Discounting and altruism to future decision-makers*

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Abstract

Can a generation's discounting of future generations' consumption utilities be interpreted as "pure" altruism towards future generations, that is, a concern that these are better off according to their likewise forward-looking preferences? It turns out that the answer is positive for many but not all discount functions used in the economics literature. In particular, traditional exponential discounting is consistent with altruism towards the next generation, and "hyperbolic" discounting of the form used by Phelps and Pollak (1968) and Laibson (1997) is consistent with altruism to all future generations. More generally, we provide one sufficient and one necessary condition for a discount function to be consistent with altruism, and we establish a recursive functional equation which defines a one-to-one mapping between discounting and altruism.

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1 Introduction

Many economics issues concern sequences of intertemporal decisions. In models of such situations, the successive decisions are usually taken either by one and the same individual or by successive generations in a dynasty. We here analyze a widely used class of time preferences for such analyses, namely preferences that can be represented by an additively separable utility function, written as the sum of discounted instantaneous utilities. Hence, the decision-maker in each period has some concern for future consumption - be it his or her own or that of future generations - and that concern is represented by a discount function. We here ask whether such intertemporal preferences are consistent with "pure" altruism towards these future decision-makers. By pure altruism we mean a concern that others are better off according to their preferences. By contrast, a decision-maker is sometimes called "paternalistically" or "impurely" altruistic if he or she cares directly about others' consumption or his or her gifts or bequests to others ("warm glow effects"), without full regard to all factors of relevance for others' well-being.[1]

Pure altruism brings in a potential interdependence among economic agents' utilities: one agent may care about another agent who in his or her turn cares about the first agent, etc. For example, let x_i be the consumption of individual i, for $i = 1, 2, 3$, and suppose preferences over allocations $x = (x_1, x_2, x_3)$ are of the following "pure altruism" form:

$$\begin{cases} U_1(x) = u(x_1) + \alpha U_2(x) \\ U_2(x) = u(x_2) + \beta U_1(x) + \gamma U_3(x) \\ U_3(x) = u(x_3) \end{cases} \tag{1}$$

where u is the common subutility function for own consumption, $\alpha \geq 0$ is a measure of 1's altruism towards 2, and likewise for $\beta \geq 0$ and $\gamma \geq 0$. Note that agent 1 indirectly cares about agent 3's consumption, via his or her altruism towards 2 and 2's altruism towards 3.

If $\alpha\beta \neq 1$, then this system of equations uniquely determines all individuals' total utilities as functions of the allocation x. We then have:

$$\begin{cases} U_1(x) = \frac{1}{1-\alpha\beta}\left[u(x_1) + \alpha u(x_2) + \alpha\gamma u(x_3)\right] \\ U_2(x) = \frac{1}{1-\alpha\beta}\left[\beta u(x_1) + u(x_2) + \gamma u(x_3)\right] \\ U_3(x) = u(x_3) \end{cases} \tag{2}$$

While preference in equations (1) are explicit in terms of pure altruism, it may appear in equations (2) that agent 1, for example, cares about 2's consumption but not about 2's caring about 1 and 3. Agent 1 may thus not appear to be purely altruistic towards 2. However, as we just saw, equations (2) are in fact consistent with pure altruism. Agent 1 does care about 2's caring for 3 - that is why 3's consumption appears in the expression for $U_1(x)$ in equations (2).

[1] For example Ray (1987) and Hori (2001) use the term "paternalistic altruism," while Andreoni (1989) use the term "impure altruism."

Recognition of the utility interdependences that may arise among altruistic economic agents goes back at least to Edgeworth (1881), who examined the effects of pure altruism on the contract curve in a two-person exchange economy. With X and Y denoting the two persons in question, Edgeworth wrote that "we might suppose that the object which X (whose own utility is P), tends - in a calm, effective moment - to maximize, is not P, but $P + \lambda\Pi$; where λ is a *coefficient of effective sympathy* [Edgeworth's italics]. And similarly Y - not of course while rushing to self-gratification, but in those regnant moments which characterise an ethical 'method' - may propose to himself as an end $\Pi + \mu P$." (op. cit. p. 53).[2]

Winter (1969), Becker (1974, 1981), Barro (1974), Arrow (1981), Pearce (1983), Bernheim, Schleifer and Summers (1985), Bernheim (1986), Ray (1987), Kimball (1987), Barro and Becker (1988), Bernheim and Bagwell (1988), Bernheim and Stark (1988), Lindbeck and Weibull (1988), Andreoni (1989), Abel and Bernheim (1991) and Bergstrom (1999) analyze in a wide range of alternative settings pure and impure altruism among economic agents.[3] For instance, the Ricardian equivalence theorem, as formulated by Barro (1974), leans heavily on pure altruism of the type studied here: each generation cares about the next generation's total utility, which in turn depends on the following generation's total utility, in an infinite chain (see Bernheim (1987) for a survey, and Andreoni (1989) for a model that allows for both pure and impure altruism). By contrast, other researchers have analyzed models of "impure" altruism, as in Lane and Mitra (1981) and Leininger (1986), where each generation cares about the next generation's consumption, or as in Andreoni (1989), where individuals care both about others' total utility and about their own contribution to others' consumption. The present analysis identifies conditions under which paternalistic altruism for others' consumption is - or is not - behaviorally equivalent to pure altruism. In his study of pure intergenerational altruism, Ray (1987) called for precisely such an investigation, and we hope to have shed some light on this issue.[4]

More exactly, we here consider a sequence of decision-makers $\tau = 0, 1, 2...$, each of whom takes some action in the corresponding period. These decision-makers could be successive generations or the successive incarnations of one individual, and the actions could be consumption-savings decisions (with or without commitment). The decision-makers are assumed to have preferences of the following forward-looking and additively separable form:

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{\infty} f(t)\, u(x_{\tau+t}) \quad . \tag{3}$$

Here $u(x_{\tau+t})$ is the instantaneous consumption utility t periods later and $f(t)$ the discount factor attached to that period. Such representations of time preferences are commonplace

[2] See Collard (1975) for an analysis of Edgeworth's treatment of altruism.

[3] See section 5 below for a discussion of the most closely related papers.

[4] "The representation of non-paternalistic functions in paternalistic form has ... been the subject of limited attention .... a systematic analysis of the relationship between these two frameworks is yet to be written, and appears to be quite a challenge, especially for models with an infinite horizon." (Ray, 1987, pp. 113-114)

in the economics literature. For example, in the seminal paper by Samuelson (1937), $f(t) = \delta^t$ for some $\delta \in (0,1)$, while in Phelps and Pollack (1968) and Laibson (1997) $f(t) = \beta\delta^t$ for some $\beta \in (0,1]$ and $\delta \in (0,1)$.[5] We will refer to the first case as *exponential discounting* and call the second *quasi-exponential discounting*.[6]

The function U_τ is usually taken to be decision-theoretic: it determines the choice made by the decision-maker in period τ, with due regard to the presence or absence of commitment possibilities. From a normative viewpoint, $U_\tau(x)$ represents the *welfare* of decision-maker τ: the higher this function value is, the "better off" is that decision-maker. The welfare or "total utility" of a decision-maker so defined does thus not stem only from current consumption but also from (the anticipation of) consumption in future periods.

We ask whether intertemporal preferences of the form (3) are consistent with the notion that each decision-maker is purely altruistic towards future decision-makers. More precisely, given preferences represented in the form (3), does there exist a nonnegative function a such that

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{\infty} a(t)\, U_{\tau+t}(x) \ ? \tag{4}$$

We will call such a function a an *altruism-weight function*, and refer to $a(t)$ as the *altruism weight* that the current decision-maker places on the welfare of his or her t'th successor.[7] In Edgeworth's (1881) words, $a(t)$ is the current decision-maker's *coefficient of effective sympathy* for the decision-maker t periods later.

We find that a function a satisfying equation (4) always exists and is unique, and that the function values can be successively determined from a relatively simple system of recursive equations (proposition 1). However, there is no guarantee, *a priori*, that all function values $a(t)$ are non-negative. A negative function value $a(t)$ means that every decision-maker is "spiteful" to the decision-maker t periods later, that is, he or she prefers allocations that make that decision-maker *worse* off (in terms of that decision-maker's

[5] In their intergenerational models, Buiter and Carmichael (1984) and Burbidge (1984) study utility functions of a similar form, but add one more term for preceding generation:

$$U_\tau(x) = u(x_\tau) + \gamma u(x_{\tau-1}) + \sum_{t=1}^{\infty} \delta^t u(x_{\tau+t}) \ .$$

Kimball (1987) and Ray (2002) analyze situations when *all* past consumption utilities are included.

[6] Such discount functions are frequently called hyperbolic or quasi-hyperbolic. We prefer the present terminology since this class of functions contain exponential (but not hyperbolic) functions as a special case.

[7] Altruistic concern for *earlier selves* seems irrelevant since earlier selves by definition are not around at the time of the decision in question, so to speak. The same holds for *earlier generations*, unless (some of) these are are still alive, as in overlapping genertaions models. Kimball (1987) analyses such a model, see section 5. Note, however, that if decision makers (successive generations or the successive incarnations of one individual) derive utility from past consumption, then even a strictly forward-looking altruistic decision maker (as in equation (4)) may rationally "invest" in the memories of future generations or his future selves. However, that falls outside the scope of the present study, see Kimball (1987) and Ray (2002) for such analyses.

preferences).

It thus seems desirable to identify conditions that rule out this possibility. It turns out that a sufficient condition for this is that all discount factors be positive and that the ratio $f(t+1)/f(t)$ between successive discount factors be non-decreasing with "time distance" t (proposition 2). This ratio reflects the decision-maker's *patience* concerning events t time units ahead in time: it is the decision maker's discount factor from period $\tau + t$ to the next, as viewed from any period τ. The condition thus requires decision makers' patience to be non-decreasing with time distance - a property that seems to conform with available empirical evidence (see for example Frederick et al. (2001)). This condition is met by standard exponential discounting; then $f(t+1)/f(t) = \delta$ for all t. More generally, it is satisfied by all quasi-exponential discount functions, and by some functions of the hyperbolic form $f(t) = 1/(\lambda + \mu t)$ used in psychology (see for example Ainslie (1992)). We also show that the condition is closely related to, but distinct from, convexity of the discount function (proposition 3).

Exponential discounting, the canonical model in economics, corresponds to one-period pure altruism: each decision-maker attaches a positive weight to his or her successor's welfare and zero weight to all other decision-makers. Such altruism have been much used in intergenerational macroeconomic models, see Barro (1974) for pioneering work. We show that exponential discounting is a boundary case in the following sense: a necessary condition for altruism towards future decision-makers is $f(t) \geq f(1)^t$ for all t. In other words, future periods should not be more heavily discounted than what is obtained by exponential extrapolation of the discounting from the present to the next period (proposition 4). Also this necessary condition seems to agree with empirical evidence. However, the condition is evidently violated by all discount functions that place positive weight on the near future but zero weight on some more distant period.

We also find that quasi-exponential discounting, in the above mentioned (β, δ)-form, corresponds to altruism of a form which is itself sort of quasi-exponential: all altruism weights are the product of a common constant factor and an exponential factor. The case $\beta = 1/2$ turns out to play a special role. For such discounting, the altruism weights are strictly exponential. By contrast, an example of discount functions that imply "spite" towards future decision-makers is obtained when the quasi-exponential discounting scheme is altered in such a way that the parameter β applies only to periods $t \geq 2$ (rather than to all periods $t \geq 1$). This can be seen to violate the necessary condition mentioned above. Moreover, hyperbolic discounting of the type used in the psychology literature turns out in many (but not all) cases to be consistent with pure altruism.

Another finding is that truncated exponential altruism, that is, exponentially declining altruism weights attached to the next $T > 1$ decision-makers, followed by zero altruism weights thereafter, is equivalent to discount functions generated from generalized Fibonacci sequences, with $T = 2$ corresponding to the classical Fibonacci sequence.[8] It turns out that the induced discount functions need not be monotonic with respect to time

[8] The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, ..., each term, after the first two, being the sum of the two preceding terms.

distance.

The remainder of the paper is organized as follows. The model is set up in section 2, and section 3 establishes the mentioned results. Section 4 analyzes a few examples, section 5 discusses related work, and section 6 concludes. An appendix at the end of the paper provides alternative proofs (using the generating-function approach) for some claims made in section 4.

2 Model

Consider a sequence of decision-makers $\tau = 0, 1, 2, ...$, where each decision-maker τ is to make some choice in period τ (and may or may not live in other periods). There may be finitely or infinitely many such decision-makers. In order to save on notation and treat the most challenging case, we henceforth presume an infinite sequence of decision-makers.[9]

Suppose, thus, that in each time period $t \in \mathbb{N} = \{0, 1, 2, ...\}$ there is a single decision-maker who takes some action $x_t \in X$, where X is the set of alternatives available in each period t. For the sake of concreteness, we will call action x_t *consumption* in period t, and by a *consumption stream* x we will mean an infinite sequence of consumption vectors x_t, $x = (x_0, x_1, ...) \in X^\infty$.

Each decision-maker τ has preferences $\succcurlyeq_\tau$ over consumption streams $x \in X^\infty$. A *preference profile* $\succcurlyeq$ for the sequence of decision-makers is thus a sequence $\langle \succcurlyeq_\tau \rangle_{\tau \in \mathbb{N}}$ of preferences, one for each decision-maker τ. We here focus on preference profiles $\langle \succcurlyeq_\tau \rangle_{\tau \in \mathbb{N}}$ for which there exist functions $U_\tau : X^\infty \to \mathbb{R}$, one for each decision-maker $\tau \in \mathbb{N}$, such that $x \succcurlyeq_\tau y$ if and only if $U_\tau (x) \geq U_\tau (y)$, where

$$U_\tau (x) = \sum_{t=0}^{\infty} f(t) u (x_{\tau+t}) \tag{5}$$

for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}_+$ with $f(0) = 1$. We will call $u(x_s)$ the *instantaneous (sub)utility* from consumption in period s, and $f(t)$ the *discount factor* that each decision-maker attaches to the instantaneous utility t periods later.[10] Hence, each decision-maker uses the same instantaneous subutility function u and discount function f. Koopmans (1960), Koopmans et al. (1964), Burness (1973) and Weibull (1985) develop axiom systems for such utility representations.

We will say that a sequence $\langle U_\tau \rangle_{\tau \in \mathbb{N}}$ of utility functions (5) is *consistent with* (additively separable) *pure altruism* if for all $\tau \in \mathbb{N}$ and $x \in X^\infty$,

$$U_\tau (x) = u (x_\tau) + \sum_{t=1}^{\infty} a (t) U_{\tau+t} (x) \ , \tag{6}$$

[9] All results are easily adapted to the case of a finite number of decision makers.

[10] It is behaviorally plausible that in real life the well-being of decision makers depends in part on their memories of past own actions or actions taken by preceding generations. See Kimball (1987) and Ray (2002) for studies of such preferences.

for some $a : \mathbb{N}_+ \to \mathbb{R}_+$, where $\mathbb{N}_+ = \{1, 2, ...\}$.[11] Here $a(t)$ will be called the *altruism weight* that the decision-maker places on the welfare or total utility of the decision-maker t periods later.

3 Results

Under what conditions is a sequence $\langle U_\tau \rangle_{\tau \in \mathbb{N}}$ defined in equation (5) consistent with pure altruism, and if it is, what are the implied altruism weights? A key result for answering this and related questions is the following observation:

Proposition 1 *If $\langle U_\tau \rangle$ satisfies equation (5) for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}$, then $\langle U_\tau \rangle$ also satisfies equation (6), where $a : \mathbb{N}_+ \to \mathbb{R}$ is the unique solution to*

$$a(t) = \begin{cases} f(1) & \text{if } t = 1 \\ f(t) - \sum_{s=1}^{t-1} f(t-s)a(s) & \text{if } t > 1 \end{cases} . \tag{7}$$

Proof: Suppose $\langle U_\tau \rangle$ satisfies equation (5) for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}$ with $f(0) = 1$. Let $a : \mathbb{N}_+ \to \mathbb{R}$ be defined by (7). Then

$$f(t) = \sum_{s=1}^{t} a(s) f(t-s) \qquad \forall t \in \mathbb{N}_+ . \tag{8}$$

Hence,

$$\begin{aligned} U_\tau(x) &= u(x_\tau) + \sum_{t=1}^{\infty} \sum_{s=1}^{t} a(s) f(t-s) u(x_{\tau+t}) = \qquad (9) \\ &= u(x_\tau) + \sum_{s=1}^{\infty} a(s) \left[\sum_{t=s}^{\infty} f(t-s) u(x_{\tau+t}) \right] \\ &= u(x_\tau) + \sum_{s=1}^{\infty} a(s) \left[\sum_{k=0}^{\infty} f(k) u(x_{\tau+s+k}) \right] = u(x_\tau) + \sum_{s=1}^{\infty} a(s) U_{\tau+s}(x) \end{aligned}$$

Since the resulting equation holds for all τ, this proves the claim.[12] **End of proof.**

[11] A negative weight attached to another individual's welfare expresses "spite" rather than "altruism." Such welfare weights appear pathological in the present context. That is the reason behind the non-negativity constraint on a. We do not deny that the excluded possibility may sometimes be psychologically relevant, but it appears not to be typical for consumers facing conventional decision problems.

[12] It is easily verified that the change of order of summation in this derivation is justified. For by assumption U_τ is a real-valued function and hence the sum in the first line converges to some $\lambda \in \mathbb{R}$. Hence, for every $\varepsilon > 0$ there exists a T_ε such that summation from $t = 1$ up to T_ε brings this partial sum within ε from λ. Given T_ε, the order of summation may be changed and the final expression is obtained by letting $\varepsilon \to 0$.

Clearly, the discount function f may be recovered from the altruism function a from equations (7):

$$f(t) = \begin{cases} a(1) & \text{if } t = 1 \\ \sum_{s=0}^{t-1} a(t-s)f(s) & \text{if } t > 1 \end{cases} \tag{10}$$

This recursive equation system, which uniquely determines f from a (recall the normalization $f(0) = 1$), essentially states that the discount factor $f(t)$ attached to consumption t periods later can be computed as that period's contribution to the altruism-weighted welfare of all interim period decision-makers. For example, the discount factor $f(2)$ attached to consumption two periods ahead equals the altruism weight placed on the decision-maker two periods ahead, $a(2)$, plus the altruism weight placed on the decision-maker one period ahead times that decision-maker's discounting over one period, $a(1) f(1)$.

It is immediate from equation (10) that if the function a is non-negative, so is f. However, as pointed out above, and seen in equation (7), a may well take negative values. For example, (7) gives $a(2) = f(2) - f^2(1)$, so in order for $a(2)$ to be negative it suffices that $f(2) < f^2(1)$. In particular, this is the case when $f(1) > 0$ and $f(2) = 0$, as in the intergenerational analyses in Lane and Mitra (1981) and Leininger (1986), where each generation's welfare is a function of its own consumption and that of its immediate descendant.[13] Another example of negative welfare weights $a(t)$ is when f is of the form $f(0) = 1$ and $f(t) = 1/(0.5 + t)$ for all $t > 0$; then $f^2(1) > f(2)$. A third example is $f(t) = 1/(1 + t^2)$ for all t, yielding $f^2(1) = 1/4$ and $f(2) = 1/5$. A fourth example is when the parameter β in the quasi-exponential (β, δ)-representation kicks in with one period's delay, that is, when $f(0) = 1$, $f(1) = \delta$ and $f(t) = \beta\delta^t$ for all $t \geq 2$. Then clearly $a(2) = (\beta - 1)\delta^2 < 0$ whenever $\beta < 1$.

For what class of discount functions f can one then guarantee that all welfare weights $a(t)$ are nonnegative? It turns out that a sufficient condition for this is that f be positive and that the associated "patience" function $g : \mathbb{N}_+ \to \mathbb{R}$, defined by $g(t) = f(t)/f(t-1)$, should be non-decreasing. This condition is clearly met by all quasi-exponential discount functions with $\beta \leq 1$. The proof of this result, based on our initial and more restrictive conjecture, was kindly provided by Ulf Persson.

Proposition 2 *Suppose $f > 0$, and let g be the associated patience function. If g is non-decreasing, then $a \geq 0$. If g is strictly increasing, then $a > 0$.*

Proof: Suppose first that g is non-decreasing. Since $a(1) = f(1)$, we have $a(1) > 0$. Suppose $a(s) \geq 0 \quad \forall s < t$. By equation (7):

[13] More generally, it is easy to verify that equation (7) gives $a(t) = (-1)^{t+1} f(1)^t$ for all $t \geq 1$ if $f(t) = 0$ for all $t > 1$.

$$\begin{aligned} a(t) &= f(t) - f(1)a(t-1) - \sum_{s=1}^{t-2} a(s)f(t-s) \qquad (11) \\ &= g(t)f(t-1) - f(1)a(t-1) - \sum_{s=1}^{t-2} g(t-s)a(s)f(t-s-1) \\ &\geq g(t)\left[f(t-1) - \sum_{s=1}^{t-2} a(s)f(t-s-1)\right] - f(1)a(t-1) \\ &= g(t)a(t-1) - f(1)a(t-1) \\ &= [g(t) - f(1)]\,a(t-1) \geq 0\ , \end{aligned}$$

where equation (7) was used again for $a(t-1)$, and where the last inequality follows from the fact that, by assumption, g is non-decreasing with $g(1) = f(1)$.

Secondly, suppose that g is strictly increasing. If $a(s) > 0\ \forall s \leq t$, then the same reasoning as above leads to $a(t) > [g(t) - g(1)]\,a(t-1) > 0$. **End of proof.**

We note that if both f and a are non-negative - as we just saw they are under the hypothesis of proposition 2 - then the altruistic weight $a(t)$ attached to the decision-maker in any period t cannot exceed the discount factor $f(t)$ attached to consumption in that period, by equation (7): $0 \leq a(t) \leq f(t)$ for all positive integers t. In particular, if $f(t)$ goes to zero as t goes to infinity, then so does $a(t)$.

Another observation is that the hypothesis in proposition 2 is closely related to, but distinct from, the condition that f be convex. We here call a function $f : \mathbb{N} \to \mathbb{R}$ convex if its piece-wise affine extension to $\mathbb{R}_+ \supset \mathbb{N}$ is convex.[14] It is easy to see that convexity is not sufficient for the altruism weights to be nonnegative. For example, any convex function with $f(0) = 1$, $f(1) = 0.2$ and $f(2) = 0.02$ has $a(2) = -0.02$. However, a seemingly slight strengthening of the hypothesis in proposition 2 effectively requires f to be convex. To see this, let $F : \mathbb{R}_+ \to \mathbb{R}$ be twice differentiable with $F(0) = 1$, $F > 0$ and $F' \leq 0$, and let $G : \mathbb{R}_+^2 \to \mathbb{R}$ be defined by

$$G(t,s) = \frac{F(t+s)}{F(t)}\ . \qquad (12)$$

Clearly, the restriction of F to $\mathbb{N}$ is a discount function f, and the associated patience function g satisfies $g(t+1) = G(t,1)$ for all $t \in \mathbb{N}$. In particular, g is non-decreasing - as required in proposition 2 - if G_1', the partial derivative of G with respect to its first argument, is nonnegative. Under the latter, somewhat more stringent hypothesis, F, and hence also f, are convex (and thus, by proposition 2, a is nonnegative):

Proposition 3 *If $F : \mathbb{R}_+ \to \mathbb{R}$ is twice differentiable with $F(0) = 1$, $F > 0$, $F' \leq 0$, and $G_1' \geq 0$, then f is convex and g non-decreasing.*

[14] For all reals t between any two integers k and $k+1$, let $F(t) = (t-k)\,f(k+1) + (k+1-t)\,f(k)$, thus defining a piece-wise affine extension of f.

Proof: By differentiation,

$$G_1'(t,s) \geq 0 \qquad \Leftrightarrow \qquad F'(t+s)\,F(t) \geq F(t+s)\,F'(t) \ . \tag{13}$$

Suppose that $G_1'(t,s) \geq 0$ for all $s > 0$. If $F'(t) = 0$, then $F'(t+s) \geq 0$ for all $s > 0$, and hence $F''(t) \geq 0$. If instead $F'(t) < 0$, then

$$\begin{aligned} F''(t) &= \lim_{s \downarrow 0} \frac{F'(t+s) - F'(t)}{s} \geq \lim_{s \downarrow 0} \frac{1}{s}\left[\frac{F(t+s)}{F(t)} - 1\right] F'(t) \\ &= \lim_{s \downarrow 0}\left[\frac{F(t+s) - F(t)}{s}\right]\frac{F'(t)}{F(t)} = \frac{[F'(t)]^2}{F(t)} > 0 \ . \end{aligned} \tag{14}$$

End of proof.

We next turn to the task of identifying a *necessary* condition for all altruism weights to be nonnegative. For this purpose, let us first briefly consider the special case when the discount function f is exponential: $f(t) = \delta^t$ for all t, for some $\delta \in (0,1)$. It is not difficult to verify that equation (7) then gives $a(1) = \delta$ and $a(t) = 0$ for all integers $t > 1$. To see this, first note that equation (7) gives $a(1) = \delta$ and $a(2) = 0$. Suppose, thus, that $a(1) = \delta$ and $a(s) = 0$ for all $s = 2, .., t-1$. Then (7) yields

$$a(t) = \delta^t - \sum_{s=1}^{t-1} \delta^{t-s} a(s) = \delta^t - \delta^{t-1}\delta = 0 \ . \tag{15}$$

By induction, $a(t) = 0$ for all positive integers t.[15]

In other words, exponential discounting is equivalent to altruism to the next decision-maker only. It is easily verified that the converse also holds: by equation (10), one readily obtains that one-period altruism implies exponential discounting: if $a(1) = \alpha \geq 0$ and $a(t) = 0$ for all $t > 1$, then $f(t) = \alpha^t$ for all t. This is not surprising: if each decision-maker attaches an altruistic weight α to the next decision-maker, and zero weight to all others, then the contribution to current welfare from the instantaneous utility t periods ahead should be the product of how much the current decision-maker cares about the next decision-maker, how much the next decision-maker cares about his successor, and so on up to the t'th decision maker.

These observations concerning exponential discounting can be used to establish a necessary condition for (nonnegative) altruism in general, namely, that the discount function should not decline faster than exponentially, as compared with its decline from the current period to the next:

Proposition 4 *If $a \geq 0$, then $f(t) \geq [f(1)]^t$ for all t.*

Proof: Suppose $a \geq 0$, and let f be the associated discount function, as defined in (10). Let $\alpha = a(1)$, and let a^* be the altruism-weight function defined by $a^*(1) = \alpha$ and $a^*(t) = 0$ for all $t > 1$. We know from the above observation that the discount function f^* associated with a^* is $f^*(t) = \alpha^t$ for all t. However, it follows from (10) that $f(t) \geq f^*(t)$ for all $t > 1$ since $a \geq a^*$. Hence, $f(t) \geq \alpha^t = [f(1)]^t$ for all $t > 1$. **End of proof**.

[15] See appendix for an alternative proof based on the generating function.

4 Examples

4.1 Exponential altruism

Suppose that the welfare weights $a(t)$ decrease exponentially over future decision-makers t. What is then the associated discount function? To study this case, let $a(t) = \alpha^t$ for some $\alpha \in (0,1)$ and all t. Equation (10) then gives $f(1) = \alpha$, $f(2) = 2\alpha^2$, and $f(3) = 4\alpha^3$. One may thus conjecture that

$$f(t) = 2^{t-1}\alpha^t \qquad \forall t > 0. \tag{16}$$

This conjecture is easily proved by induction.[16] Suppose, thus, that $f(s) = 2^{s-1}\alpha^s$ for all $s = 1, 2, .., t$, for some positive integer t. Then (10) gives

$$\begin{aligned} f(t+1) &= \alpha^{t+1} + \sum_{s=1}^{t} 2^{s-1}\alpha^s\alpha^{t+1-s} = \alpha^{t+1}\left[1 + \sum_{s=1}^{t} 2^{s-1}\right] \\ &= \alpha^{t+1}\left[1 + (2^t - 1)\right] = 2^t\alpha^{t+1}, \end{aligned} \tag{17}$$

establishing (16).

Substituting (16) in (5) we obtain

$$U_\tau(x) = u(x_\tau) + \frac{1}{2}\sum_{t=1}^{\infty}(2\alpha)^t u(x_{\tau+t}) \ . \tag{18}$$

Hence, exponential altruism is equivalent to quasi-exponential discounting with $\beta = 1/2$.

4.2 Quasi-exponential discounting

We found that exponential altruism weights imply a special case of quasi-exponential discounting. What altruism weights correspond to quasi-exponential discounting more generally? Suppose, thus, that $f(0) = 1$ and $f(t) = \beta\delta^t$ for all positive integers. Then $a(1) = \beta\delta$ and $a(2) = \beta(1-\beta)\delta^2$. More generally:

$$a(t) = \beta(1-\beta)^{t-1}\delta^t \qquad \forall t \geq 1. \tag{19}$$

To show this, suppose $a(s) = \beta(1-\beta)^{s-1}\delta^s$ for $s = 1, 2, .., t$, for some positive integer t. Then (10) gives

[16] In the appendix we provide an alternative proof using the generating function.

$$\begin{aligned} a(t+1) &= \beta\delta^{t+1} - \beta^2\delta^{t+1}\sum_{s=1}^{t}(1-\beta)^{s-1} \\ &= \beta\delta^{t+1} - \beta^2\delta^{t+1}\sum_{s=0}^{t-1}(1-\beta)^{s} \\ &= \beta\delta^{t+1} - \beta^2\left(\frac{1-(1-\beta)^t}{\beta}\right)\delta^{t+1} \\ &= \beta(1-\beta)^t\delta^{t+1}. \end{aligned} \tag{20}$$

This establishes (19) by induction in t.[17]

Not surprisingly, one-period altruism is obtained when $\beta = 1$, that is, under exponential discounting. By contrast, when $\beta \neq 1$, then every altruism weight can be written as the product of a common constant and an exponential factor:

$$U_\tau(x) = u(x_\tau) + \gamma\sum_{t=1}^{\infty}\alpha^t U_{\tau+t}(x) \ , \tag{21}$$

where $\gamma = \beta/(1-\beta)$ and $\alpha = (1-\beta)\delta$. In particular, we obtain exponential altruism when $\beta = 1/2$, in accordance with subsection 4.1.

Angeletos et al (2001) made the following estimate of the parameter pair (β, δ) in the Phelps-Pollak-Laibson model, based on annual US data: $\beta = 0.55$ and $\delta = 0.96$. The associated altruism weights are thus $\gamma \approx 1.22$ and $\alpha \approx 0.43$.

4.3 Hyperbolic discounting

Empirical studies of temporal preferences suggest that the discount function f should be hyperbolic from period 1 onwards. Hence, Ainslie (1992), following Herrnstein (1981) and Mazur (1987), suggests $f(t) = (\lambda + \mu t)^{-1}$ for all $t \geq 1$, for some $\lambda, \mu > 0$ (op. cit. eq. (3.7)). A related class of hyperbolic discount functions, given by $f(t) = (1 + \mu t)^{-\gamma}$ for $\mu, \gamma > 0$, was suggested by Loewenstein and Prelec (1992). As a generalization of both, suppose $f(t) = (\lambda + \mu t)^{-\gamma}$ for all $t \geq 1$, where $\lambda, \mu, \gamma > 0$ are such that $\lambda + \mu \geq 1$ (so that $f(1) \leq f(0)$, where $f(0) = 1$ by assumption).

It is easily verified that the associated patience function g is non-decreasing if and only if $(\lambda + \mu)^2 \geq \lambda + 2\mu$.[18] In particular, all discount functions with $\lambda = 1$ (thus all those of the Loewenstein and Prelec variety) satisfy this condition and are hence consistent

[17] See appendix for an alternative proof using the generating function.

[18] To see this, suppose $f(0) = 1$ and $f(t) = (\lambda + \mu t)^{-\gamma}$ for all $t \geq 1$. Then $g(2) \geq g(1)$ if and only if $(\lambda + \mu)^2 \geq \lambda + 2\mu$. Moreover, $g'(t) \geq 0$ for all $t \geq 1$.

with altruism towards future decision-makers.[19] By contrast, some others are not, as was noted in section 3. There, we had $\lambda = 0.5$ and $\mu = \gamma = 1$, hence satisfying the monotonicity condition for f but not that for g. Indeed in the present class of hyperbolic discount functions, the sufficient condition in proposition 2 is in fact also necessary: if $(\lambda + \mu)^2 < \lambda + 2\mu$, then $a(2) < 0$.[20] Proposition 2 is thus sharp within this class of discount functions.

Here proposition 4 is sharp as well: in the present class of discount functions, the necessary condition for (nonnegative) altruism in proposition 4 is equivalent to the condition that $(\lambda + \mu)^t \geq \lambda + \mu t$ for all t, which, in its turn, is equivalent to $(\lambda + \mu)^2 \geq \lambda + 2\mu$.[21]

4.4 Finite-horizon exponential altruism

We conclude by examining the intermediate case between one-period altruism and exponential altruism, namely when the altruism weight decreases exponentially over a finite number of decision-makers, beyond which all weights are zero. What are the corresponding discount functions? Before answering this question in general, let us first briefly consider the special case when altruism stretches only to the next two decision-makers.

4.4.1 The special case $T = 2$

Suppose that $a(1) = \alpha$, $a(2) = \alpha^2$, and $a(t) = 0$ for all $t > 2$. >From (10) we then obtain $f(0) = 1$, $f(1) = \alpha$, $f(2) = 2\alpha^2$, $f(3) = 3\alpha^3$, $f(4) = 5\alpha^4$, $f(5) = 8\alpha^5$... In fact, it is easily verified that all discount factors $f(t)$ can be written in the form $f(t) = m(t)\alpha^t$, where $m : \mathbb{N} \to \mathbb{N}$ satisfies the recursive equation

$$m(t) = m(t-1) + m(t-2) \quad \text{for all } t \geq 2 \tag{22}$$

with initial values $m(0) = m(1) = 1$. Hence, each integer $m(t)$, after the first two, is the sum of the two integers immediately preceding it. The sequence $\langle m(t) \rangle$ is the so-called *Fibonacci sequence*: 1,1,2,3,8,13... It is well-known that the solution to the difference equation (22) is

$$m(t) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{t+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{t+1}. \tag{23}$$

[19] So far we have been unable to obtain a closed-form representation of the altruistic weights corresponding to hyperbolic discounting. However, it is not difficult to derive a closed-form representation of the generating function of the altruism function a associated with the hyperbolic discount $f(t) = (1+t)^{-1}$, see appendix.

[20] This follows from (7), which gives $a(2) = f(2) - [f(1)]^2$, a difference which is nonnegative precisely when g is non-decreasing.

[21] To see this, note that

$$\frac{\partial}{\partial t} \left[(\lambda + \mu)^t - \lambda - \mu t \right] \geq 0$$

for all $t \geq 2$, granted $\lambda, \mu \geq 0$ and $\lambda + \mu \geq 1$, as assumed above.

Moreover, the ratio between two successive numbers of the Fibonacci sequence converge to the so-called golden number (also called the golden section) η,

$$\lim_{t\to\infty}\frac{m(t+1)}{m(t)}=\eta=\frac{1+\sqrt{5}}{2}. \tag{24}$$

As expected, the discount function f satisfies the necessary condition of proposition 4: $f(t)=m(t)\alpha^t \geq \alpha^t = f(1)^t$ for all t, since $m(t)\geq 1$ for all $t\geq 1$. However, it violates the sufficient condition in proposition 2: $g(3)=3\alpha/2<2\alpha=g(2)$.

4.4.2 The general case

Having studied the special case of altruism to the next two decision-makers, we now turn to the general case of an arbitrary finite altruism horizon. More exactly, let $T\geq 2$, and suppose $a(t)=\alpha^t$, for all $t\leq T$ and for some $\alpha\in(0,1)$, and suppose $a(t)=0$ for all $t>T$. By a slight generalization of the above study of the special case $T=2$, it can be verified that the associated discount function, f_T, then can be written as $f_T(t)=m_T(t)\alpha^t$, where

$$m_T(t)=\sum_{s=1}^{\min\{t,T\}} m_T(t-s) \tag{25}$$

for all positive integers t, and $m_T(0)=1$. Under equation (25), we have

$$\begin{aligned} f_T(t) &= \alpha^t \sum_{s=1}^{\min\{t,T\}} m_T(t-s)=\alpha^t \sum_{s=1}^{\min\{t,T\}} \alpha^{s-t} f_T(t-s) \\ &= \sum_{s=1}^{\min\{t,T\}} \alpha^s f_T(t-s). \end{aligned}$$

Setting $a(s)=\alpha^s$ for all positive integers $s\leq T$ and otherwise $a(s)=0$, we obtain (10).

It follows that, for any finite horizon T,

$$1\leq m_T(t)\leq m_{T+1}(t)\leq 2^{t-1} \tag{26}$$

and

$$\alpha^t\leq f_T(t)\leq f_{T+1}(t)\leq \frac{1}{2}(2\alpha)^t \tag{ft}$$

for all t. Hence, the longer the altruism horizon T, the higher the discount factor given to each future instantaneous utility. As expected, the necessary condition in proposition 4 is then satisfied by all f_T since, as seen above, $f_2(t)\geq f(1)^t$.

Moreover, it follows from a result for recursive equations that the ratio between the m_T-weights assigned to two consecutive periods t and $t+1$ converges as t goes to infinity (see e.g. Weisstein, 1999), for any given $T>1$:

$$\lim_{t\to\infty}\frac{f_T(t+1)}{f_T(t)} = \alpha\lim_{t\to\infty}\frac{m_T(t+1)}{m_T(t)} = \alpha\eta_T \ , \tag{27}$$

where η_T is the unique solution in the interval $(1,+\infty)$ to the equation $x = 2 - x^{-T}$. Hence, for each $T > 1$, the discount function is *asymptotically exponential*, with discount factor $\alpha\eta_T$, where η_T is the golden number when $T = 2$, and $\eta_T \to 2$ when $T \to \infty$ (thus establishing continuity at $T = \infty$, see section 4.1 above).

Note, however, that the induced discount function need not be monotonic. In fact, for all $T \geq 2$ and $\alpha > 1/2$: $f_T(1) < f_T(2) < f_T(0)$. Note also that for any finite horizon T, $g_T(T+1) < g_T(T)$, where $g_T : \mathbb{N}_+ \to \mathbb{R}$ is defined as in proposition 2. The hypothesis in proposition 2 is thus violated when the altruism horizon is finite and exceeds 1.[22] (Recall, however, that the hypothesis is met when $T = 1$, in which case we have exponential discounting).

5 Related literature

The closest work we have found are Zeckhauser and Fels (1968), Kimball (1987) and Hori (2001). Zeckhauser and Fels (1968) studied a class of altruistic intergenerational preferences, and, like here, derived the corresponding representation based on each generation's utility from its own consumption. In our notation, they studied preferences represented by utility functions U_τ satisfying the following recursive equation:

$$U_\tau(x) = u(x_\tau) + bU_\tau(x) + d\sum_{t=1}^{\infty} a^t U_{\tau+t}(x), \tag{29}$$

for some $a, b, d \geq 0$. They found exponential discounting to be incompatible with this "pure altruism" equation: in our notation, $f(t) = \delta^t$ implies $b = 1$ which in turn makes $U_\tau(x)$ undetermined in (29). As seen above, however, their claim is not valid in our framework; exponential discounting *is* consistent with altruism, namely altruism towards the next generation (and no other generation).

Much like here, Kimball (1987) analyzed the utility interdependence that may arise when generations are altruistic towards each other. Unlike here, though, he studied the more general case when the current generation cares not only about future generations' consumption but also about that of past generations.[23] In our notation, Kimball thus

[22] It follows from (25) that $m_T(t) = 2m_T(t-1)$ for all $t \in \{2, 3, ...T\}$ and $m_T(T+1) = 2m_T(T) - 1$. Hence,

$$g_T(T+1) = \frac{2m_T(T)-1}{m_T(T)}\alpha < 2a = g_T(T). \tag{28}$$

[23] However, unlike us, he did not allow for altruism towards future generations beyond the first, see below.

focused on preferences of the form

$$U_\tau(x) = \sum_{t=-\infty}^{+\infty} f(t) u(x_{\tau+t}), \tag{30}$$

where $f : \mathbb{Z} \to \mathbb{R}_+$ (with $\mathbb{Z}$ denoting the set of all integers), and asked whether this is consistent with pure altruism towards one's immediate successor and descendant. Formally: the question was whether there exist scalars $\alpha^+, \alpha^- > 0$ such that, for all consumption streams x and generations τ,

$$U_\tau(x) = u(x_\tau) + \alpha^+ U_{\tau+1}(x) + \alpha^- U_{\tau-1}(x). \tag{31}$$

Kimball (1987) identified necessary and sufficient conditions on f for this, and also identified conditions under which f is exponential on $\mathbb{N} \subset \mathbb{Z}$. From the above analysis we know that if f vanishes for all negative integers and is exponential on $\mathbb{N}$, then equation (31) is satisfied for $\alpha^+ = f(1)$ and $\alpha^- = 0$. By taking the limit $\alpha^- \downarrow 0$ in Kimball's model, one obtains exponential discounting.[24] Hence, our results are complementary to those of Kimball (1987), and agree with his along the boundary between the parametric domains of the two models.

Hori (2001) identifies sufficient conditions under which mutual altruistic utility dependence among finitely many economic agents can be resolved. Our framework differs from his in that we have an infinite number of economic agents. However, our analysis is simpler since in our model the utility dependence goes in one direction only.

6 Conclusions

We started out by asking if discounting of future instantaneous utilities is consistent with "pure" altruism towards future decision-makers. We identified a recursive functional equation which establishes a one-to-one relationship between discount factors and altruistic weights attached to future generations or future selves. We saw that some discount functions used in the literature are consistent with altruism towards one's future selves or future generations, while others are not. We also established a sufficient condition, and a necessary condition, for consistency in this respect. These conditions are met by the quasi-exponential discounting models currently under investigation in the macroeconomics literature (see for example. Laibson (1997), Barro (1999), Krusell and Smith (1999), Laibson and Harris (2001) and Angeletos et al (2001)), as well as by many of the hyperbolic discounting models in the psychology literature (see for example Herrnstein (1981), Mazur (1987) and Ainslie (1992)).

>From a behavioral viewpoint, however, all the discounting models discussed here seem quite restrictive. See, for example, Frederick, Loewenstein and O'Donoghue (2001),

[24] Set the two free parameters, a and d in his proposition 2, equal to zero. Then, as $\alpha^- \downarrow 0$: his parameters $\lambda \to +\infty$, $\mu \to \alpha^+$, and $f(t) \to (\alpha^+)^t$ for all $t \in \mathbb{N}$ and $f(t) \to 0$ otherwise.

Kahneman (2000) and Rubinstein (2001) for more general approaches to intertemporal choice. We hope, however, that our study has shed some light on a somewhat wider path than the well-trodden but narrow path of exponential discounting.

7 Appendix

The sums in equations (7) and (10) are convolutions, and convolutions are known to be conveniently analyzed by means of the Fourier transform and the generating function. We here briefly show how the generating function can be used in the present context.

Recall first that the generating function φ of a function $f : \mathbb{N} \to \mathbb{R}$ is the function $\varphi : D \to \mathbb{R}$ defined by $\varphi(x) = \sum_{t=0}^{+\infty} f(t) x^t$, where the domain $D \subset \mathbb{R}$ is the set of numbers $x \in \mathbb{R}$ for which the series converges. The usefulness of the generating function (and, likewise, the Fourier-transform) emanates from the following two facts. First, one may recover the underlying function f from its generating function φ by taking the derivatives of φ at $x = 0$: $\varphi(0) = f(0)$, $\varphi'(0) = f(1)$, ..., $\varphi^{(n)}(0) = n!f(n)$ for all positive integers n. Secondly, the generating function of the convolution of two functions equals the product of the functions' generating functions.

In the present context, the latter property allows us to solve (10) for the two generating functions. However, the domain of a is $\mathbb{N}_+$ instead of $\mathbb{N}$, so it is convenient to first define the function $b : \mathbb{N} \to \mathbb{R}$ as the "minus-one" shift of $a : \mathbb{N}_+ \to \mathbb{R}$ by setting $b(t) = a(t+1)$ for all $t \in \mathbb{N}$. Let φ be the generating function of the discount function f, and let ψ be the generating function of the "minus-one" shift b of the altruism-weight function a.

Equation (10) can now be written as

$$f(t+1) = \sum_{s=0}^{t} b(s) f(t-s) \qquad \forall t \in \mathbb{N}. \tag{32}$$

The right-hand side is the convolution of f and b, and the left-hand side is the "plus-one" shift of f. Using well-known properties of generating functions, we obtain the following equation, by taking the generating functions of both sides:

$$\frac{1}{x}\left[\varphi(x) - \varphi(0)\right] = \varphi(x)\psi(x) \ . \tag{33}$$

Hence, since $f(0) = 1$, we obtain

$$\varphi(x) = \frac{1}{1 - x\psi(x)} \qquad \text{and} \qquad \psi(x) = \frac{\varphi(x) - 1}{x\varphi(x)}. \tag{34}$$

Suppose, for instance, that f is given and we look for a. Then $a(t+1) = b(t) = \psi^{(t)}(0)/t!$ for all positive integers, where

$$\psi(x) = \frac{\varphi(x) - 1}{x\varphi(x)} = \left[f(1) + f(2)x + f(3)x^2 + ...\right]/\varphi(x) \ . \tag{35}$$

7.1 Proof of equation (15)

Suppose, thus, that $f(t) = \delta^t$ for all t, where $\delta \in (0,1)$. Then

$$\varphi(x) = 1 + \sum_{t=1}^{\infty} \delta^t x^t = 1 + \delta x \sum_{t=0}^{\infty} (\delta x)^t = 1 + \frac{\delta x}{1 - \delta x} \ . \tag{36}$$

Hence, by (34),

$$\psi(x) = \delta = \sum_{t=0}^{\infty} a(t+1) x^t. \tag{37}$$

Consequently, $a(1) = \delta$, and $a(t) = 0$ $\forall t \geq 2$.

7.2 Proof of equation (19)

Suppose, thus, that $f(0) = 1$ and $f(t) = \beta\delta^t$ for all positive integers, where $\beta, \delta \in (0,1)$. Then

$$\varphi(x) = 1 + \sum_{t=1}^{\infty} \beta\delta^t x^t = 1 + \beta\delta x \sum_{t=0}^{\infty} (\delta x)^t = 1 + \frac{\beta\delta x}{1 - \delta x} \ . \tag{38}$$

Hence, by (34),

$$\psi(x) = \frac{\beta\delta}{1 - (1-\beta)\delta x} = \beta\delta \sum_{t=0}^{\infty} [(1-\beta)\delta x]^t = \sum_{t=0}^{\infty} a(t+1) x^t. \tag{39}$$

Consequently,

$$a(t) = \frac{\beta}{1-\beta} [(1-\beta)\delta]^t \qquad \forall t \in \mathbb{N}_+ \ . \tag{40}$$

7.3 The generating function of a hyperbolic discount function

Suppose $f(t) = (t+1)^{-1}$ for all t. Then

$$\varphi(x) = -\frac{1}{x} \ln(1-x), \tag{41}$$

and hence, by (34), the generating function ψ of b, the minus-one shift of a, is

$$\begin{aligned} \psi(x) &= \frac{1}{x} + \frac{1}{\ln(1-x)} = \frac{x + \ln(1-x)}{x \ln(1-x)} \\ &= \frac{\frac{1}{2} + \frac{1}{3}x + \frac{1}{4}x^2 + O(x^3)}{1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + O(x^4)} \ . \end{aligned} \tag{42}$$

Not having found a closed-form expression for, b, the "inverse" of this generating function, we just note that $a(t+1) = b(t) = \psi^{(t)}(0)/t!$ for all $t \in \mathbb{N}$.

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