Optimal Age-Based Portfolios with Stochastic Investment Opportunity Sets

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Abstract

In an environment with stocks and short-term debt, random changes in the risk-reward frontier produce hedging demands for equities, implying that portfolio policies supporting optimal life-cycle consumption are rarely mean-variance efficient. Pursuing optimal life-cycle portfolio policies is technologically feasible but it represents a significant burden for individuals and financial firms acting as fiduciaries. As a result, investors often rely on relatively simple investment heuristics, most often age-based portfolio policies that rebalance the investor's portfolio as a function of age alone. We find that (i) the welfare losses associated with these policies are often negligible, so that the trade-off between first-best policies and simpler optimal age-based policies likely favors the approximate policy, and that (ii) not only do initial age-based portfolios display the same overall pattern as first-best portfolios but they are also always within the same order of magnitude.

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1 Introduction

In an environment with stocks and short-term debt, random changes in the risk-reward frontier produce hedging demands for equities (e.g., Detemple, Garcia, and Rindisbacher, 2003) so that portfolio policies supporting optimal life-cycle consumption are rarely mean-variance efficient.[1] Pursuing optimal life-cycle portfolio policies is technologically feasible but it represents a significant burden for individuals and financial firms acting as fiduciaries. As a result, investors often rely on relatively simple investment heuristics, most often age-based portfolio policies that rebalance the investor's portfolio as a function of age alone. If the welfare losses associated with such investment heuristics are not disproportionate, the trade-off between optimality and simplicity may favor the approximate policy. While this is not usually the case of one-size-fits-all age-based rules – such those embodied in target-date retirement funds (Bodie and Treussard, 2006)[2] – from which serious welfare losses may arise, many *optimal* age-based policies are sufficiently close to the first-best optimum for individuals to consider the trade-off.

This article addresses the following questions. (i) In an environment with stochastic parameters, can age-based policies produce welfare levels in the neighborhood of the first-best optimum? (ii) How close are optimal (i.e., welfare maximizing) age-based portfolio weights to first-best portfolio weights? We find that (i) the welfare losses associated with these policies are often negligible, so that the trade-off between first-best policies and simpler optimal age-based policies likely favors the approximate policy, and that (ii) not only do initial age-based portfolios display the same overall pattern as first-best portfolios but they are also always within the same order of magnitude. In addition, we show that stock holdings under the first-best policy and the optimal age-based policy systematically decrease in risk aversion. Moreover, both under the first-best and under the age-based policy, stock holdings increase as the procyclicality of the interest rate process weakens. Finally, we find that the volatility of stock returns precision – a factor that *per se* does not affect first-best portfolios – systematically affects the optimal age-based portfolio exposure to equity risk. Specifically, increased volatility of stock returns precision depresses equity holdings and often reduces them below the levels suggested by first-best optimal portfolios. This effect is caused by the pre-programmed nature of age-based portfolios and the natural lock-in effect that it engenders relative to flexible first-best policies.

The remainder of the paper is organized as follows. Section 2 presents the financial environment, the life-cycle optimization problem, and the corresponding first-best portfolio policies. Section 3 introduces the class of age-based portfolio policies and describes the properties of optimal age-based portfolios in relation to first-best policies. Section 4 concludes. The Appendix collects a few basic rules of Malliavin calculus used to derive explicit first-best portfolio policies as well as the Matlab code for all numerical work contained in the article.

[1] As first noted in Merton (1971 and 1973), for constant relative risk aversion preferences, a mean-variance portfolio policy is optimal only if (i) the investor has logarithmic preferences or (ii) financial markets are characterized by constant or deterministic time-varying parameters.

[2] Bodie and Treussard (2006) analyze the welfare implications of adopting target-date retirement funds (TDRFs) in a constant parameter model with human capital risk. They find that TDRFs are approximately optimal for some individuals but that they are very inappropriate for others, especially for significantly risk-averse individuals whose risky human capital is positively correlated with equity markets.

2 The Model

This Section describes the financial environment, the life-cycle optimization problem, and the corresponding first-best portfolio policies. The solution methodology for first-best policies is that of Cox and Huang (1989). The mathematical machinery of Malliavin calculus that we employ is exposed in Nualart (1995) and in Detemple, Garcia, and Rindisbacher (2003). The Appendix collects a few basic rules of Malliavin calculus used to derive explicit first-best portfolio policies in Section 2.2.

2.1 Financial Markets

The financial markets are characterized by two securities: an equity index and a locally risk-free asset. The former obeys

$$dS_t = S_t \left(\mu_t dt + \sigma_t dW_t\right), \tag{1}$$

in which μ_t and σ_t are respectively the stochastic drift and volatility of the stock index. The latter follows

$$dB_t = B_t r_t dt, \tag{2}$$

where r_t is the instantaneous risk-free rate, which evolves according to

$$dr_t = \kappa_r \left(\bar{r} - r_t\right) dt + \sigma_r r_t dW_t. \tag{3}$$

In Eq. (3), κ_r is the mean reversion parameter, $\bar{r}$ is the long-run mean, and σ_r is the volatility parameter. The risk-free rate is said to be *procyclical*, relative to the stock index, if the volatility parameter σ_r is positive and it is referred to as *countercyclical* if σ_r is negative. The model nests the assumption of a constant spot rate $r_t = r$, which corresponds to $\sigma_r = 0$.

The market price of risk or Sharpe ratio, θ_t, is

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t} \tag{4}$$

and obeys

$$d\theta_t = \kappa_\theta \left(\bar{\theta} - \theta_t\right) dt + \sigma_\theta \theta_t dW_t. \tag{5}$$

In the above, κ_θ is the parameter of mean reversion, $\bar{\theta}$ is the long-run mean, and σ_θ is the volatility parameter of the process. A constant Sharpe ratio obtains by setting $\sigma_\theta = 0$.

The inverse of the stochastic equity volatility is referred to as the *precision process* $y_t = \sigma_t^{-1}$ (Chacko and Viceira, 2005) and it follows

$$dy_t = \kappa_y \left(\bar{y} - y_t\right) dt + \sigma_y y_t dW_t.^{3} \tag{6}$$

Whenever either $\sigma_r \neq 0$ or $\sigma_\theta \neq 0$, or both, the *instantaneous investment opportunity set* – or capital allocation line – varies stochastically over time.[4] The investment opportunity

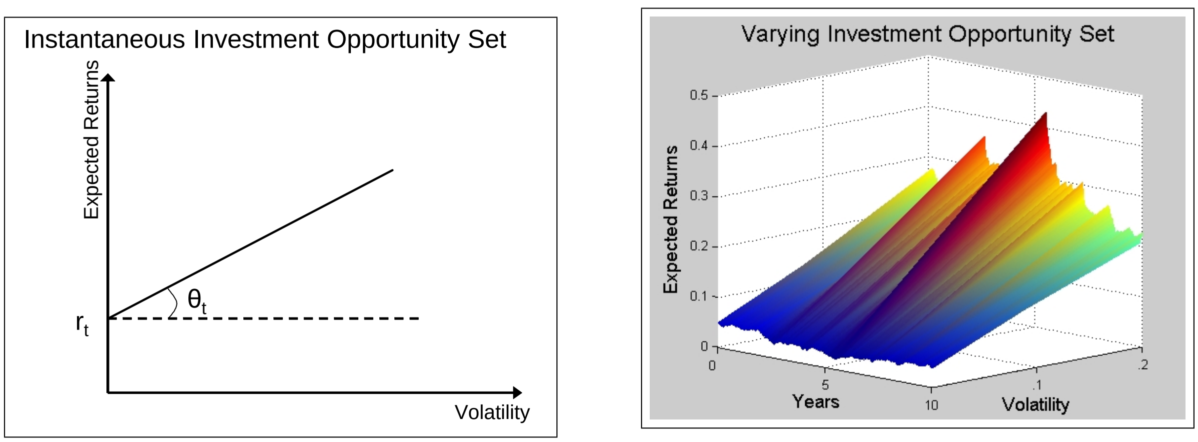
[3] While Chacko and Viceira (2005) model variances and inverse variances, this paper models standard deviations and inverse standard deviations (precision processes).

[4] For a classic treatment of investment opportunity sets, see Bodie, Kane, and Marcus (2005, esp. Ch. 8).

set is a complete description of the risk-reward trade-off faced by individual investors at any point in time. This trade-off is depicted in Figure 1, Panel (a). Panel (b) illustrates the notion of a stochastically changing investment opportunity set by tracing its intercept – the spot rate, whose evolution is given by Eq. (3) – and its slope – the market price of risk, which follows Eq. (5) – over a decade of simulated changes.

Figure 1: Investment Opportunity Sets

Panel (a) Panel (b)



The implications of a stochastic investment opportunity set on the optimal asset demand of life-cycle investors were first studied by Merton (1971, 1973). In particular, the individual corrects his financial portfolio in an attempt to hedge adverse fluctuations in future interest rates and market prices of risk. Accordingly, the individual's demand for assets that perform well during adverse fluctuations increases relative to the Markowitz mean-variance efficient demand. Similarly, asset classes performing poorly during adverse events are less demanded relative to mean-variance demands. Subsection 2.2 provides a mathematical representation for these hedging demand components.

2.2 Optimal Consumption and Supporting First-Best Portfolio

An individual has an investment horizon of T years. His initial wealth is equal to k and his preferences are represented by the Constant Relative Risk Aversion (CRRA) utility function over terminal wealth, X_T,

$$E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right], \tag{7}$$

where γ is the constant parameter of relative risk aversion.

In this environment, the optimal terminal wealth maximizes Eq. (7) subject to the Arrow-Debreu budget constraint

$$E\left[\xi_T X_T\right] \leq k, \tag{8}$$

where

$$\xi_T = e^{\left\{-\int_0^T \left(r_v+\frac{\theta_v^2}{2}\right)dv-\int_0^T \theta_v dW_v\right\}} \tag{9}$$

is the state-price density function. Optimality requires that

$$X_T^* = \frac{k}{E\left(\xi_T^\rho\right)} \xi_T^{\frac{-1}{\gamma}}, \tag{10}$$

in which $\rho = 1 - \frac{1}{\gamma}$. Substituting Eq. (10) into Eq. (7) yields the individual's ex-ante welfare level under the optimal policy, or *value function*,

$$\frac{k^{1-\gamma}}{1-\gamma} \left[E\left(\xi_T^\rho\right)\right]^\gamma . \tag{11}$$

Equation (11) corresponds to a certainty-equivalent terminal wealth – the *certain* dollar amount providing the individual with the same utility as the *random* wealth produced by the optimal life-cycle policy – equal to

$$CE(\gamma, k) = k \left[E\left(\xi_T^\rho\right)\right]^{\frac{\gamma}{1-\gamma}} . \tag{12}$$

Furthermore, the optimal fraction of wealth invested in the risky asset at time t, π_t, is

$$\pi_t^* = \frac{\theta_t}{\sigma_t \gamma} \underbrace{- \frac{\rho}{\sigma_t} \frac{E_t\left[\xi_{t,T}^\rho \int_t^T \mathcal{D}_t r_v dv\right]}{E_t\left[\xi_{t,T}^\rho\right]}}_{IR\ Hedge} \underbrace{- \frac{\rho}{\sigma_t} \frac{E_t\left[\xi_{t,T}^\rho \int_t^T \mathcal{D}_t \theta_v \left(\theta_v dv + dW_v\right)\right]}{E_t\left[\xi_{t,T}^\rho\right]}}_{MPR\ Hedge}, \tag{13}$$

where $\mathcal{D}_t$ is the Malliavin derivative operator.[5] In addition to the standard instantaneous mean-variance component, the investor's demand for the equity index contains an interest-rate (IR) hedging term and a market-price-of-risk (MPR) hedging term. The interest-rate hedge is negative whenever the spot rate, r_t, is procyclical ($\sigma_r > 0$) and it is positive whenever r_t is countercyclical ($\sigma_r < 0$), its sign being a direct consequence of the sign of the Malliavin derivative, $\mathcal{D}_t r_v$. On the other hand, the MPR hedge cannot be signed with certainty because of the stochastic component in the integral $\int_t^T \mathcal{D}_t \theta_v \left(\theta_v dv + dW_v\right)$. The two Malliavin derivatives follow

$$d\mathcal{D}_t r_v = -\kappa_r \mathcal{D}_t r_v dv + \sigma_r \mathcal{D}_t r_v dW_v, \tag{14}$$

subject to the initial condition $\mathcal{D}_t r_t = r_t \sigma_r$ and

$$d\mathcal{D}_t \theta_v = -\kappa_\theta \mathcal{D}_t \theta_v dv + \sigma_\theta \mathcal{D}_t \theta_v dW_v, \tag{15}$$

subject to the initial condition $\mathcal{D}_t \theta_t = \theta_t \sigma_\theta$.[6]

Figure 2 and Figure 3 below depict comparative statics of the hedging demands in Eq. (13) for an individual whose investment horizon T is ten years. Equations (14) and (15) serve

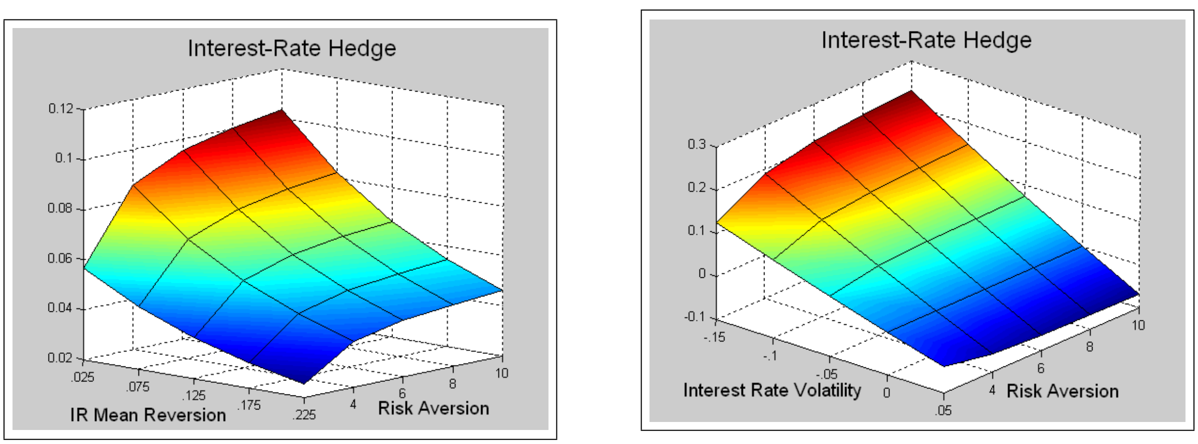
[5] Detemple, Garcia, and Rindisbacher (2003) prove the validity of Eq. (13). Treussard (2005) reviews the Malliavin methodology and emphasizes its economic intepretation while deriving portfolio policies in a stocks-and-bonds environment.

[6] The derivations of the dynamics for $\mathcal{D}_t r_v$ and $\mathcal{D}_t \theta_v$ are contained in the Appendix.

in the Monte Carlo simulations of the IR and MPR hedges.[7] The benchmark calibration for our comparative statics is the following. The stock volatility is $\sigma_t = .2$, its approximate historical long-run estimate for the U.S. market. The interest rate process is calibrated according to $r_0 = .05$, $\bar{r} = .05$, and $\sigma_r = -.05$, values consistent with U.S. historical data over the past century (e.g., Bodie, Kane, and Marcus, 2006, Chapter 5) and econometric work suggesting countercyclical tendencies in interest rates (e.g., Detemple, Garcia, and Rindisbacher, 2003). The market price of risk is calibrated according to $\theta_0 = .2$, $\bar{\theta} = .2$, and $\sigma_\theta = .1$, which is consistent with the estimates of MacKinlay (1995) and the evidence of Detemple, Garcia, and Rindisbacher (2003) that market prices of risk tend to be more volatile than interest rates. The mean reversion parameters are set at $\kappa_r = \kappa_\theta = .1$, implying half lives of five to seven years for both IR and MPR shocks. Historically, U.S. interest-rate shocks appear to be more persistent than market-price-of-risk shocks. However, in this article, equal persistence is chosen as a starting point for comparative statics. Additional numerical results with alternative calibrations, which are not reported in this article, suggest that none of the paper's conclusions are qualitatively altered by changes in parameter values.[8]

Figure 2: Interest-Rate Hedge Comparative Statics

Panel (a) Panel (b)



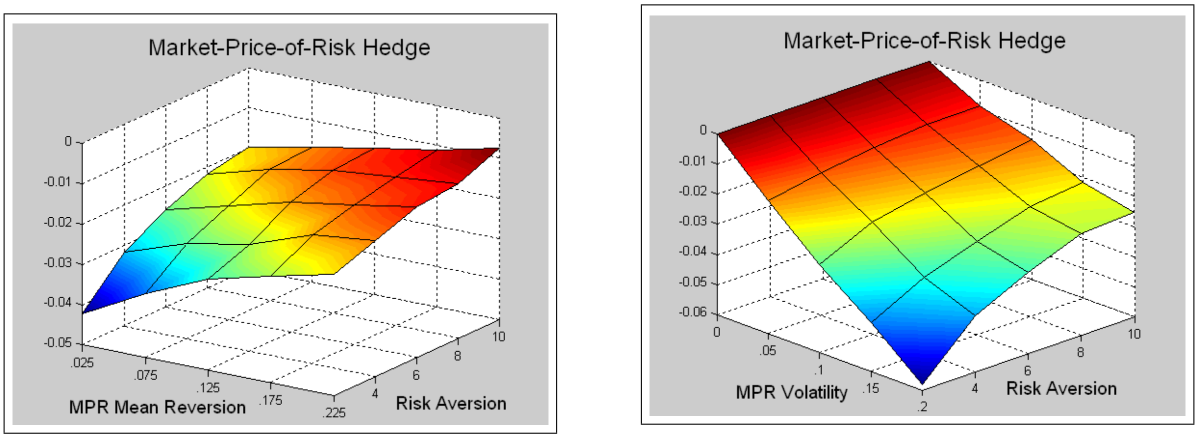
Calibration: The investment horizon is $T = 10$ years, discretized into $D = 100$ annual time steps. The benchmark calibration is: $\sigma_t = .2$; $r_0 = .05$; $\theta_0 = .2$; $\kappa_r = .1$; $\bar{r} = .05$; $\sigma_r = -.05$; $\kappa_\theta = .1$; $\bar{\theta} = .2$; $\sigma_\theta = .1$. 10,000 paths are simulated.

[7] For a textbook treatment of Monte Carlo simulations in the context of financial engineering, see Glasserman (2003).

[8] The Appendix includes Matlab code for all numerical work in this article, which can be used to replicate the results under alternative calibrations.

Figure 3: Market-Price-of-Risk Hedge Comparative Statics

Panel (a) Panel (b)



Calibration: The investment horizon is $T = 10$ years, discretized into $D = 100$ annual time steps. The benchmark calibration is: $\sigma_t = .2$; $r_0 = .05$; $\theta_0 = .2$; $\kappa_r = .1$; $\bar{r} = .05$; $\sigma_r = -.05$; $\kappa_\theta = .1$; $\bar{\theta} = .2$; $\sigma_\theta = .1$. 50,000 paths are simulated.

Figure 2 presents comparative statics for the magnitude of IR hedging demand for equities as a fraction of wealth. Panel (a) shows that, holding the interest rate volatility constant, the investor optimally demands increasing equity exposure as interest rate shocks become more persistent (smaller values of κ_r). This reflects the assumed negative correlation between equity returns and unexpected changes in interest rates. Indeed, since local adverse perturbations are more detrimental under higher persistence regimes, the individual requires a higher compensation from equity markets. In turn, beneficial interest rate shocks generate smaller gains, or even losses, from equity holdings. Similarly, as the investor becomes increasingly risk averse (larger values of γ), an enhanced hedging motive leads to higher equity holdings. Panel (b) shows that the IR hedging demand for stocks rises markedly as the economy displays significantly more volatile and contercyclical spot rates ($\sigma_r < 0$, and larger $|\sigma_r|$ values). When $\sigma_r = 0$ interest-rate risk vanishes and the hedging demand is identically equal to zero. When $\sigma_r > 0$ (procyclical spot rate movements) interest-rate risk depresses equity demand by 5% to 7% for individuals whose risk aversion γ is in the range $[4, 10]$.

Figure 3 contains comparative statics for the market-price-of-risk hedging demand for equities as a fraction of the individual's total portfolio. In most cases, the hedge against stochastic changes in MPR reduces the individual's overall demand for equities by 2% to 6% of total wealth. Panel (a) shows that, holding constant the MPR volatility at $\sigma_\theta = .1$, the investor optimally increases (in absolute terms) his short-sale hedge position when shocks become less transitory (smaller values of κ_θ). In addition, as the investor becomes more risk averse, the absolute value of the MPR hedge decreases. Furthermore, the MPR short-sale position is accentuated by higher volatilities (larger values of σ_θ). At $\sigma_\theta = 0$ there is no uncertainty and the MPR hedging demand is identically equal to zero. As a technical note, more Monte Carlo simulations – 50,000 paths instead of 10,000 paths as for the IR hedge – were used to improve the convergence properties weakened by the stochastic component in the MPR hedge, $\int_t^T \mathcal{D}_t\theta_v \left(\theta_v dv + dW_v\right)$. A smaller number of paths would generate minor non-monotonicities in the MPR-hedge surfaces.

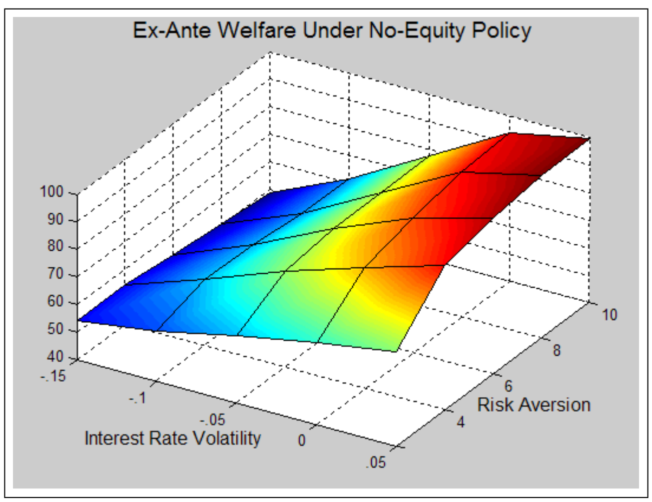
Overall, the magnitude of the IR hedge tends to exceed that of the MPR hedge so that the net departure from mean-variance efficiency is often driven by the IR hedge. As described above, this leads to larger equity positions in economies with countercyclical IR movements and reduced equity positions in environments with procyclical IR shocks.

Before describing the properties of optimal age-based portfolios in relation to first-best policies, we present Figure 4, which tabulates the welfare implications of the *no-equity investment rule*. The no-equity rule consists of continuously rolling over the individual's wealth, X_t, in short bonds, so that the wealth process obeys

$$dX_t = r_t X_t dt, \tag{16}$$

subject to the initial condition $X_0 = k$. In particular, Figure 4 depicts the ratio of the no-equity certainty equivalent to the optimal certainty equivalent, the latter's being provided in closed form by Eq. (12). This natural benchmark allows us to assess the appropriateness of the present model to conduct ex-ante welfare analysis.

Figure 4: Ex-Ante Welfare Properties of No-Equity Rule



Calibration: The investment horizon is $T = 40$ years, discretized into $D = 100$ annual time steps. The calibration for the IR process is $r_0 = \bar{r} = .05$ and $\kappa_r = .1$. 50,000 paths are simulated.

The welfare implications of the no-equity investment policy vary significantly across interest rate volatilities and individual risk aversions. Under highly countercyclical interest rate regimes ($\sigma_r = -.15$), the welfare losses incurred by individuals adopting this policy are close to 50% for all degrees of risk aversion considered. On the other hand, the no-equity rule is approximately optimal for many degrees of risk aversion under procyclical interest rates ($\sigma_r = .05$). These wide differences in welfare levels suggest that the model is appropriate to conduct meaningful ex-ante welfare analysis.

3 Optimal Age-Based Portfolio Policies

In Section 2 we provided the analytical expression for first-best portfolio policies and we analyzed the properties of the resulting hedging components. Our Appendix contains a prototypical algorithm to obtain precise values for first-best policies via Monte Carlo simulations. However, the computational burden associated with the implementation of the first-best program may be sufficiently costly for individual investors and their fiduciaries to rely on relatively simple approximations, such as age-based portfolio policies. If the welfare losses associated with such investment heuristics are not disproportionate, the trade-off between optimality and simplicity may favor the approximate policy. While this is not usually the case of one-size-fits-all age-based rules – such those embodied in TDRFs (Bodie and Treussard, 2006) – from which serious welfare losses may arise, many *optimal* age-based policies are sufficiently close to the first-best optimum for individuals to consider the trade-off.

While the optimal policy is uniquely determined by Eq. (13), there are infinitely many non-optimal investment policies available to investors. We concentrate on typical age-based portfolio policies, π_t^L, which are *linear* in age and take the form

$$\pi_t^L = \pi_0 \left(\frac{T-t}{T} \right) + \pi_T \left(\frac{t}{T} \right), \tag{17}$$

where the control variables π_0 and π_T are the initial and the terminal fractions of the portfolio invested in equities, respectively. For any policy given by Eq. (17), the individual's wealth evolves according to

$$dX_t = X_t \left[r_t + \left[\pi_0 \left(\frac{T-t}{T} \right) + \pi_T \left(\frac{t}{T} \right) \right] \sigma_t \theta_t \right] dt + X_t \left[\sigma_t \left[\pi_0 \left(\frac{T-t}{T} \right) + \pi_T \left(\frac{t}{T} \right) \right] \right] dW_t, \tag{18}$$

so that

$$X_T(\pi_0, \pi_T) = k \beta_{1,T}(\pi_0, \pi_T) \beta_{2,T}(\pi_0, \pi_T) \beta_{3,T}(\pi_0, \pi_T) \tag{19}$$

is the investor's terminal wealth, in which

$$\beta_{1,T}(\pi_0, \pi_T) = e^{\left\{ \int_0^T r_t dt \right\}} e^{\left\{ \pi_0 \int_0^T \left(\frac{T-t}{T} \right) \sigma_t \theta_t dt \right\}} e^{\left\{ \pi_T \int_0^T \left(\frac{t}{T} \right) \sigma_t \theta_t dt \right\}}, \tag{20}$$

$$\beta_{2,T}(\pi_0, \pi_T) = e^{\left\{ -\frac{\pi_0^2}{2} \int_0^T \sigma_t^2 \left(\frac{T-t}{T} \right)^2 dt \right\}} e^{\left\{ -\frac{\pi_T^2}{2} \int_0^T \sigma_t^2 \left(\frac{t}{T} \right)^2 dt \right\}} e^{\left\{ -\pi_0 \pi_T \int_0^T \sigma_t^2 \left(\frac{T-t}{T} \right) \left(\frac{t}{T} \right) dt \right\}}, \tag{21}$$

and

$$\beta_{3,T}(\pi_0, \pi_T) = e^{\left\{ \pi_0 \int_0^T \left(\frac{T-t}{T} \right) \sigma_t dW_t \right\}} e^{\left\{ \pi_T \int_0^T \left(\frac{t}{T} \right) \sigma_t dW_t \right\}}. \tag{22}$$

This representation of the investor's terminal wealth, $X_T(\pi_0, \pi_T)$, is valuable in that it permits an efficient optimization algorithm. More specifically, it allows to hold a unique set of Monte Carlo simulations constant throughout the numerical optimization, thereby reducing significantly the computational burden.

3.1 Properties of Optimal Age-Based Portfolios

In our study of optimal age-based portfolios, we consider two alternative regimes for the stock-returns precision process y_t: a stable regime ($\sigma_y = .05$) and a volatile regime ($\sigma_y = .15$). These values for σ_y represent extremes in the spectrum of parameter estimates reported in Chacko and Viceira (Table 1, 2005). A standard application of Ito's Lemma reveals that the mean-reversion parameter for precision $\kappa_y = .1$ and the long-run mean $\bar{y} = 5$ are also within the range of point estimates reported by Chacko and Viceira. Considering two precision regimes allows us to illustrate how certain modeling assumptions, which were irrelevant in the derivation of the first-best policy, do matter for second-best or optimal age-based policies.

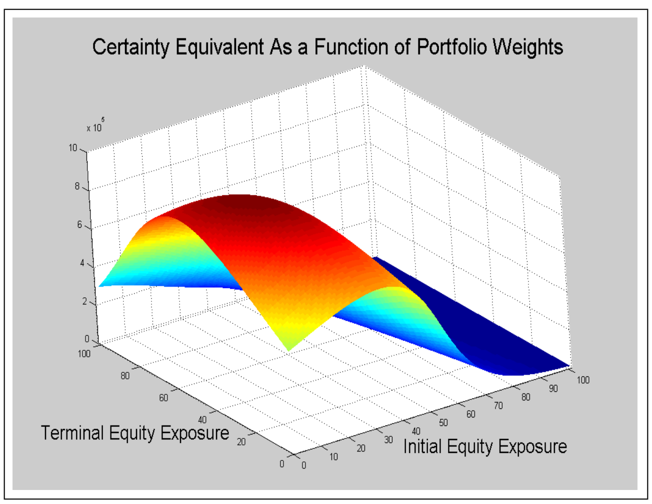
Within the class of age-based policies specified by Eq. (17), optimal portfolio policies solve the optimization problem

$$\max_{\pi_0, \pi_T} E\left[\frac{(X_T(\pi_0, \pi_T))^{1-\gamma}}{1-\gamma}\right], \tag{23}$$

where $X_T(\pi_0, \pi_T)$ is given by Eq. (19).

We solve the maximization problem in Eq. (23) numerically: we conduct a grid search over the set $\{(\pi_0, \pi_T) | \pi_0 \in [0, 1] \text{ and } \pi_T \in [0, 1]\}$, which produces interior solutions for nearly all triples $(\gamma, \sigma_r, \sigma_y)$.[9] The grid search may be followed or replaced by local numerical optimization such as a Newton-Raphson algorithm.

Figure 5: Objective Function in Equation (23)



The objective function depicted in Figure 5 is not globally concave but displays a unique maximum. In general, the objective function may be characterized by numerous local maxima, requiring added care in the numerical algorithm to ensure that these are not accidentally reached. Consequently, a global search mechanism is recommended.

Table 1 below presents the ex-ante welfare implications of following optimal age-based portfolio policies under both the stable and the volatile precision regimes. Since the welfare

[9] Tables 1 and 2 below demonstrate that the restriction to long-only positions is not severely binding.

losses associated with these policies are often negligible, the trade-off between first-best policies and simpler optimal age-based policies likely favors the approximate policy.

Table 1:
Ex-Ante Welfare of Optimal Age-Based Policy Under Stable and Volatile Precision Regime

IR Volatility	-0.15	-0.1	-0.05	0	0.05
			Risk Aversion 2		
Stable Precision	0.98266	0.973	0.96823	0.97117	0.97753
Volatile Precision	0.9317	0.9203	0.9185	0.92739	0.94358
			Risk Aversion 4		
Stable Precision	0.99352	0.98757	0.98474	0.98761	0.99215
Volatile Precision	0.96738	0.95416	0.95437	0.96644	0.98193
			Risk Aversion 6		
Stable Precision	0.98953	0.9895	0.98926	0.9925	0.99571
Volatile Precision	0.97438	0.96259	0.96431	0.97849	0.99179
			Risk Aversion 8		
Stable Precision	0.98353	0.98894	0.99116	0.99492	0.99713
Volatile Precision	0.97516	0.96527	0.96855	0.98438	0.99557
			Risk Aversion 10		
Stable Precision	0.97746	0.98758	0.99215	0.99628	0.99655
Volatile Precision	0.9735	0.96555	0.97049	0.988	0.99597

Calibration: The investment horizon is $T = 40$ years, discretized into $D = 100$ annual time steps. IR process: $r_0 = \bar{r} = .05$ and $\kappa_r = .1$. MPR process: $\theta_0 = \bar{\theta} = .2$, $\kappa_\theta = .1$, and $\sigma_\theta = .1$. Precision process: $y_0 = \bar{y} = 5$, $\kappa_y = .1$, and $\sigma_y = .15$ (Volatile Precision) or $\sigma_y = .05$ (Stable Precision). 50,000 paths are simulated.

Table 1 shows that the welfare properties of optimal age-based portfolio policies are relatively satisfactory, all reaching welfare levels in excess of 90%. We observe that increased volatility in returns precision systematically reduces ex-ante welfare, mostly by .02 to .05. It is noteworthy that the lowest welfare levels are associated with $\gamma = 2$. This result suggests that the lack of flexibility of the age-based portfolio policies is most disutilitarian to individuals who normally prefer higher equity-risk levels under first-best conditions. Indeed, while the first-best permits to revise portfolio weights at every instant as the underlying economy evolves, age-based rules force the investor to choose the initial and terminal portfolio weights at time $t = 0$ unconditionally on future states of the economy. This leads to a reduction in equity exposure due to the desire not to be locked-in during swings in volatilities. Table 2 below quantifies this effect.

Table 2: Current First-Best Optimal (FB) Portfolio, Mean-Variance (MV) Portfolio, Optimal Initial and Terminal Equity Exposure of Age-Based Policies Under Two Stock Return Precision Regimes

					Table Legend					
					Risk Aversion					
FB/MV Weights	**FB**	**MV**	**FB**	**MV**	**FB**	**MV**	**FB**	**MV**	**FB**	**MV**
Stable Precision	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**
Volatile Precision	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**	**Initial**	**Terminal**
IR Volatility	**-0.15**		**-0.1**		**-0.05**		**0**		**0.05**	
					Risk Aversion 2					
FB/MV Weights	**64.16%**	**50.00%**	**57.85%**	**50.00%**	**51.85%**	**50.00%**	**45.98%**	**50.00%**	**40.50%**	**50.00%**
Stable Precision	**65.00%**	**48.00%**	**57.00%**	**45.00%**	**48.00%**	**43.00%**	**41.00%**	**39.00%**	**35.00%**	**38.00%**
Volatile Precision	**57.00%**	**37.00%**	**49.00%**	**34.00%**	**41.00%**	**31.00%**	**34.00%**	**29.00%**	**29.00%**	**28.00%**
					Risk Aversion 4					
FB/MV Weights	**49.43%**	**25.00%**	**40.43%**	**25.00%**	**30.85%**	**25.00%**	**22.14%**	**25.00%**	**14.88%**	**25.00%**
Stable Precision	**54.00%**	**30.00%**	**42.00%**	**26.00%**	**30.00%**	**22.00%**	**19.00%**	**18.00%**	**11.00%**	**16.00%**
Volatile Precision	**48.00%**	**23.00%**	**36.00%**	**20.00%**	**25.00%**	**16.00%**	**16.00%**	**13.00%**	**9.00%**	**12.00%**
					Risk Aversion 6					
FB/MV Weights	**45.11%**	**16.67%**	**34.58%**	**16.67%**	**24.48%**	**16.67%**	**14.69%**	**16.67%**	**6.25%**	**16.67%**
Stable Precision	**50.00%**	**25.00%**	**38.00%**	**20.00%**	**24.00%**	**16.00%**	**12.00%**	**12.00%**	**4.00%**	**9.00%**
Volatile Precision	**45.00%**	**19.00%**	**33.00%**	**15.00%**	**20.00%**	**12.00%**	**10.00%**	**8.00%**	**3.00%**	**7.00%**
					Risk Aversion 8					
FB/MV Weights	**42.75%**	**12.50%**	**31.80%**	**12.50%**	**20.65%**	**12.50%**	**10.95%**	**12.50%**	**2.26%**	**12.50%**
Stable Precision	**48.00%**	**22.00%**	**35.00%**	**18.00%**	**21.00%**	**13.00%**	**9.00%**	**9.00%**	**0.00%**	**6.00%**
Volatile Precision	**44.00%**	**17.00%**	**31.00%**	**13.00%**	**18.00%**	**9.00%**	**8.00%**	**6.00%**	**0.00%**	**4.00%**
					Risk Aversion 10					
FB/MV Weights	**41.49%**	**10.00%**	**30.43%**	**10.00%**	**18.87%**	**10.00%**	**8.89%**	**10.00%**	**0.03%**	**10.00%**
Stable Precision	**47.00%**	**20.00%**	**34.00%**	**16.00%**	**20.00%**	**11.00%**	**7.00%**	**7.00%**	**0.00%**	**3.00%**
Volatile Precision	**43.00%**	**16.00%**	**30.00%**	**12.00%**	**17.00%**	**8.00%**	**6.00%**	**5.00%**	**0.00%**	**2.00%**

Calibration: The investment horizon is $T = 40$ years, discretized into $D = 100$ annual time steps. IR process: $r_0 = \bar{r} = .05$ and $\kappa_r = .1$. MPR process: $\theta_0 = \bar{\theta} = .2$, $\kappa_\theta = .1$, and $\sigma_\theta = .1$. Precision process: $y_0 = \bar{y} = 5$, $\kappa_y = .1$, and $\sigma_y = .15$ (Volatile Precision) or $\sigma_y = .05$ (Stable Precision). 50,000 paths are simulated.

Table 2 summarizes the properties of first-best optimal portfolios at time zero, mean-variance efficient portfolios at the initial date, and optimal age-based portfolio weights at the initial and at the terminal dates, under both the stable and the volatile precision regimes. Table 2 reveals that stock holdings under the first-best policy and the optimal age-based policy systematically decrease in risk aversion. For example, under the first-best, and for σ_r equal to $-.05$, initial equity exposures range from 51.85% ($\gamma = 2$) to 18.87% ($\gamma = 10$). Correspondingly, initial equity exposures decrease from 48% ($\gamma = 2$) to 20% ($\gamma = 10$) for the age-based policy under the stable regime and from 41% ($\gamma = 2$) to 17% ($\gamma = 10$) under the volatile regime.

Moreover, both under the first-best and under the age-based policy, stock holdings increase as the procyclicality of the interest rate process weakens. For an individual with a degree of risk aversion equal to four, the first-best equity exposure at time zero ranges from 14.88% to 49.43% as σ_r varies from 0.05 to -0.15, while the initial age-based portfolio weight ranges from 11% to 54% under the stable regime and from 9% to 48% under the volatile regime. Thus, not only do initial age-based portfolios display the same overall pattern as first-best portfolios but they are also always within the same order of magnitude.

While first-best terminal portfolios are stochastic, and thus unknown at the initial date, we consider initial mean-variance weights as a proxy for their counterpart at the terminal date. The use of this proxy is motivated by Eq. (13), which, when evaluated at T, reduces to the Markowitz component. However, initial mean-variance portfolios are not necessarily equal to expected mean-variance portfolios at maturity because of volatility effects and, therefore, they represent a biased estimate for age-based terminal weights.

Finally, we note that the volatility of stock returns precision – a factor that *per se* does not affect first-best portfolios – systematically affects the optimal age-based portfolio exposure to equity risk. Specifically, increased volatility of stock returns precision depresses equity holdings and often reduces them below the levels suggested by first-best optimal portfolios. This effect is caused by the pre-programmed nature of age-based portfolios and the natural lock-in effect that it engenders relative to flexible first-best policies.

4 Conclusion

In an environment with stocks and short-term debt, random changes in the risk-reward frontier produce hedging demands for equities (e.g., Detemple, Garcia, and Rindisbacher, 2003) so that portfolio policies supporting optimal life-cycle consumption are rarely mean-variance efficient. Pursuing optimal life-cycle portfolio policies is technologically feasible but it represents a significant burden for individuals and financial firms acting as fiduciaries. As a result, investors often rely on relatively simple investment heuristics, most often age-based portfolio policies that rebalance the investor's portfolio as a function of age alone. We find that (i) the welfare losses associated with these policies are often negligible, so that the trade-off between first-best policies and simpler optimal age-based policies likely favors the approximate policy, and that (ii) not only do initial age-based portfolios display the same overall pattern as first-best portfolios but they are also always within the same order of magnitude. In addition, we show that stock holdings under the first-best policy and the optimal age-based policy systematically decrease in risk aversion. Moreover, both under the first-best and under the age-based policy, stock holdings increase as the procyclicality of the interest rate process weakens. Finally, we find that increased volatility of stock returns precision depresses equity holdings and often reduces them below the levels suggested by first-best optimal portfolios. This effect is caused by the pre-programmed nature of age-based portfolios and the natural lock-in effect that it engenders relative to flexible first-best policies.

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5 Appendix A: Malliavin Calculus

5.1 Basic Rules

These formulas appear in Detemple, Garcia, and Rindisbacher (2003). They are compiled in the form of an Appendix for ease. The Brownian Motion is taken to be one-dimensional. The generalization to the multi-dimensional case is straight forward.

1. (Wiener Integral) If $h(t)$ is a non-stochastic function of time and $F(W) = \int_0^T h(s)dW(s)$, then $\mathcal{D}_t F(W) = h(t)$.
2. (Riemann Integral) If $h(t)$ is a function of time and of the path of the Brownian Motion up to time t and $F(W) = \int_0^T h(s)ds$, then $\mathcal{D}_t F(W) = \int_t^T \mathcal{D}_t h(s)ds$.
3. (Ito Integral) If $h(t)$ is a path-dependent function of time and $F(W) = \int_0^T h(s)dW(s)$, $\mathcal{D}_t F(W) = h(t) + \int_t^T \mathcal{D}_t h(s)dW(s)$.
4. (Chain Rule) If $F = (F_1, ..., F_n)'$ are Brownian functionals and $\phi(F)$ is a differentiable function of F, then $\mathcal{D}_t \phi(F) = \Sigma_{i=1}^n \phi_i'(F)\mathcal{D}_t F_i(W)$, where $\phi_i'(F)$ is the derivative with respect to the ith argument of ϕ.
5. (Commuting Operators) The Conditional Expectation operator E_s and the Derivative Operator $\mathcal{D}_v$ commute.

5.2 An Application: The Dynamics of $\mathcal{D}_t r_v$ and $\mathcal{D}_t \theta_v$

Consider

$$dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r r_t dW_t, \tag{24}$$

which in integral form gives

$$r_v = r_0 + \kappa_r \int_0^v (\bar{r} - r_s) ds + \sigma_r \int_0^v r_s dW_s. \tag{25}$$

Applying the Malliavin operator, it becomes

$$\mathcal{D}_t r_v = \mathcal{D}_t r_0 + \kappa_r \int_0^v (\mathcal{D}_t \bar{r} - \mathcal{D}_t r_s) ds + \sigma_r \int_0^v \mathcal{D}_t r_s dW_s + \sigma_r r_t, \tag{26}$$

which simplifies to

$$\mathcal{D}_t r_v = -\kappa_r \int_0^v \mathcal{D}_t r_s ds + \sigma_r \int_0^v \mathcal{D}_t r_s dW_s + \sigma_r r_t, \tag{27}$$

using rules of Malliavin calculus. In differential form, one obtains

$$d\mathcal{D}_t r_v = -\kappa_r \mathcal{D}_t r_v dv + \sigma_r \mathcal{D}_t r_v dW_v, \tag{28}$$

subject to the initial condition $\mathcal{D}_t r_t = \sigma_r r_t$. An identical derivation applies to produce $\mathcal{D}_t \theta_v$.

6 Appendix B: Matlab Code

6.1 First-Best Hedging Demands

6.1.1 Interest-Rate Hedge

```
function [IRH]=IRhedge(GAM,KR,SR)
    D=100;
    P=10000;
    W=randn(P,10*D);
    gam=GAM;
    kr=KR;
    rb=.05;
    sr=SR;
    kth=.1;
    thb=.2;
    sth=.1;
    %%
    RB=rb*ones(P,1);
    THB=thb*ones(P,1);
    SQB=1*ones(P,1);
    SDRB=0*ones(P,1);
    DRB=sr*rb*ones(P,1);
    for i=1:10*D
    RE=RB+kr*(rb-RB)*(1/D)+sr*RB.*W(1:P,i)*((1/D)^.5);
    THE=THB+kth*(thb-THB)*(1/D)+sth*THB.*W(1:P,i)*((1/D)^.5);
    SQE=SQB-SQB.*(RB*(1/D)+THB.*W(1:P,i)*((1/D)^.5));
    DRE=DRB-kr*DRB*(1/D)+sr*DRB.*W(1:P,i)*((1/D)^.5);
    SDRE=SDRB+DRB*(1/D);
    RB=RE;
    THB=THE;
    SQB=SQE;
    DRB=DRE;
    SDRB=SDRE;
    end
    rho=1-1/gam;
    IRH=-(rho/.2)*((SQE.^rho)'*SDRE)/((SQE.^rho)'*ones(P,1));
```

6.1.2 Market-Price-of-Risk Hedge

```
function [THH]=THhedge(GAM,KTH,STH)
    D=100;
    P=50000;
    W=randn(P,10*D);
    gam=GAM;
    kr=.1;
    rb=.05;
    sr=-.05;
    kth=KTH;
    thb=.2;
    sth=STH;
    %%
    RB=rb*ones(P,1);
    THB=thb*ones(P,1);
    SQB=1*ones(P,1);
    DTHB=thb*sth*ones(P,1);
    SDTHB=0*ones(P,1);
    for i=1:10*D
    RE=RB+kr*(rb-RB)*(1/D)+sr*RB.*W(1:P,i)*((1/D)^.5);
    THE=THB+kth*(thb-THB)*(1/D)+sth*THB.*W(1:P,i)*((1/D)^.5);
    SQE=SQB-SQB.*(RB*(1/D)+THB.*W(1:P,i)*((1/D)^.5));
    DTHE=DTHB-kth*DTHB*(1/D)+sth*DTHB.*W(1:P,i)*((1/D)^.5);
    SDTHE=SDTHB+THB.*DTHB*(1/D)+DTHB.*W(1:P,i)*((1/D)^.5);
    RB=RE;
    THB=THE;
    SQB=SQE;
    DTHB=DTHE;
    SDTHB=SDTHE;
    end
    rho=1-1/gam;
    THH=-(rho/.2)*((SQE.^rho)'*SDTHE)/((SQE.^rho)'*ones(P,1));
```

6.2 Optimal Age-Based Portfolio Policies

6.2.1 Components of Wealth Process

```
function [BRO,I1O,I2O,I3O,I4O,I5O,I6O,I7O,SQO]=WelfAn(KR,SR,KTH,STH,P)
    D=100;
    YRS=40;
    %IR Process
    kr=KR;
    rb=.05;
    sr=SR;
    %Theta Process
    kth=KTH;
    thb=.2;
    sth=STH;
    %Y Process
    ky=.1;
    yb=5;
    sy=.15;
    %%
    RB=rb*ones(P,1);
    THB=thb*ones(P,1);
    SQB=1*ones(P,1);
    YB=yb*ones(P,1);
    BRB=1*ones(P,1);
    I1B=0*ones(P,1);
    I2B=0*ones(P,1);
    I3B=0*ones(P,1);
    I4B=0*ones(P,1);
    I5B=0*ones(P,1);
    I6B=0*ones(P,1);
    I7B=0*ones(P,1);
    WB=randn(P,1);
    for i=1:YRS*D
    WE=randn(P,1);
    RE=RB+kr*(rb-RB)*(1/D)+sr*RB.*WB*((1/D)^.5);
    THE=THB+kth*(thb-THB)*(1/D)+sth*THB.*WB*((1/D)^.5);
    SQE=SQB-SQB.*(RB*(1/D)+THB.*WB*((1/D)^.5));
    YE=YB+ky*(yb-YB)*(1/D)+sy*YB.*WB*((1/D)^.5);
    BRE=BRB+BRB.*RB*(1/D);
    I1E=I1B+((YRS*D+1-i)/(YRS*D))*(THB./YB)*(1/D);
    I2E=I2B+((i-1)/(YRS*D))*(THB./YB)*(1/D);
    I3E=I3B+(((YRS*D+1-i)/(YRS*D))^2)*(YB.^(-2))*(1/D);
    I4E=I4B+(((i-1)/(YRS*D))^2)*(YB.^(-2))*(1/D);
    I5E=I5B+((i-1)/(YRS*D))*((YRS*D+1-i)/(YRS*D))*(YB.^(-2))*(1/D);
    I6E=I6B+((YRS*D+1-i)/(YRS*D))*(WB./YB)*((1/D)^.5);
    I7E=I7B+((i-1)/(YRS*D))*(WB./YB)*((1/D)^.5);
    RB=RE;
    THB=THE;
    SQB=SQE;
    YB=YE;
    BRB=BRE;
    I1B=I1E;
    I2B=I2E;
    I3B=I3E;
    I4B=I4E;
    I5B=I5E;
    I6B=I6E;
    I7B=I7E;
    WB=WE;
    end
    BRO=BRE;
    I1O=I1E;
    I2O=I2E;
    I3O=I3E;
    I4O=I4E;
    I5O=I5E;
    I6O=I6E;
    I7O=I7E;
    SQO=SQE;
```

6.2.2 Grid Construction

```
P=50000;
gam=[2:2:10];
G=length(gam);
sr=[-.15:.05:.05];
S=length(sr);
BRO=zeros(P,S);
I1O=zeros(P,S);
I2O=zeros(P,S);
I3O=zeros(P,S);
I4O=zeros(P,S);
I5O=zeros(P,S);
I6O=zeros(P,S);
I7O=zeros(P,S);
SQO=zeros(P,S);
pi0=[0:.01:1];
PI0=length(pi0);
piT=[0:.01:1];
PIT=length(piT);
CE1=zeros(PI0,PIT);
pi1=zeros(G,S);
pi2=zeros(G,S);
CE=zeros(G,S);
for s=1:S [BRO(1:P,s),I1O(1:P,s),I2O(1:P,s),I3O(1:P,s),I4O(1:P,s),I5O(1:P,s),I6O(1:P,s),I7O(1:P,s),SQO(1:P,s)]
=WelfAn(.1,sr(s),.1,.1,P);
end
```

6.2.3 Grid Optimization

```
g=5;
s=5;
rho=1-1/gam(g);
CEOPT=100000*((SQO(1:P,s).^rho)'*ones(P,1)/P)^(-1/rho);
for a=1:PI0
for b=1:PIT
TW=100000*BRO(1:P,s).*exp(pi0(a)*I1O(1:P,s)).*exp(piT(b)*I2O(1:P,s)).*exp((-(pi0(a)^2)/2)*I3O(1:P,s))
.*exp((-(piT(b)^2)/2)*I4O(1:P,s)).*exp((-(pi0(a)*piT(b)))*I5O(1:P,s)).*exp(pi0(a)*I6O(1:P,s)).*exp(piT(b)*I7O(1:P,s));
CE1(a,b)=1000000*(((TW.^(1-gam(g)))'*ones(P,1)/P)^(1/(1-gam(g))))/CEOPT;
end
end
[BESTpiT,INDICBESTpiT]=max(CE1,[],2);
[BESTpi0,INDICBESTpi0]=max(BESTpiT);
pi1(g,s)=pi0(INDICBESTpi0);
pi2(g,s)=piT(INDICBESTpiT(INDICBESTpi0));
CE(g,s)=BESTpi0/1000000;
```