical to those presented earlier, formulas are provided without proofs.

3.2 Risk-Neutral Value-at-Risk

Before considering fixed-quantile instruments under the risk-neutral measure, a clarification of the concept of quantile in this context is necessary. Since values of the risk-neutral density function are prices, quantiles (denoted $\tilde{\alpha}$) are fractions of a dollar. The price threshold $\tilde{S}_T^*$ associated with $\tilde{\alpha}$ marks an economically-meaningful delimitation point in the space of stock prices. In particular, the total current cost of securing a unit of consumption in all states associated with realizations of the stock price below $\tilde{S}_T^*$ is exactly equal to $\tilde{\alpha}$ discounted at the risk-free rate r.

The fixed-quantile threshold under the risk-neutral measure is

$$\tilde{S}_T^* = S \exp\left\{(r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\tilde{\alpha})\right\},$$

where the tilde emphasizes the use of the risk-neutral measure.

The present value of the corresponding $\tilde{\alpha} - Value - at - Risk$ is

$$\begin{aligned} PV(VaR_{\tilde{\alpha}}(T)) &= \exp(-rT)\left[VaR_{\tilde{\alpha}}\right] \\ &= S\left[1 - \exp\left\{(r - r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\tilde{\alpha})\right\}\right] \\ &= S\left[1 - \exp\left\{-\frac{1}{2}\sigma^2 T - \sigma\sqrt{T}\Phi^{-1}(1-\tilde{\alpha})\right\}\right]. \end{aligned}$$

Over long horizons, the $O(T)$ term dominates. Over such long horizons, as the maturity date increases, the shortfall at the threshold value $\tilde{S}_T^*$[17] can only widen. As can be seen in the expression for the present value of Value-at-Risk based on Q-quantiles, this prevents the long-run decreases and sign reversions that characterize P-measure instruments.

Over shorter horizons, one must consider again the underlying economics to understand possible local non-monotonicities associated with the second term. When $\tilde{\alpha} \leq .5$, the monotonic increase in the shortfall at $\tilde{S}_T^*$ is reinforced. Indeed, all else being equal, the cost of securing consumption units in the extreme tails grows over time relative more "central" values of the distribution. This is caused by a volatility effect[18]. Thus, all else equal, the purchase of \$1-insurance contracts for the worst possible stock-price realizations adds up to (a small) $\tilde{\alpha}$ relatively more quickly as the horizon increases. This depresses $\tilde{S}_T^*$ farther away from the value of the risk-free investment. When $\tilde{\alpha} > .5$ and $\Phi^{-1}(1-\tilde{\alpha})$ is nega-

[17] Remember that the shortfall at the threshold value $\tilde{S}_T^*$ is defined as the gap between the value of the current stock price accrued at the risk-free rate and this threshold $\tilde{S}_T^*$.

[18] Breeden and Litzenberger's (1978, esp. p.637) comparative statics provide valuable intuition. The most relevant equation in this respect is their Equation 17

$$\frac{\delta \log P}{\delta T} = -r - \frac{1}{2T} - \frac{d_2}{\sigma\sqrt{T}}(r - \delta - \frac{\sigma^2}{2}) + \frac{d_2^2}{2}$$

where d_2 corresponds to the traditional notation in the Black-Scholes formula. The authors note that the square term dominates "for payoff levels of aggregate wealth that are extremely high or low relative to the current level (p.637)."

tive, a short-run non-monotonicity appears. It corresponds to the relatively large portion of insurance contracts related to "central" realizations in the basket of insurance contracts purchased. Since the cost of these tends to decrease relative to those in the tails, this leads $\tilde{S}_T^*$ to increase relatively fast over the short run. This particular behavior does not endanger any of the arguments above. These dynamics stem directly from the underlying life-cycle Arrow-Debreu model and, unlike P-measure instruments, eventually, the measure will monotonically increase with the time horizon[19].

3.3 Risk-Neutral Conditional Tail Expectation

Now, turn to the Conditional Tail Expectation under the risk-neutral measure. Remember that the Conditional Tail Expectation under the actual probability distribution is the average loss in the $\alpha\%$ worst cases. It is difficult to endow this well-defined mathematical construct with economic and financial meaning. On the other hand, the present value of the Conditional Tail Expectation statistic under the risk-neutral measure[20]

$$PV(CTE_{\tilde{\alpha}}(T)) = S\left[1 - \frac{1}{\tilde{\alpha}}\Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\tilde{\alpha})\right)\right]$$

is monotonically increasing and can be interpreted within finance theory.

This instrument compares the normalized insurance-value of a bond to that of a "stripped"

[19] Much of the argument in this last paragraph may seem to indicate to the careless reader that in the long-run, one can omit variance consideration since only the "mean" term matters. Correspondingly, the short-run would be the territory of volatility. This could not be further from the truth. Indeed, this sort of reasoning finds its roots in the very same place as the fallacy exposed by Bodie (1995). This is the fallacy borne by the dynamics of P-measure risk instruments. The fact that Q-measure instruments are eventually increasing is a testimony to the economic importance of volatility in the long-run.

[20] This risk-neutral Conditional Tail Expectation uses the stock-price threshold based on the risk-neutral distribution. Also, the expectation is taken with respect to the risk-neutral distribution.

stock. The "stripped" stock is a contract that pays the value of the stock whenever the stock is worth less than the threshold value $\tilde{S}_T^*$. The normalization makes the comparison feasible by scaling both values to a per-insurance-level basis. Consider the following, where $\tilde{\phi}(.)$ indicates the risk-neutral density function.

$$\begin{aligned}
E^Q(S_T \quad | \quad S_T \leq \tilde{S}_T^*) &= \frac{1}{\tilde{\alpha}} \int_{-\infty}^{\log \tilde{S}_T^*} \underbrace{S_T}_{payoff} \underbrace{\tilde{\phi}(\log S_T)}_{risk-neutral\ probability\ density} d\log S_T \\
&= \exp(rT) \times \frac{\underbrace{\int_{-\infty}^{\log \tilde{S}_T^*} \underbrace{S_T}_{payoff} \underbrace{\exp(-rT)\tilde{\phi}(\log S_T)}_{price\ density} d\log S_T}_{insurance-value\ of\ "stripped"\ stock}}{\underbrace{\tilde{\alpha}}_{insurance\ level}} \\
&= \exp(rT) \times \frac{\underbrace{S\Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)}_{insurance-value\ of\ "stripped"\ stock}}{\underbrace{\tilde{\alpha}}_{insurance\ level}}.
\end{aligned}$$

The corresponding present value is

$$PV(E^Q(S_T \mid S_T \leq \tilde{S}_T^*)) = \frac{\underbrace{S\Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)}_{insurance-value\ of\ "stripped"\ stock}}{\underbrace{\tilde{\alpha}}_{insurance\ level}}.$$

This is the per-insurance-level insurance-value of the "stripped" stock.

CTE compares the value of the stock invested in the risk-free bond, $S\exp(rT)$, to the expected value above. When discounted back to the present, the instrument becomes

$$PV(CTE_{\tilde{\alpha}}(T)) = \frac{\underbrace{S}_{insurance-value\ of\ bond}}{\underbrace{1}_{insurance\ level\ of\ bond}} - \frac{\underbrace{S\Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)}_{insurance-value\ of\ "stripped"\ stock}}{\underbrace{\tilde{\alpha}}_{insurance\ level\ of\ "stripped"\ stock}}.$$

This provides a clear financial meaning to the risk-neutral Conditional Tail Expectation. It is the relative insurance-value of a bond investment to that of a stock investment. Note that for any insurance level $\tilde{\alpha} < 1$, the insurance-value of the stripped stock is strictly lower than that of the bond. The difference between those monotonically increases with the horizon. Also, these insurance-values are equal *only* when the horizon is null or when the insurance level of the "stripped" stock is exactly equal to 1. It is never the case that the insurance-value of the stock is greater than that of the bond. This may be viewed as an alternative – although less transparent – proof to Bodie's (1995) that stocks are riskier than bonds in the long-run.

3.4 Risk-Neutral Return Guarantee (QRG)

The Q-measure Rate Guarantee is a contingent claim that guarantees the return on the risk-free asset for all states corresponding to stock-price realizations below value $\tilde{S}_T^*$. It displays most of the characteristics of a simple put option. The only distinction is that the highest value of the stock at which the contract can be exercised, $\tilde{S}_T^*$, remains uniformly below the "strike price," $S\exp(rT)$. This results from $\tilde{\alpha}$ being fixed in the presence of a volatility effect affecting percentiles (including the median) for lognormal processes. The formula is

$$\begin{aligned} QRG_{\tilde{\alpha}}(T) &= S\left\{\Phi\left(-\Phi^{-1}(1-\alpha)\right) - \Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)\right\} \\ &= S\left\{\alpha - \Phi\left(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha)\right)\right\} \end{aligned}$$

The value of this contract is monotonically increasing with the horizon, which makes it an unambiguously valid risk measure to life-cycle investors. Furthermore, unlike all other measures considered above, it is a least-cost insurance contract that may be used by financial firms to shield themselves from the risk of their equity positions. Thus, this contract – or a contract very closely related – may be a universally-valid risk instrument; it may be used by financial firms to inform private investors as well as to provide a measure of capital adequacy at all horizons. Fixed-quantile QRG measures fail to accomplish to the latter in a uniform manner for all time horizons. The Bodie-Merton-Perold Put is a simple variant of the above contract. This universal risk measure is the subject of the next Section.

4 A Universally-Valid Risk Measure: The Bodie-Merton-Perold Put.

The preceding Sections show that P-measure instruments are not valid statistics to inform private investors of risk exposures. Their Q-measure counterparts are uniformly valid for this purpose. This results from the latters' status as bearers of economic meaning within Arrow-Debreu theory. Thus, no *universally* valid risk measure may come out of the P-measure family[21]. Before investigating which risk measure is universally valid, one needs to explicitly determine what makes an instrument appropriate in the context of capital adequacy. Consider the definition of *risk capital* given by Merton and Perold (1993).

[21] Perhaps, one may think that despite being uniformly invalid risk statistics to life-cycle investors, P-measure instruments may appropriately deal with the issue of risk adequacy. However, despite being widely used in the financial community, P-measure instruments are inherently inappropriate tools to guarantee a firm's financial integrity. The VaR and CTE measures *abandon* the left-tail, leaving firms open the possibility of stock-price realizations that would wipe them out. This is true for any choice of quantile short of $\alpha = 1$. As Matz puts it, quantiles are just "a matter of scaling and therefore meaningless (p. 19)." The dynamic PRG allows future payments by the contract holder to take place. This invalidates its use for the purpose of guaranteeing the integrity of the firm.

> “Risk capital is defined as the smallest amount that can be invested to insure the value of the firm’s net assets against a loss in value relative to the risk-free investment (p.217)[22].”

Perhaps, one may wish to define risk capital in an alternative manner. Then, one of two things would happen. First, the alternative definition may yield a dollar figure that is greater than the "smallest amount" of the Merton-Perold definition. This would imply a technological inefficiency in the production of the implicit insurance contract. In this case, the alternative measure of risk capital is effectively the sum of two terms: (1) the cost of shielding oneself from the risk (or *true* risk capital) and (2) the burden of the technological inefficiency. The second possibility is that the alternative definition of risk capital yields an amount strictly lower than the "smallest amount" required to shield oneself from the risk. Necessarily, additional funds would then be needed under certain conditions to cover the gap between the realized value of the equity and the value of the risk-free asset. These additional funds would be *de facto* risk capital. Therefore, Merton and Perold’s (1993) definition is the *only* possible definition of risk capital. This implies that risk capital *exists* and is *unique* whether or not the firm actually provides it for the purpose of managing risk or even is aware of it.

Therefore, the payoffs of the least-cost insurance contract used for the determination of risk capital must systematically make up for any shortfall that may occur. The QRG’s payoffs are exactly the amount of shortfall over the coverage region $(-\infty, \tilde{S}_T^*]$. However,

[22] Consider a bank that receives a one-hundred-dollar one-year time deposit. Suppose the interest rate is 3% annually. Suppose the bank invests the $100 received in an equity index. Today, the bank’s net assets are exactly zero. Risk capital, in this case, is the least amount necessary to insure that the bank’s net assets in a year are worth no less than zero. This is the value of a put option on the equity index with a strike price of $103.

because of the volatility term $\frac{-1}{\sigma^2}$, the threshold value $\tilde{S}_T^*$ diverges from $S\exp(rT)$. This implies that there is no particular quantile $\tilde{\alpha}$ that can be used to make QRG appropriate for capital risk determination at all investment horizons. For any particular horizon, the relevant quantile under the risk-neutral measure is

$$\begin{aligned}
\tilde{\alpha}_T &= \Pr(S_T \leq S\exp(rT) \mid S) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log S + rT - \log S - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \exp\left\{ \frac{-1}{2} (Y_T)^2 \right\} dY_T \\
&= \Phi\left(\frac{1}{2}\sigma\sqrt{T} \right).
\end{aligned}$$

Therefore, the appropriate instrument for risk capital provision *at all horizons* corresponds to a varying quantile under the risk-neutral measure.

The implementation of measures that require varying quantiles under the risk-neutral probability distribution may appear a daunting task. However, this measure is in fact a simple put option already proposed in the literature. The varying quantile corresponds to a put option with strike price $S\exp(rT)$. This is the Bodie-Merton-Perold Put. The value of this put option is

$$BMP(T) = S\left\{ \Phi\left(\frac{1}{2}\sigma\sqrt{T} \right) - \Phi\left(-\frac{1}{2}\sigma\sqrt{T} \right) \right\}.$$

This risk measure is the unique valid instrument to determine risk capital for all horizons. Bodie (1995) also shows that this put option captures the true time-dynamics of risk and is a relevant statistic to private investors. Therefore, the Bodie-Merton-Perold Put is the unique universally-valid risk measure.

5 Conclusion

Risk measures relying on fixed quantiles under the actual probability distribution, such as Value-at-Risk and Conditional Tail Expectation, may decrease with the investment horizon. These time dynamics reflect a fallacy inherent to their lack of economic content. Such risk measures are therefore inappropriate statistics for framing private investment decisions. P-measure instruments are also invalid for the determination of capital adequacy for all quantiles and horizons. On the other hand, risk-neutral fixed-quantile instruments are relevant for the purpose of informing investors of risk exposures. This is caused by their economic content within Arrow-Debreu theory. When risk capital is appropriately defined, one concludes that fixed-quantile measures cannot be universally valid for the provision of risk capital. A unique risk-neutral varying-quantile measure is valid for both purposes. This is the Bodie-Merton-Perold Put. Its universal validity and simplicity makes it a natural choice for an industry standard in risk measurement.

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