A Study of Inaction in Investment Games via the Early Exercise Premium Representation

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Abstract

This paper examines strategic investment in the context of a duopolistic continuous-time real options game. Our contribution is twofold, economic and methodological. The former is the recognition that, under fixed costs of investment and time-to-build, the firm pays a fraction of the implicit strike price to its competitor in the form of transferred foregone consumer demand. The latter is the introduction of the early exercise premium representation as a valuable device for the characterization of optimal exercise policies in real options games. We find that positive capital depreciation, technology improvement, and harm effects to the low-technology producer are not sufficient to generate equilibria characterized by action.

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1 Introduction

Since the seminal work of Fudenberg and Tirole (1985) on the optimal adoption of new technologies by competing firms under conditions of certainty, a significant research effort has been devoted to the analysis of optimal strategic investment decisions in the presence of uncertainty. Because (i) the optimal timing of project initiation resembles the optimal exercise of an American option and (ii) firm interactions tend to play a crucial role in the adoption of new technologies, this literature has merged the option-pricing methodology with the game-theoretical paradigm to analyze what is best viewed as strategic capital replacement games. Huisman, Kort, Pawlina and Thijssen (2003) offer a representative treatment of real options within the game-theory framework: in particular, they extend the work of Smets (1991) by applying a method involving symmetric mixed strategies and they review previous models featuring asymmetric firms as well as decreasing uncertainty over time. Grenadier (1996) develops an equilibrium model for the optimal investment timing of two symmetric firms competing in the real estate market. Working backwards in a dynamic programming fashion, Grenadier determines the optimal exercise policies for his duopoly model and provides a rational explanation for overbuilding in the real estate market. Huisman and Kort (1998) adapt the Stenbacka and Tombak (1994) framework to analyze technology adoption assuming a stochastic time-to-build delay. They consider dispersed versus joint equilibria in the case of endogenous firms' roles and they find that the profit stream belonging to the preemption equilibrium can be so low that both firms would be better off never exercising their capital-replacement option. For a standard textbook treatment of real investment under uncertainty, the reader is referred to Dixit and Pindyck (1994). Also, the survey article by Boyer, Gravel and Lasserre (2004) reviews the recent contributions to the literature on strategic investment games.

The isomorphism between a firm's real investment projects and a set of financial positions in American options permits the application of some of the most advanced techniques developed in contingent claims analysis. The American option valuation problem was first studied as an optimal stopping time problem by Samuelson (1965) and McKean (1965). Widely acknowledged shortcomings of the stopping time approach are its lack of direct insights into the shape of the immediate exercise boundary and the fact that it does not lead to efficient numerical procedures. Nonetheless, building on this approach, Kim (1990), Jacka (1991) and Carr, Jarrow, and Myneni (1992) derive a powerful decomposition of the American option value that emphasizes the premium attached to early exercise. In particular, the early exercise premium (EEP) representation expresses an American option as the combination of a European option and the right to exercise the option early. This representation is central to the analysis of American-style derivatives in the recent book by Detemple (2006).

This paper considers an economy populated by two symmetric firms, each holding a unique capital-replacement option over an infinite horizon. Until one of the two firms exercises its capital-replacement option, both firms operate the same technology. Upon exercising its investment option, a firm is endowed with a full stock of higher-quality productive capital, which depreciates over time at a constant rate. A second mover disadvantage is introduced additively in the revenues of the firm that still operates the old technology.

The contribution of this paper is twofold, economic and methodological. The economic contribution is the recognition that, under fixed costs of investment and time-to-build, a firm's exercise of its capital-replacement option leads not only to a temporary "loss of output associated with the acquisition and installation of new capital goods (Cooper et al. 1999, p. 923)" but also to a significant temporary reallocation of the firm's revenues to its competitor. Upon exercising its investment option, the firm pays a fraction of the implicit strike price to its competitor in the form of transferred foregone consumer demand. The methodological contribution is the introduction of the EEP representation as a valuable device for the characterization of optimal exercise policies in real options games.

The aim of our article is the analysis of *perpetual inaction* along the equilibrium path. This is motivated by our belief that the correct place to begin the thorough study of strategic interaction and its determinants is the derivation of a set of economic and mathematical conditions under which both firms optimally retain their investment option forever. Departures from these benchmark conditions are necessary to deliver active strategic interactions, as those observed daily in the real world.

We find that (i) in the absence of capital depreciation, despite the value of the follower's future revenues upon exercise approaching infinity, it is never optimal for the follower to exercise its investment option and (ii) even in the presence of a strictly positive depreciation rate, and despite the technology improvement, the follower chooses never to exercise the American option, leading to perpetual inaction on the part of both firms. Furthermore, a second mover disadvantage that is *not* proportional to the firm's revenues is crucial to obtain a non-degenerate investment policy on the part of the follower. In this context, the EEP formula produces a recursive integral equation, which can be used to study the characteristics of the follower's optimal exercise boundary. Taking as given the follower's optimal response function, the leader's policy equation is analytically intractable. However, a specific case brings our analysis to a definitive conclusion: in the extreme case in which harm effects to the low-technology producer grow so large that immediate reaction is optimal for the follower, the leader's policy reduces to a closed form expression. The application of EEP analysis to this closed form solution shows that the leader optimally delays investment forever, thereby producing a perpetual inaction equilibrium.

The structure of this paper is as follows. In Section 2 we describe the basic model: the stochastic process for market demand as well as the Markovian revenue equations are introduced. Also, we emphasize the relation of our model to the Grossman and Helpman (1991) model of sequential quality improvements. In Section 3 we illustrate the derivation of the trigger boundaries for both firms via the EEP representation. We analyze the case of no capital depreciation as well as that of a strictly positive depreciation rate. In Section 4 special attention is devoted to the study of harm effects to the follower's revenues. In Section 5 we summarize the results of the paper. Mathematical derivations are collected in an Appendix (Section 6).

2 The Model

In this Section we describe the basic model: first we introduce the stochastic process for market demand and the Markovian revenue equations, and then we relate our model to the Grossman and Helpman (1991) model of sequential quality improvements.

Consider an infinite-horizon economy populated by two firms. At time $t = 0$, firm i – $i \in \{1, 2\}$ – produces $y_i = \kappa$ units of output, which depreciate at a constant obsolescence rate $\delta \geq 0$. At time t_i, firm i initiates its capital-replacement process, which leads the new capital of quality λ ($\lambda > 1$ is a quality factor common across firms) to be operational at time $t_i + \Delta$, where Δ is a strictly positive time-to-build delay. At time $t_i + \Delta$, firm i is endowed with κ, a full stock of higher-quality productive capital. Beyond this point, each firm's capital stock depreciates at rate δ. We denote $y_{i,t} = k_{i,t}$ as firm i's output capacity at time t. The aggregate quantity produced is given by

$$Y_t = \Sigma_{i=1}^2 y_{i,t}. \tag{1}$$

The inverse demand function for this commodity is given by

$$P_t = X_t D(Y_t), \tag{2}$$

where $\frac{\partial D(Y)}{\partial Y} < 0$ and X_t is an exogenous demand-shock process that obeys

$$dX_t = \alpha X_t dt + \sigma X_t dz_t. \tag{3}$$

We account for quality improvement by assuming that the adoption of the new technology allows the firm to sell its output at a per-unit price equal to λP_t. As we shall see, this feature of the model is pivotal in proving the consistency of our assumptions with Grossman and Helpman's results. In this economy, the risk-free rate is a constant $r > 0$.

Until one of the two firms exercises its capital-replacement option, both firms operate the same technology. Suppose that, without loss of generality, firm 1 is the first firm to exercise its investment option, i.e. firm 1 is the Stackelberg[1] leader. At time t_1, firm 1 suspends its production operations and pays a fixed cost I.

The revenue for firm 1, $R_1(t) = k_{1,t} P_t$, follows

$$R_1(t) = \begin{bmatrix} k_{1,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & 0 \leq t < t_1 \\ 0, & t_1 \leq t < t_1 + \Delta \\ k_{1,t}\left[\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & t_1 + \Delta \leq t < t_2 \\ k_{1,t}\left[\lambda X_t D(k_{1,t})\right], & t_2 \leq t < t_2 + \Delta \\ k_{1,t}\left[\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & t_2 + \Delta \leq t < \infty \end{bmatrix}. \tag{4}$$

[1] Stackelberg (1934) proposes a dynamic model of duopoly in which a dominant firm (or leader) moves first and a subordinate firm (or follower) moves second. For a classic textbook treatment of the Stackelberg game the reader is referred to Myerson (2004, p. 187) and Fudenberg and Tirole (1991, pp. 67-69).

On the other hand, the revenue for firm 2, $R_2(t) = k_{2,t}P_t$, follows

$$R_2(t) = \begin{bmatrix} k_{2,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & 0 \le t < t_1 \\ k_{2,t}\left[X_t D(k_{2,t})\right], & t_1 \le t < t_1 + \Delta \\ k_{2,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & t_1 + \Delta \le t < t_2 \\ 0, & t_2 \le t < t_2 + \Delta \\ k_{2,t}\left[\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & t_2 + \Delta \le t < \infty \end{bmatrix}. \tag{5}$$

Equation (5) embodies our paper's economic recognition that, under fixed costs of investment and time-to-build, a firm's exercise of its capital-replacement option leads not only to a temporary "loss of output associated with the acquisition and installation of new capital goods (Cooper et al. 1999, p. 923)" but also to a significant temporary reallocation of the firm's revenues to its competitor. Upon exercising its investment option, the firm pays a fraction of the implicit strike price to its competitor in the form of transferred foregone consumer demand. This effect is captured by the fact that $k_{2,t}\left[X_t D(k_{2,t})\right] > k_{2,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right]$, under the assumption that $D(Y)$ is monotonically decreasing in Y.

2.1 Relation to Grossman and Helpman (1991)

Grossman and Helpman develop a model of repeated product improvements in a continuum of sectors, in which each product follows a stochastic progression up a quality ladder. Consumers share a common intertemporal utility function, which they maximize subject to an intertemporal budget constraint. At every point in time the maximization occurs in two stages: first, the consumer allocates her flow of spending to maximize the instantaneous utility for given prices and then she chooses the time pattern of spending to maximize her intertemporal utility function. To solve the static problem, *the consumer optimally selects a bundle of products with equal quality-adjusted prices.*[2] In their model, progress is not uniform across sectors and in equilibrium the distribution of qualities evolves over time.

The following Proposition demonstrates the consistency of our assumptions with the results of Grossman and Helpman's model.

Proposition 1 *The revenue equations, Eq. (4) and Eq. (5), are consistent with the Grossman and Helpman model.*

Proof. Recall that, in our model, until one of the two firms exercises its capital-replacement option, both firms operate the same technology. Upon exercising its investment option, a firm is endowed with a full stock of productive capital whose quality is $\lambda > 1$. According to Eq. (2) the price of the higher-quality producer (the leader) is $\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})$, while the price of the lower-quality firm (the follower) is $X_t D(\Sigma_{i=1}^2 k_{i,t})$. It follows that

$$\underbrace{\frac{\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})}{\lambda}}_{\text{quality-adjusted price for high-quality producer}} = \underbrace{\frac{X_t D(\Sigma_{i=1}^2 k_{i,t})}{1}}_{\text{quality-adjusted price for low-quality producer}} . \tag{6}$$

[2] See Grossman and Helpman (1991, pp. 45-46) for mathematical derivations.

Equation (6) proves that *quality-adjusted prices are equal across firms*, which is consistent with the results of Grossman and Helpman's model of quality improvements. ■

Having completed the presentation of our model, we can now exploit the early exercise premium representation to study the characteristics of optimal firm policies.

3 Characterizing Firm Policies Via the EEP Representation

In this Section we illustrate the derivation of the trigger boundaries for both firms via the EEP representation. We analyze the case of no capital depreciation as well as that of a strictly positive depreciation rate.

Since we assume that firm 1 is the first to invest in a new capital stock, we begin by determining the optimal policy for firm 2 in a backward induction manner. We then turn to firm 1's problem. This solution method follows the standard Stackelberg mechanism for subgame perfect equilibria, imposing sequentiality.[3]

Consider the problem of firm 2. Since firm 1 has already exercised its capital-replacement option, we focus on times $t \geq t_1$, the investment initiation time for firm 1. For any such t, firm 2's value is equal to the value of one of the following two options: (i) an option to initiate the capital-replacement process between t_1 and $t_1 + \Delta$, whose value is denoted by F, and (ii) an option that allows to initiate the investment process beyond time $t_1 + \Delta$, whose value is denoted by W. If unexercised at time $t_1 + \Delta$, the F option expires and is exchanged with the W option. Recall from Eq. (5) that the revenue stream for firm 2 will be equal to $k_{2,t}\left[X_t D(k_{2,t})\right]$ over the time interval $t_1 \leq t < t_1 + \Delta$ and it will be equal to $k_{2,t}\left[X_t D(\Sigma_{i=1}^{2} k_{i,t})\right]$ beyond $t_1 + \Delta$ until the capital-replacement process is initiated.

3.1 Firm 2's Optimal Behavior Beyond $t_1 + \Delta$

Suppose that firm 1 has already begun to produce using its new capital stock. The *strategy space* for firm 2 is $\mathcal{S}_{t_1+\Delta,\infty}$, the set of stopping times that take values between $t_1 + \Delta$ and ∞. For any $t_2 \in \mathcal{S}_{t_1+\Delta,\infty}$, the value of the follower's capital-replacement option is given by

$$W_t(EP, H, t_2) = \tilde{E}_t\left[\int_t^{t_2} e^{\{-r(v-t)\}} dH_v + e^{\{-r(t_2-t)\}}\left[EP_{t_2} - I\right]\right], \tag{7}$$

where $\tilde{E}_t$ is the expectation operator with respect to the Cox and Ross (1976) risk-neutral measure. In Eq. (7) dH_t is the flow of instantaneous revenues received until the time of

[3] Exogenous sequentiality in Stackelberg real option games is a frequent assumption: the reader can find numerous examples in the survey article of Boyer, Gravel and Lasserre (2004). Recent articles with exogenous and endogenous firm roles are Huisman and Kort (1998) and Wu (2006).

exercise,[4] which is equal to

$$
\begin{aligned}
dH_t &= k_{2,t}\left[X_t D(\Sigma_{i=1}^{2} k_{i,t})\right] dt \\
&= X_t \kappa e^{\{-\delta t\}}\left(\kappa e^{\{-\delta(t-t_1-\Delta)\}} + \kappa e^{\{-\delta t\}}\right)^{-\gamma} dt \\
&= \kappa^{1-\gamma} X_t e^{\{\delta(\gamma-1)t\}}\left(e^{\{\delta(t_1+\Delta)\}} + 1\right)^{-\gamma} dt, \qquad (8)
\end{aligned}
$$

and EP_t is the value of the future flow of revenues earned beyond time $t_2 + \Delta$ when the option to replace the firm's capital is exercised at time $t = t_2$. In what follows, we assume a specific functional form for the market demand $D(Y_t)$. In particular, we adopt the form $D(Y_t) = Y_t^{-\gamma}$, where γ is the constant inverse elasticity parameter taking values in the interval $(0, 1)$.[5] Under this assumption, we derive the value of the future flow of revenues conditional on immediate exercise, which is

$$
EP_t = \lambda \kappa^{1-\gamma} X_t \frac{e^{\{\delta\gamma t\}}\left(e^{\{\delta t_1\}} + e^{\{\delta t\}}\right)^{-\gamma}}{\delta(1-\gamma)}.\text{[6]} \qquad (9)
$$

For the analysis of strategic firm behavior as that embedded in Eq. (7), this paper relies on the early exercise premium (EEP) representation for American options as exposed in Detemple (2006, p. 3 and pp. 41-43). The early exercise premium representation is parametric in that it expresses the American option value as a function of its unknown optimal exercise boundary. However, using the fact that immediate exercise is optimal when the boundary is reached, the EEP formula produces a recursive integral equation, which can be used to study the characteristics of the boundary. More specifically the EEP representation formula provides a decomposition of the American contingent claim into its European counterpart and the right to exercise the option early.[7]

Accordingly, Eq. (7) can be represented as the sum of a European option and the early exercise premium, denoted EEP_t. Hence, Eq. (7) becomes

$$
W_t(EP, H, t_2) = \lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} dH_v + e^{\{-r(T-t)\}}\left[EP_T - I\right]^+\right] + EEP_t, \qquad (10)
$$

[4] The flow of instantaneous revenues prior to investment, dH_t, represents the dividends flow from the option. It can be rewritten in cumulative form as

$$
H_t = \int_{t_1}^{t_1+\Delta} k_{2,v} X_v D(k_{2,v}) dv + \int_{t_1+\Delta}^{t} k_{2,v} X_v D(\Sigma_{i=1}^{2} k_{i,v}) dv.
$$

[5] The restriction on the parameter γ is imposed to guarantee that, as the intial capital stock depreciates, firms will potentially have an incentive to invest in the new technology. Indeed, it is straightforward to show that, unless $\gamma \in (0, 1)$, firm revenues approach the geometric Brownian motion $\frac{X_t}{2}$ $(\gamma = 1)$ or explode $(\gamma > 1)$, which represent degenerate cases.

[6] The mathematical derivation of EP_t is contained in the Appendix.

[7] An alternative decomposition is the delayed exercise premium (DEP) representation, which emphasizes the gains from waiting to exercise. The DEP representation, developed by Carr, Jarrow, and Myneni (1992), expresses the value of the American option as the sum of the payoffs upon immediate exercise and the additional value of waiting.

in which

$$EEP_t = \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r \left[EP_v - I \right] dv - dA_v - dH_v \right) \right], \tag{11}$$

where τ_v is the first time at which exercise becomes optimal. Detemple (2006, p. 43) provides valuable intuition for the components of the EEP_t. The local gains from early exercise are given by Eq. (11): $r\left[EP_v - I\right]dv$ is the amount of interest collected over time if one invests immediately and places the proceeds in the riskless account, $-dA_v$ is the loss incurred by exercising early due to foregone expected appreciation in the payoffs, and $-dH_v$ is the loss due to foregone cash flows earned prior to investment. In the next Lemma we derive the explicit expression for the expected appreciation in payoffs dA_t.

Lemma 1 *The expected appreciation in future revenues dA_t is equal to $EP_t\left[\alpha + \gamma\delta \frac{e^{\{\delta(t_1-t)\}}}{e^{\{\delta(t_1-t)\}}+1}\right] dt$.*

Proof. We follow the proof of Theorem 34, p. 76 in Detemple (2006). A standard application of Ito's Lemma to Eq. (9) yields

$$\begin{aligned} dEP_t &= d\left[\lambda\kappa^{1-\gamma} X_t \frac{e^{\{\delta\gamma t\}}\left(e^{\{\delta t_1\}} + e^{\{\delta t\}}\right)^{-\gamma}}{\delta\left(1-\gamma\right)}\right] \\ &= \underbrace{EP_t\left[\alpha + \gamma\delta \frac{e^{\{\delta(t_1-t)\}}}{e^{\{\delta(t_1-t)\}}+1}\right] dt}_{dA_t} + \sigma EP_t dz_t, \end{aligned} \tag{12}$$

where $\alpha = r$ under the risk-neutral measure. ■

Armed with an explicit formula for each of the components entering Eq. (11), we can state an interesting result on the optimal behavior of firm 2 when the rate of depreciation δ is equal to zero. The next Proposition exemplifies how the adoption of the EEP representation allows for the immediate detection of degenerate investment policies.

Proposition 2 *If the rate of depreciation δ is equal to zero, then (i) the value of the firm's future revenues upon exercise approaches $+\infty$ and (ii) the follower never exercises its investment option.*

Proof.

(i) Let

$$\xi_{t,v} = \exp\left\{-\left[r + \frac{1}{2}\theta^2\right](v-t) - \theta(z_v - z_t)\right\} \tag{13}$$

be the state-price density (Vasicek, 1977), in which $\theta = \frac{\alpha - r}{\sigma}$ is the market price of risk. Then, under no depreciation, EP_t is

$$EP_t = E_t\left[\int_{t+\Delta}^{\infty} \xi_{t,v}\kappa\left[\lambda X_v D(2\kappa)\right] dv\right] = \lambda 2^{-\gamma}\kappa^{1-\gamma} X_t [\int_{t+\Delta}^{\infty} 1 dv] \to +\infty \tag{14}$$

(ii) Under the risk-neutral measure and with no depreciation,

$$
\begin{aligned}
EEP_t &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - dA_v - dH_v \right) \right] \\
&= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(-rIdv - dH_v \right) \right], \qquad (15)
\end{aligned}
$$

where the second equality is obtained by using the result in Lemma 1 and setting $\delta = 0$. Because $(-rIdv - dH_v)$ is always negative, the American option will never be exercised prior to its maturity date T, for $T \to +\infty$. See Detemple (2006, p. 43).

■

Proposition 2 reveals a perhaps counterintuitive result: despite the fact that the value of the payoffs upon exercise approaches infinity, the follower optimally chooses perpetual inaction when $\delta = 0$. The EEP representation produces this conclusion readily by showing that there can be no local gains from early exercise. This provides evidence of the EEP approach's value as a detection mechanism for degenerate equilibria in real options games.

The result stated in Proposition 2 (ii) is directly related to Merton's (1973) demonstration that it is never optimal to exercise an American call option on a non-dividend-paying asset prior to its maturity date. In particular, the local gains from early exercise reduce to $-rIdv$, in our notation, since the option studied by Merton bears no dividends. The question remains as to whether the introduction of capital depreciation is sufficient to generate an incentive for the follower to invest with a positive probability. This is the object of Proposition 3.

Proposition 3 *In the case of a strictly positive depreciation rate δ, despite the quality factor $\lambda > 1$, the follower chooses never to initiate the capital-replacement process.*

Proof. For any $T > t_1 + \Delta$,

$$
\begin{aligned}
EEP_t &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - EP_v \left[r + \gamma\delta \frac{e^{\{\delta(t_1 - v)\}}}{e^{\{\delta(t_1 - v)\}} + 1} \right] dv - dH_v \right) \right] \\
&= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \phi_v dv \right], \qquad (16)
\end{aligned}
$$

where $\phi_v = -rI - EP_v \left[\frac{\gamma\delta e^{\{\delta(t_1 - v)\}}}{e^{\{\delta(t_1 - v)\}} + 1} + \frac{\delta(1-\gamma)\left(e^{\{\delta(t_1+\Delta)\}} + 1\right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}}\right)^{-\gamma}} \right]$ is always negative. The second equality is obtained by re-expressing dH_v as a function of EP_v. In particular, Eq. (8) and Eq. (9) yield

$$
dH_v = EP_v \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1\right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}}\right)^{-\gamma}} dv. \qquad (17)
$$

■

Equation (12) provides the insight necessary to understand Proposition 3. In particular, dA_t can be rewritten as $EP_t \left[\alpha - \left(-\gamma\delta \frac{e^{\{\delta(t_1 - t)\}}}{e^{\{\delta(t_1 - t)\}} + 1} \right) \right] dt$, in which $\left(-\gamma\delta \frac{e^{\{\delta(t_1 - t)\}}}{e^{\{\delta(t_1 - t)\}} + 1} \right)$ acts as a deterministic *negative* dividend rate on the underlying investment payoffs. This proves that capital depreciation is not sufficient to guarantee a non-degenerate behavior on the part of the follower. This result suggests that the introduction of a second mover disadvantage is required to create adequate incentives for the follower to exercise its investment option.

Proposition 3 implies that the American investment option held by the follower can be valued as a perpetual European call option whose price is determined by the following Proposition.

Proposition 4 *In the case of a strictly positive depreciation rate δ, and a quality factor $\lambda > 1$, the investment option held by the follower can be treated as a perpetual European option, whose value is*

$$W_t(EP, H, \infty) = \frac{X_t \kappa^{1-\gamma 1-\gamma} e^{\{\delta(\gamma-1)t\}}}{(1-\gamma)\,\delta} \left[\left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma} + \lambda e^{\{\delta t\}} \left(\frac{e^{\{\delta(t_1-t)\}} + 1}{e^{\{\delta t_1\}} + e^{\{\delta t\}}} \right)^{\gamma} \right]. \quad (18)$$

Proof. See Appendix for details. ■

As shown in the Appendix, $W_t(EP, H, \infty)$ can be expressed as

$$W_t(EP, H, \infty) = EP_t \left(e^{\{\delta(t_1-t)\}} + 1 \right)^{\gamma} \left[\frac{e^{\{\delta(\gamma-1)t\}} \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda} + 1 \right] > EP_t,$$

which allows us to relate our finding to the well-established result that, under the assumption of no payouts to the common stock over the life of the contract, the price of a perpetual warrant equals the price of the common stock. This result was obtained in different environments by Samuelson (1965), Samuelson and Merton (1969), Black and Scholes (1973) and Merton (1973). In our case, because (i) dH_t acts as a stochastic positive dividend rate on the option and (ii) a positive depreciation rate translates into a negative dividend rate on the underlying investment opportunity EP_t, we find that the perpetual European option *exceeds* the value of the flow of revenues upon exercise, EP_t.

Propositions 3 and 4 complete the analysis of firm 2's optimal behavior beyond $t_1 + \Delta$. In the next Subsection we consider the follower's optimal behavior in the time interval $[t_1, t_1 + \Delta]$ in which the leader shuts down its operations to install the new capital goods. Recall that the follower's option to initiate the capital-replacement process between t_1 and $t_1 + \Delta$, denoted by F, will be exchanged for $W_{t_1+\Delta}(EP, H, \infty)$ if no investment is undertaken by $t_1 + \Delta$. Thus, the minimum value of the F option at time $t_1 + \Delta$ will be $W_{t_1+\Delta}(EP, H, \infty)$. With this in place, we can turn to the question of whether (and when) firm 2 would replace its capital prior to (or exactly at) $t_1 + \Delta$.

3.2 Firm 2's Optimal Behavior Prior to $t_1 + \Delta$

Suppose that firm 1 has not begun to produce with the newly-purchased stock of capital. The strategy space for firm 2 is $\mathcal{S}_{t_1, t_1+\Delta}$, the set of stopping times that take values between t_1 and $t_1 + \Delta$. For any $t_2 \in \mathcal{S}_{t_1, t_1+\Delta}$, the value of the follower's capital-replacement option is given by

$$
\begin{aligned}
F_t(EP,H,t_2) &= \tilde{E}_t\left[\int_t^{t_2} e^{\{-r(v-t)\}}dH_v + e^{\{-r(t_2-t)\}}\left[EP_{t_2}-I\right]\right] \\
&= \tilde{E}_t\left[\int_t^{t_1+\Delta} e^{\{-r(v-t)\}}dH_v + e^{\{-r(t_1+\Delta-t)\}}\max\left[EP_{t_1+\Delta}-I, W_{t_1+\Delta}(EP,H,\infty)\right]\right] \\
&\quad +EEP_t, \qquad (19)
\end{aligned}
$$

in which

$$
dH_t = k_{2,t}\left[X_t D(k_{2,t})\right]dt = X_t\kappa e^{\{-\delta t\}}\left(\kappa e^{\{-\delta t\}}\right)^{-\gamma}dt = \kappa^{1-\gamma}X_t e^{\delta(\gamma-1)t}dt \qquad (20)
$$

and EP_t is given by Eq. (9), as a consequence of the time-to-build delay. This in turn implies that the expected appreciation in future revenues dA_t is given by $EP_t\left[\alpha+\gamma\delta\frac{e^{\{\delta(t_1-t)\}}}{e^{\{\delta(t_1-t)\}}+1}\right]dt$, as proved in Lemma 1 above. Proposition 5 below exploits the EEP approach to determine the follower's investment policy prior to $t_1+\Delta$. Proposition 6 examines optimal behavior exactly at time $t_1+\Delta$.

Proposition 5 *In the case of a strictly positive depreciation rate δ the follower chooses never to exercise the option $F_t(EP,H,t_2)$ prior to its maturity date $t_1+\Delta$.*

Proof. For any $t_1+\Delta$,

$$
\begin{aligned}
EEP_t &= \tilde{E}_t\left[\int_t^{t_1+\Delta} e^{\{-r(v-t)\}}1_{\{\tau_v=v\}}\left(r\left[EP_v-I\right]dv - EP_v\left[r+\gamma\delta\frac{e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}}+1}\right]dv - dH_v\right)\right] \\
&= \tilde{E}_t\left[\int_t^{t_1+\Delta} e^{\{-r(v-t)\}}1_{\{\tau_v=v\}}\phi_v dv\right], \qquad (21)
\end{aligned}
$$

where $\phi_v = -rI - EP_v\left[\frac{\gamma\delta e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}}+1}\right] - X_v\left(\kappa e^{-\delta v}\right)^{1-\gamma}$ is always negative. ■

The F option is not priced since it does not enter future derivations nor does it suggest further links to the option theory literature, as did the W option. The reader can easily value the option, if so desired, by following the lines of the proof of Proposition 4 in the Appendix.

Since Propositions 3 and 5 show that firm 2 will never optimally exercise the capital replacement-option prior to and beyond $t_1+\Delta$, it remains to be shown whether investment occurs at $t_1+\Delta$ precisely.

Proposition 6 *Firm 2 will never exercise its capital-replacement option.*

Proof. In Proposition 3 we prove that, for $t > t_1+\Delta$, firm 2 will never exercise the option $W_t(EP,H,\infty)$. Also, in Proposition 5 we demonstrate that, for $t_1 \le t < t_1+\Delta$, firm 2 will never exercise the option $F_t(EP,H,t_2)$. It remains to show whether firm 2 initiates its investment process exactly at time $t_1+\Delta$. Exercise will occur only if

$$
\max\left[EP_{t_1+\Delta}-I, W_{t_1+\Delta}(EP,H,\infty)\right] = EP_{t_1+\Delta}-I.
$$

However, from Proposition 4, we know that

$$
\begin{aligned}
W_{t_1+\Delta}(EP,H,\infty) &= X_t\kappa^{1-\gamma}e^{\{(\gamma-1)\delta(t_1+\Delta)\}}\frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{(1-\gamma)\,\delta} + EP_{t_1+\Delta}\left(e^{\{-\delta\Delta\}}+1\right)^{\gamma} \\
&> X_t\kappa^{1-\gamma}e^{\{(\gamma-1)\delta(t_1+\Delta)\}}\frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{(1-\gamma)\,\delta} + EP_{t_1+\Delta} \\
&> EP_{t_1+\Delta} - I, \qquad (22)
\end{aligned}
$$

which implies that firm 2 will never initiate its capital-replacement process. ■

Proposition 6 could be directly derived by recognizing that, because the value of the perpetual European option $W_{t_1+\Delta}(EP,H,\infty)$ exceeds $EP_{t_1+\Delta}$ as stated in Proposition 4, $W_{t_1+\Delta}(EP,H,\infty)$ must be in excess of the net payoff upon exercise $EP_{t_1+\Delta} - I$.

Having determined the optimal policy for firm 2, we now turn to firm 1's problem.

3.3 Firm 1's Optimal Investment Behavior

The previous Subsections have shown that firm 2 will never initiate its capital-replacement process, regardless of firm 1's investment policy t_1. This simplifies significantly the optimal stopping-time problem of the leader, since firm 1 need not consider strategic interactions while optimizing. The leader firm holds a perpetual American option with a strike price equal to I and a dividend rate equal to

$$
dH_t = k_{1,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right]dt = \kappa e^{\{-\delta t\}}X_t\left(2\kappa e^{\{-\delta t\}}\right)^{-\gamma}dt = 2^{-\gamma}\kappa^{1-\gamma}X_t e^{\{\delta(\gamma-1)t\}}dt. \qquad (23)
$$

Upon exercising the option, it receives

$$
EP_t - I, \qquad (24)
$$

where

$$
EP_t = E_t\left[\int_{t+\Delta}^{\infty}\xi_{t,v}k_{1,v}\left[\lambda X_v D(\Sigma_{i=1}^2 k_{i,v})\right]dv\right] = \lambda\kappa^{1-\gamma}X_t\frac{e^{\{\delta\gamma(t+\Delta)\}\left(e^{\{\delta(t+\Delta)\}}+1\right)^{-\gamma}}}{\delta\,(1-\gamma)}.^{8} \qquad (25)
$$

The strategy space for firm 1 is $\mathcal{S}_{0,\infty}$, the set of stopping times that take values between 0 and ∞. For any $t_1 \in \mathcal{S}_{0,\infty}$, the value of the leader's capital-replacement option is given by

$$
\begin{aligned}
G_t(EP,H,t_1) &= \tilde{E}_t\left[\int_t^{t_1} e^{\{-r(v-t)\}}dH_v + e^{\{-r(t_1-t)\}}\left[EP_{t_1} - I\right]\right] \\
&= \lim_{T\to\infty}\tilde{E}_t\left[\int_t^{T} e^{\{-r(v-t)\}}dH_v + e^{\{-r(T-t)\}}\left[EP_T - I\right]^{+}\right] + EEP_t. \qquad (26)
\end{aligned}
$$

The following Lemma derives the expected appreciation in future revenues that will be used to quantify potential local gains from early exercise.

[8] The mathematical derivation of EP_t is contained in the Appendix.

Lemma 2 *The expected appreciation in future revenues is* $dA_t = EP_t \left[\alpha + \delta\gamma \left(\frac{e^{\{-\delta(t+\Delta)\}}}{1+e^{\{-\delta(t+\Delta)\}}}\right)\right] dt.$

Proof. Applying Ito's Lemma to Equation (25) yields

$$dEP_t = \underbrace{EP_t \left[\alpha + \delta\gamma \left(\frac{e^{\{-\delta(t+\Delta)\}}}{1+e^{\{-\delta(t+\Delta)\}}}\right)\right] dt}_{dA_t} + \sigma EP_t dz_t. \tag{27}$$

■

Having now acquired all the elements necessary for the implementation of the EEP representation, we sign the local gains associated with the G option and we derive the optimal investment policy for firm 1.

Proposition 7 *Firm* 1 *will not exercise its capital-replacement option.*

Proof. Before proceeding with the proof let us rearrange Eq. (23) to express dH_t as a function of EP_t. We obtain

$$dH_t = EP_t \frac{2^{-\gamma}\delta(1-\gamma)e^{\{\delta(\gamma-1)t\}}}{\lambda e^{\{\delta\gamma(t+\Delta)\}}(e^{\{\delta(t+\Delta)\}}+1)^{-\gamma}} dt. \tag{28}$$

Using Eq. (28) the early exercise premium for the leader firm becomes

$$\begin{aligned} EEP_t &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - EP_v \left[r + \delta\gamma \left(\frac{e^{\{-\delta(v+\Delta)\}}}{1+e^{\{-\delta(v+\Delta)\}}}\right)\right] dv \right)\right] \\ &\quad - \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(EP_v \frac{2^{-\gamma}\delta(1-\gamma)e^{\{\delta(\gamma-1)v\}}}{\lambda e^{\{\delta\gamma(v+\Delta)\}}(e^{\{\delta(v+\Delta)\}}+1)^{-\gamma}} \right) dv \right] \\ &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \phi_v dv, \right] \end{aligned}$$

which leads to perpetual inaction since

$$\phi_v = -rI - EP_v \left[\delta\gamma \left(\frac{e^{\{-\delta(v+\Delta)\}}}{1+e^{\{-\delta(v+\Delta)\}}}\right) + \frac{2^{-\gamma}\delta(1-\gamma)e^{\{\delta(\gamma-1)v\}}}{\lambda e^{\{\delta\gamma(v+\Delta)\}}(e^{\{\delta(v+\Delta)\}}+1)^{-\gamma}}\right] < 0.$$

■

Interestingly, even though (i) productive capital continuously depreciates at a strictly positive rate δ, (ii) investing in new capital yields a revenue enhancing quality improvement $\lambda > 1$, and (iii) firm 1 is unencumbered with strategic considerations, since the follower optimally chooses to never install new capital goods, firm 1 finds it optimal to perpetually delay the adoption of the new technology. This result suggests that the introduction of a second mover disadvantage is required to create adequate incentives for both firms to exercise their investment option. The introduction of such a harm effect to the follower's revenues allows us to illustrate another use of the EEP representation in strategic real option games. The next Section modifies the model described in Section 2 to account for harm effects.

4 An Amended Model with Harm Effects

Grenadier (1996) develops an equilibrium model for the optimal investment timing of two symmetric firms competing in the real estate market, in which he accounts for the degree of "harm" to the follower's revenues caused by the leader's adoption of the improved technology. Along the same line of reasoning, this Section introduces a constant additive harm effect $h > 0$ in the revenue of firm 2 as a penalty for producing with the old technology. This penalty may be motivated as part of a consumer retention effort via advertising. While the revenue equations for firm 1 are unchanged, the revenue equations for firm 2 are lessened by h in the time interval $[t_1 + \Delta, t_2]$, starting when the leader re-enters the market after the acquisition and installation of the new capital goods. Equation (29) contains the revenues for firm 2 in the amended model.

$$R_2(t) = \begin{bmatrix} k_{2,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & 0 \le t < t_1 \\ k_{2,t}\left[X_t D(k_{2,t})\right], & t_1 \le t < t_1 + \Delta \\ k_{2,t}\left[X_t D(\Sigma_{i=1}^2 k_{i,t})\right] - h, & t_1 + \Delta \le t < t_2 \\ 0, & t_2 \le t < t_2 + \Delta \\ k_{2,t}\left[\lambda X_t D(\Sigma_{i=1}^2 k_{i,t})\right], & t_2 + \Delta \le t < \infty \end{bmatrix}. \tag{29}$$

In this amended environment we exploit the EEP approach to measure local gains from early exercise and, in particular, (i) we derive a sufficient condition for the follower to invest with positive probability in equilibrium, (ii) we demonstrate the relation of our results with Grenadier's and (iii) we obtain a non-linear recursive integral equation that can be used to obtain the follower's optimal trigger boundary.

4.1 Firm 2's Optimal Behavior Beyond $t_1 + \Delta$ Including Harm Effects

Suppose that firm 1 has already begun to produce using its new capital stock. Accordingly, firm 2's capital-replacement option is given by Eq. (10). While the instantaneous revenue flow is altered to be consistent with Eq. (29), and it obeys

$$\begin{aligned} dH_t &= \left[X_t k_{2,t} D(\Sigma_{i=1}^2 k_{i,t}) - h\right] dt \\ &= \left[\kappa^{1-\gamma} X_t e^{\{\delta(\gamma-1)t\}} \left(e^{\{\delta(t_1+\Delta)\}} + 1\right)^{-\gamma} - h\right] dt, \end{aligned} \tag{30}$$

the value of future payoffs upon exercise, EP_t, is unaffected by the harm effect and thus is given by Equation (9). This in turn implies that the expected appreciation in payoffs, dA_t, is given by Lemma 1. The following two Propositions derive a closed form expression for the EEP, which is used to provide a sufficient condition for perpetual inaction on the part of both firms.

Proposition 8 *For any maturity date T, $T \to \infty$, the early exercise premium EEP_t takes the form*

$$EEP_t = \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \phi_v dv \right],$$

where

$$\phi_v = h - rI - EP_v \left[\frac{\gamma \delta e^{\{\delta(t_1 - v)\}}}{e^{\{\delta(t_1-v)\}} + 1} + \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}} \right)^{-\gamma}} \right]. \tag{31}$$

Proof. Rearranging Eq. (30) to express dH_t as a function of EP_t yields

$$dH_t = \left[EP_t \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta t\}} \left(e^{\{\delta t_1\}} + e^{\{\delta t\}} \right)^{-\gamma}} - h \right] dv,$$

which is substituted in Eq. (11) to obtain

$$\begin{aligned} EEP_t &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} (r \left[EP_v - I \right] dv - EP_v \left[r + \gamma\delta \frac{e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}} + 1} \right] dv) \right] \\ &\quad - \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(EP_v \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}} \right)^{-\gamma}} - h \right) dv \right] \\ &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \phi_v dv \right], \end{aligned}$$

in which

$$\phi_v = h - rI - EP_v \left[\frac{\gamma \delta e^{\{\delta(t_1 - v)\}}}{e^{\{\delta(t_1-v)\}} + 1} + \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}} \right)^{-\gamma}} \right].$$

■

Proposition 9 *A sufficient condition for perpetual inaction by both firms is that the constant harm effect h be less than or equal to rI.*

Proof. Consider Eq. (31) under the condition that $h \le rI$. It follows directly that ϕ_v is always less than $-EP_v \left[\frac{\gamma \delta e^{\{\delta(t_1 - v)\}}}{e^{\{\delta(t_1-v)\}} + 1} + \frac{\delta (1-\gamma) \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta v\}} \left(e^{\{\delta t_1\}} + e^{\{\delta v\}} \right)^{-\gamma}} \right]$, for all v. Since this last term is always non-positive, we find that, beyond time $t_1 + \Delta$, there are no local gains from early exercise and firm 2's EEP is worthless. Perpetual inaction for *both* firms follows from Propositions 3, 5, 6, and 7. ■

In what follows, we further discuss our assumption of an *additive* rather than a *proportional* harm effect in relation to Grenadier's (1996) model. We show via the EEP representation how a multiplicative harm effect delivers a degenerate firm behavior in our model while it does not in Grenadier's simpler environment.

Remark 1 *In the presence of a multiplicative harm effect, Propositions 3, 5, 6, and 7 obtain and deliver perpetual inaction on the part of both firms. Indeed, under the alternative assumption of proportional harm effects, Eq. (30) becomes*

$$
\begin{aligned}
dH_t &= \left[(1-h)\, X_t k_{2,t} D(\Sigma_{i=1}^2 k_{i,t})\right] dt \\
&= (1-h)\, \kappa^{1-\gamma} X_t e^{\{\delta(\gamma-1)t\}} \left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} dt \\
&= EP_t \frac{(1-h)\,\delta\,(1-\gamma)\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta t\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma}} dt,
\end{aligned}
$$

in which the last equality is the result of standard algebraic manipulations to rewrite dH_t as a function of EP_t. Correspondingly, Eq. (11) becomes

$$
\begin{aligned}
EEP_t &= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v=v\}} (r\,[EP_v - I]\, dv - EP_v \left[r + \gamma\delta \frac{e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}}+1}\right] dv)\right] \\
&\quad - \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v=v\}} \left(EP_v \frac{(1-h)\,\delta\,(1-\gamma)\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta v\}}\left(e^{\{\delta t_1\}}+e^{\{\delta v\}}\right)^{-\gamma}}\right) dv\right] \\
&= \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v=v\}} \phi_v dv\right],
\end{aligned}
$$

in which

$$
\phi_v = -rI - EP_v \left[\frac{\gamma\delta e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}}+1} + \frac{(1-h)\,\delta\,(1-\gamma)\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta v\}}\left(e^{\{\delta t_1\}}+e^{\{\delta v\}}\right)^{-\gamma}}\right] < 0.
$$

The previous Remark motivates our adoption of an additive harm effect, which has significantly different implications than its multiplicative counterpart. However, this does not apply to Grenadier's model since, with constant revenues prior to investment, multiplicative and additive harm effects are equivalent, the proof of which is the object of the following Remark.

Remark 2 ***Relation to Grenadier (1996).*** *Grenadier develops a strategic real options game, which he solves via the partial differential equation (PDE) approach. In order to obtain the follower's investment policy he solves the differential equation*

$$
0 = \frac{1}{2}\sigma^2 X^2 W''(X) + \mu X W'(X) - rW(X) \tag{32}
$$

subject to the following optimality conditions

$$
W(X_F) + \frac{(1-h)\,R}{r} = \left[\frac{D(2)}{r-\alpha} e^{-(r-\alpha)\Delta}\right] X_F - I \tag{33}
$$

$$
W'(X_F) = \left[\frac{D(2)}{r-\alpha} e^{-(r-\alpha)\Delta}\right] \tag{34}
$$

$$
W(0) = 0. \tag{35}
$$

Equation (33) is the value-matching condition, which requires that the value of the option upon exercise be equal to the value of the net payoff, Eq. (34) is the smooth-pasting or high-contact condition (Merton, 1973), which requires that marginal gains be equated, and Eq. (35) is an absorbing barrier preventing arbitrage. We adapt Grenadier's notation to ours with the exception of $D(Q)$, which is a deterministic inverse demand function with $Q \in \{1,2\}$ representing the number of new production facilities and X_F, which is the constant trigger boundary for the follower. R is the constant flow of revenues earned by the follower until exercise. It is trivial to show that Eq. (33) can be re-expressed as

$$W(X_F) + \frac{R}{r} - h' = \left[\frac{D(2)}{r-\alpha} e^{-(r-\alpha)\Delta}\right] X_F - I, \tag{36}$$

where $h\prime = \frac{hR}{r}$ is the additive harm effect corresponding to h. Hence, in Grenadier's simpler environment, considering an additive rather than proportional harm effect leaves the problem unaltered.

Proposition 10 below derives a pseudo closed form solution for the value of the capital-replacement option held by the follower beyond $t_1 + \Delta$, which is instrumental in obtaining a recursive integral equation for the trigger boundary.

Proposition 10 *If $h > rI$, the value of the American capital-replacement option held by firm 2 beyond time $t_1 + \Delta$ is given by*

$$\begin{aligned}
& W_t(EP, H, t_2) \\
= \quad & EP_t \left[\frac{\left(e^{\{\delta(t_1+\Delta)\}} + 1\right)^{-\gamma}}{\lambda e^{\{\delta t\}} \left(e^{\{\delta t_1\}} + e^{\{\delta t\}}\right)^{-\gamma}} + \left(e^{\{\delta(t_1-t)\}} + 1\right)^{\gamma} \right] - \frac{h}{r} \\
& + (h - rI) \left[\lim_{T\to\infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi \left(\frac{\ln\left(\frac{EP_t}{B_v}\right) + \mu_{v-t}}{\sigma\sqrt{v-t}} \right) \right] dv \right] \\
& - \lim_{T\to\infty} \int_t^T EP_t e^{\{-r(v-t)\}} \left(\frac{e^{\{\delta(t_1-t)\}} + 1}{e^{\{\delta(t_1-v)\}} + 1} \right)^{\gamma} \Phi \left(\frac{\ln(\frac{EP_t}{B_v}) + \mu_{v-t} + \sigma^2(v-t)}{\sigma\sqrt{v-t}} \right) dv, \quad (37)
\end{aligned}$$

with $\mu_{v-t} = \left(r - \frac{1}{2}\sigma^2\right)(v-t) - \gamma \ln\left(\frac{e^{\{\delta(t_1-v)\}}+1}{e^{\{\delta(t_1-t)\}}+1}\right)$ and B_v is the immediate exercise boundary.

Proof. A pseudo closed form solution is derived for the value of the early exercise premium as a function of the unknown exercise boundary B_v as in Kim (1990), Jacka (1991), and Carr-Jarrow-Myneni (1992). See Appendix for details. ■

Equation (37) represents the American option value in terms of the unknown optimal exercise boundary B_v, thereby making it a pseudo rather than an actual closed form solution. By using the fact that immediate exercise is optimal when the boundary is reached, the EEP formula produces a recursive integral equation for the trigger boundary, which is derived in Theorem 1.

Theorem 1 *For any given T, $T \to \infty$, the immediate exercise boundary B solves the recursive non-linear integral equation*

$$\begin{aligned} B_t - I &= B_t \left[\frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta t\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma}} + \left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma} \right] - \frac{h}{r} \\ &\quad + (h - rI)\left[\lim_{T\to\infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi\left(\frac{\ln\left(\frac{B_t}{B_v}\right)+\mu_{v-t}}{\sigma\sqrt{v-t}} \right) \right] dv \right] \\ &\quad - \lim_{T\to\infty} \int_t^T B_t e^{\{-r(v-t)\}} \left(\frac{e^{\{\delta(t_1-t)\}}+1}{e^{\{\delta(t_1-v)\}}+1} \right)^{\gamma} \Phi\left(\frac{\ln(\frac{B_t}{B_v})+\mu_{v-t}+\sigma^2(v-t)}{\sigma\sqrt{v-t}} \right) dv \end{aligned} \tag{38}$$

where the terminal boundary is $B_T = I$.

Proof. The result follows directly from Proposition 31 and Corollary 34 of Detemple (2006), which state that the terminal boundary is given by the maximum of the strike price and the strike price multiplied by the ratio of the risk-free rate to the implicit dividends yield on the underlying asset, i.e., in our context,

$$B_T = \max\left[I, \frac{r}{-\frac{\gamma\delta e^{\{\delta(t_1-t)\}}}{e^{\{\delta(t_1-t)\}}+1}} I \right] = I. \tag{39}$$

■

Equation (38) can be solved numerically via a recursive algorithm that makes use of the terminal value given by Eq. (39). The limiting trigger frontier (or immediate exercise boundary) can be approximated by iterating with increasingly large T.

4.2 Firm 2's Optimal Behavior Prior to $t_1 + \Delta$ Including Harm Effects

Because the harm effect penalizes firm 2 only after firm 1 re-enters the market, the problem of optimal exercise of the capital-replacement option prior to firm 1's re-entry is unchanged relative to Subsection 3.2. Therefore, we find that firm 2 will never exercise its capital replacement option prior to time $t_1 + \Delta$.

4.3 Firm 1's Optimal Behavior Including Harm Effects

As was the case in Subsection 3.3, firm 1 holds a perpetual American option with a strike price I and an option dividend rate equal to

$$dH_t = X_t k_{1,t} D(\Sigma_{i=1}^2 k_{i,t})dt = 2^{-\gamma}\kappa^{1-\gamma} X_t e^{\{\delta(\gamma-1)t\}}dt. \tag{40}$$

Upon exercising the option, firm 1 receives

$$EP_t - I, \tag{41}$$

where

$$
\begin{aligned}
EP_t &= \lambda E_t \left[\int_{t+\Delta}^{t_2} \xi_{t,v} k_{1,v} \left[X_v D(\Sigma_{i=1}^2 k_{i,v}) \right] dv + \int_{t_2}^{t_2+\Delta} \xi_{t,v} k_{1,v} \left[X_v D(k_{1,v}) \right] dv \right] \\
&\quad + \lambda E_t \left[\int_{t_2+\Delta}^{\infty} \xi_{t,v} k_{1,v} \left[X_v D(\Sigma_{i=1}^2 k_{i,v}) \right] dv \right] \\
&= \frac{\lambda \kappa^{2-\gamma} \left(\kappa e^{\{\delta(t+\Delta)\}} + 1 \right)^{1-\gamma} e^{\{\delta(t+\Delta)\}}}{1-\gamma} X_t E_t \left[\int_{t+\Delta}^{t_2} \xi_{t,v} e^{\left\{ (\alpha - \frac{1}{2}\sigma^2)(v-t) + \sigma(Z_v - Z_t) \right\}} e^{\{-(2-\gamma)\delta v\}} dv \right] \\
&\quad + \lambda \frac{\kappa^{2-\gamma} e^{\{\delta(2-\gamma)(t+\Delta)\}}}{1-\gamma} X_t E_t \left[\int_{t_2}^{t_2+\Delta} \xi_{t,v} e^{\{-(2-\gamma)\delta v\}} e^{\left\{ (\alpha - \frac{1}{2}\sigma^2)(v-t) + \sigma(Z_v - Z_t) \right\}} dv \right] \\
&\quad + \lambda \frac{\kappa^{2-\gamma} e^{\{(2-\gamma)\delta\Delta + \delta t\}}}{1-\gamma} X_t E_t \left[(e^{\{\delta t\}} + e^{\{\delta t_2\}})^{1-\gamma} \int_{t_2+\Delta}^{\infty} \xi_{t,v} e^{\{-(2-\gamma)\delta v\}} e^{\left\{ (\alpha - \frac{1}{2}\sigma^2)(v-t) + \sigma(Z_v - Z_t) \right\}} dv \right].
\end{aligned} \tag{42}
$$

For any of firm 1's option-exercise policies $t_1 \in \mathcal{S}_{0,\infty}$, where $\mathcal{S}_{0,\infty}$ is the set of stopping times that take values between 0 and ∞, the value of its capital-replacement option is given by

$$
\begin{aligned}
G_t(EP, H, t_1) &= \tilde{E}_t \left[\int_t^{t_1} e^{\{-r(v-t)\}} dH_v + e^{\{-r(t_1-t)\}} \left[EP_{t_1} - I \right] \right] \\
&= \lim_{T \to \infty} \tilde{E}_t \left[\int_t^{T} e^{\{-r(v-t)\}} dH_v + e^{\{-r(T-t)\}} \left[EP_T - I \right]^+ \right] + EEP_t. \qquad (43)
\end{aligned}
$$

While Eq. (42) is analytically intractable because t_2 is a random variable preventing the passage of the expectation operator into the integrals, a specific case is sufficient to bring our analysis to a definitive conclusion. In particular, we consider the extreme case in which the harm effect h is so large ($h \to \infty$) that exercising the option at time $t_1 + \Delta$ is optimal also surely. The opposite limiting case occurs as h goes to zero, and $t_2 \to \infty$ with probability one, for which Propositions 3, 5, 6, and 7 obtain.

Consider Eq. (37) as h is made increasingly large. For any given boundary B, $W_t(EP, H, t_2)$ can be rewritten as

$$
\begin{aligned}
& W_t(EP, H, t_2) \\
&= EP_t \left[\frac{\left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{\lambda e^{\{\delta t\}} \left(e^{\{\delta t_1\}} + e^{\{\delta t\}} \right)^{-\gamma}} + \left(e^{\{\delta(t_1 - t)\}} + 1 \right)^{\gamma} \right] \\
&\quad - rI \left[\lim_{T \to \infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi \left(\frac{\ln\left(\frac{EP_t}{B_v} \right) + \mu_{v-t}}{\sigma \sqrt{v-t}} \right) \right] dv \right] \\
&\quad - \lim_{T \to \infty} \int_t^T EP_t e^{\{-r(v-t)\}} \left(\frac{e^{\{\delta(t_1-t)\}} + 1}{e^{\{\delta(t_1-v)\}} + 1} \right)^{\gamma} \Phi \left(\frac{\ln(\frac{EP_t}{B_v}) + \mu_{v-t} + \sigma^2(v-t)}{\sigma\sqrt{v-t}} \right) dv \\
&\quad + h \left(\left[\lim_{T \to \infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi \left(\frac{\ln\left(\frac{EP_t}{B_v} \right) + \mu_{v-t}}{\sigma\sqrt{v-t}} \right) \right] dv \right] - \frac{1}{r} \right),
\end{aligned} \tag{44}
$$

whose first three terms are bounded for any $t_2 \in \mathcal{S}_{t_1+\Delta,\infty}$. We focus on the last term in Eq. (44) under the assumption that B_v is not identically equal to zero, so that

$$\left(\lim_{T\to\infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi \left(\frac{\ln\left(\frac{EP_t}{B_v}\right) + \mu_{v-t}}{\sigma\sqrt{v-t}} \right) \right] dv - \frac{1}{r} \right) < \lim_{T\to\infty} \int_t^T \left[e^{\{-r(v-t)\}} \right] dv - \frac{1}{r} = 0,$$

which implies that

$$\frac{\partial W}{\partial h} \mid_{\bar{B}} < 0. \tag{45}$$

Recall that firm 2 optimally chooses to exercise its option at time $t_1 + \Delta$ whenever

$$max[W_{t_1+\Delta}, EP_{t_1+\Delta} - I] = EP_{t_1+\Delta} - I, \tag{46}$$

as in Proposition 6. In light of Eq. (45), Eq. (46) can be forced to obtain by selecting a sufficiently large harm effect. Indeed, the value of the option is reduced by a large h down to the point at which firm 2 prefers to exercise immediately, which is equivalent to setting $B_{t_1+\Delta} = 0$. Under these conditions, Eq. (42) becomes

$$\begin{aligned} EP_t &= \frac{\lambda\kappa^{2-\gamma} e^{\{\delta(2-\gamma)(t+\Delta)\}} X_t}{1-\gamma} E_t \left[\int_{t+\Delta}^{t+2\Delta} \xi_{t,v} e^{\{-(2-\gamma)\delta v\}} e^{\left\{(\alpha - \frac{1}{2}\sigma^2)(v-t) + \sigma(Z_v - Z_t)\right\}} dv \right] \\ &\quad + \frac{\lambda\kappa^{2-\gamma} e^{\{(2-\gamma)\delta\Delta + \delta t\}} X_t}{1-\gamma} E_t \left[(e^{\{\delta t\}} + e^{\{\delta(t+\Delta)\}})^{1-\gamma} \int_{t+2\Delta}^{\infty} \xi_{t,v} e^{\{-(2-\gamma)\delta v\}} e^{\left\{(\alpha - \frac{1}{2}\sigma^2)(v-t) + \sigma(Z_v - Z_t)\right\}} dv \right] \\ &= -\frac{\lambda\kappa^{2-\gamma} X_t}{\delta(2-\gamma)(1-\gamma)} e^{\{\delta(2-\gamma)(t+\Delta)\}} \left[e^{\{-(2-\gamma)\delta(t+2\Delta)\}} - e^{\{-(2-\gamma)\delta(t+\Delta)\}} \right] \\ &\quad + \frac{\lambda\kappa^{2-\gamma} X_t}{\delta(2-\gamma)(1-\gamma)} e^{\{(2-\gamma)\delta\Delta + \delta t\}} (e^{\{\delta t\}} + e^{\{\delta(t+\Delta)\}})^{1-\gamma} e^{\{-(2-\gamma)\delta(t+2\Delta)\}} \\ &= \frac{\lambda\kappa^{2-\gamma} X_t}{\delta(2-\gamma)(1-\gamma)} \left[1 - e^{\{-\delta(2-\gamma)\Delta\}} \left[1 - (1 + e^{\{\delta\Delta\}})^{1-\gamma} \right] \right]. \end{aligned} \tag{47}$$

A standard application of Ito's Lemma to Eq. (47) yields

$$dEP_t = \underbrace{EP_t \alpha dt}_{=dA_t} + EP_t \sigma dz_t. \tag{48}$$

Substituting dA_t in Eq. (11) shows that there can be no local gains from early exercise and that perpetual inaction obtains.

5 Conclusion

In recent years a significant research effort has been devoted to the analysis of optimal strategic investment decisions in the presence of uncertainty. Because (i) the optimal timing of project initiation resembles the optimal exercise of an American option and (ii) firm interactions tend to play a crucial role in the adoption of new technologies, this literature has merged the option-pricing methodology with the game-theoretical paradigm to analyze what is best viewed as strategic capital replacement games. The isomorphism between a firm's real investment projects and a set of financial positions in American options permits the application of some of the most advanced techniques developed in contingent claims analysis, such as the early exercise premium (EEP) representation, which expresses an American option as the combination of a European option and the right to exercise the option early.

This paper considers an economy populated by two symmetric firms, each holding a unique capital-replacement option over an infinite horizon. In addition, a second mover disadvantage is introduced additively in the revenues of the firm that still operates the old technology. The contribution of this paper is twofold, economic and methodological. The economic contribution is the recognition that, under fixed costs of investment and time-to-build, the firm pays a fraction of the implicit strike price to its competitor in the form of transferred foregone consumer demand. The methodological contribution is the introduction of the EEP representation as a valuable device for the characterization of optimal exercise policies in real options games.

We find that (i) in the absence of depreciation, despite the value of the follower's future revenues upon exercise approaching infinity, the follower never exercises its investment option and (ii) even in the presence of a strictly positive depreciation rate, and despite the technology improvement, the follower chooses never to exercise the American option, leading to perpetual inaction on the part of both firms. Furthermore, a second mover disadvantage that is *not* proportional to the firm's revenues is crucial to obtain a non-degenerate investment policy on the part of the follower. In this context, the EEP formula produces a recursive integral equation, which can be used to study the characteristics of the follower's optimal exercise boundary. Taking as given the follower's optimal response function, the leader's policy equation is analytically intractable. However, a specific case brings our analysis to a definitive conclusion: in the extreme case in which harm effects to the low-technology producer grow so large that immediate reaction is optimal for the follower, the leader's policy reduces to a closed form expression. The application of EEP analysis to this closed form solution shows that the leader optimally delays investment forever, thereby producing a perpetual inaction equilibrium.

References

[1] Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81, 1973.

[2] Boyer, M., E. Gravelle, and P. Lasserre, "Real Options and Strategic Competition: A Survey," Working Paper, 2004.

[3] Cox, J., and S. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3, 1976.

[4] Carr, P, R. Jarrow, and R. Myneni, "Alternative Characterization of American Put Options," *Mathematical Finance*, 2, 1992.

[5] Cooper, R., J. Haltiwanger and L. Power, "Machine Replacement and the Business Cycle: Lumps and Bumps," *American Economic Review*, 89, 4, 1999.

[6] Detemple, J., American-Style Derivatives: Valuation and Computation, Chapman and Hall, 2006.

[7] Dixit, A., and R. Pindyck, Investment under Uncertainty, Princeton University Press, Princeton, NJ, 1994.

[8] Fudenberg, D., and J. Tirole, Game Theory, MIT Press, Cambridge, MA, 1991.

[9] Fudenberg, D., and J. Tirole, Preemption and Rent Equalization in the Adoption of New Technology, *Review of Economic Studies*, 52, 1985.

[10] Grenadier, S., "The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets," *Journal of Finance*, 51, 5, 1996.

[11] Grossman, G. and E. Helpman, "Quality Ladders in the Theory of Growth," *Review of Economic Studies*, 58, 1991.

[12] Huisman, K., P. Kort, G. Pawlina, and J. Thijssen, "Strategic Investment under Uncertainty: Merging Real Options with Game Theory," Working Paper, Tilburg University, 2003.

[13] Huisman, K., and P. Kort, "A Further Analysis on Strategic Timing of Adoption of New Technologies under Uncertainty," CentER Discussion Paper 9803, Tilburg University, 1998.

[14] Jacka, S, "Optimal Stopping and the American Put," *Mathematical Finance*, 1, 1991.

[15] Kim, I., "The Analytic Valuation of American Options," *Review of Financial Studies,* 3, 1990.

[16] McKean, H., "A Free Boundary Problem For the Heat Equation Arising From a Problem In Mathematical Economics," *Industrial Management Review*, 6, 1965.

[17] Merton, R., "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 1973.

[18] Myerson, R., Game Theory: Analysis of Conflict, Harvard University Press, 2004.

[19] Samuelson, P., "Rational Theory of Warrant Pricing," *Industrial Management Review*, 6, 1965.

[20] Samuelson, P. and R. Merton, "A Complete Model of Warrant Pricing that Maximizes Utility," *Sloan Management Review*, 1969.

[21] Smets, F., "Exporting versus FDI: The Effect of Uncertainty, Irreversibilities and Strategic Interactions.," Working Paper, Yale University, 1991.

[22] Stackelberg, H., "Marktform und Gleichgewicht," 1934.

[23] Stenbacka, R., and M. Tombak, "Strategic Timing Of Adoption of New Technologies Under Uncertainty," *International Journal of Industrial Organization*, 12, 1994.

[24] Vasicek, O., "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 1977.

[25] Wu, J., "Credible Capacity Preemption in a Duopoly Market Under Uncertainty," Working Paper, 2006.

6 Appendix

This Subsection contains lengthy proofs and mathematical derivations used in the body of the article.

6.1 Derivation of Firm 2's EP_t

$$
\begin{aligned}
EP_t &= E_t\left[\int_{t+\Delta}^{\infty} \xi_{t,v} k_{2,v}\left[\lambda X_v D(\Sigma_{i=1}^2 k_{i,v})\right] dv\right] \\
&= \lambda X_t E_t[\int_{t+\Delta}^{\infty} e^{\left\{-\left[r+\frac{1}{2}\theta^2\right](v-t)-\theta(z_v-z_t)\right\}} \kappa e^{\{-\delta(v-t-\Delta)\}} \\
&\quad \cdot e^{\left\{\left[\alpha-\frac{1}{2}\sigma^2\right](v-t)+\sigma(z_v-z_t)\right\}}\left(\kappa e^{\{-\delta(v-t_1-\Delta)\}}+\kappa e^{\{-\delta(v-t-\Delta)\}}\right)^{-\gamma} dv] \\
&= \lambda\kappa^{1-\gamma} e^{\{\delta t+(1-\gamma)\delta\Delta\}} X_t \int_{t+\Delta}^{\infty} e^{\{-\delta v\}}\left(e^{\{-\delta(v-t_1)\}}+e^{\{-\delta(v-t)\}}\right)^{-\gamma} dv \\
&= \lambda\kappa^{1-\gamma} e^{\{\delta t+(1-\gamma)\delta\Delta\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma} X_t \int_{t+\Delta}^{\infty} e^{\{\delta(\gamma-1)v\}} dv \\
&= -\frac{\lambda\kappa^{1-\gamma} X_t}{\delta(\gamma-1)} e^{\{\delta t+\delta(1-\gamma)\Delta\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma} e^{\{\delta(\gamma-1)(t+\Delta)\}} \\
&= \lambda\kappa^{1-\gamma} X_t \frac{e^{\{\delta\gamma t\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma}}{\delta(1-\gamma)}.
\end{aligned}
$$

6.2 Proposition 2

If the rate of depreciation $\delta = 0$, then (i) the value of the firm's future revenues upon exercise approaches $+\infty$ and (ii) the firm never exercises.

Proof.

(i) Let

$$\xi_{t,v} = \exp\left\{-\left[r + \frac{1}{2}\theta^2\right](v-t) - \theta(z_v - z_t)\right\} \tag{49}$$

be the state-price density, in which $\theta = \frac{\alpha - r}{\sigma}$ is the market price of risk. Then, under no depreciation, EP_t is

$$\begin{aligned} EP_t &= E_t\left[\int_{t+\Delta}^{\infty} \xi_{t,v}\lambda\kappa X_v D(2\kappa) dv\right] \\ &= \lambda 2^{-\gamma}\kappa^{1-\gamma} E_t\left[\int_{t+\Delta}^{\infty} \xi_{t,v} X_v dv\right] \\ &= \lambda 2^{-\gamma}\kappa^{1-\gamma} X_t E_t[\int_{t+\Delta}^{\infty} e^{\left\{-\left[r+\frac{1}{2}\theta^2\right](v-t)-\theta(z_v-z_t)\right\}} e^{\left\{\left[\alpha-\frac{1}{2}\sigma^2\right](v-t)+\sigma(z_v-z_t)\right\}} dv] \\ &= \lambda 2^{-\gamma}\kappa^{1-\gamma} X_t E_t[\int_{t+\Delta}^{\infty} e^{\left\{\left[\alpha-\frac{1}{2}\sigma^2-r-\frac{1}{2}\theta^2\right](v-t)+(\sigma-\theta)(z_v-z_t)\right\}} dv] \\ &= \lambda 2^{-\gamma}\kappa^{1-\gamma} X_t[\int_{t+\Delta}^{\infty} 1 dv] \to +\infty \end{aligned} \tag{50}$$

(ii) Under the risk-neutral measure and with no depreciation,

$$\begin{aligned} EEP_t &= \lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - dA_v - dH_v\right)\right] \\ &= \lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - rEP_v dv - dH_v\right)\right] \\ &= \lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(-rIdv - dH_v\right)\right]. \end{aligned} \tag{51}$$

Because $(-rIdv - dH_v)$ is always negative, the American option will never be exercised prior to its maturity date by Corollary 22 of Detemple (2006).

■

6.3 Proposition 4

In the case of a strictly positive depreciation rate δ, and a quality factor $\lambda > 1$, the option can be treated as a pure European option, whose value is given by

$$W_t(EP, H, t_2) = \frac{X_t \kappa^{1-\gamma} e^{\{\delta(\gamma-1)t\}}}{(1-\gamma)\delta} \left[\left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma} + \lambda e^{\{\delta t\}} \left(\frac{e^{\{\delta(t_1-t)\}} + 1}{e^{\{\delta t_1\}} + e^{\{\delta t\}}} \right)^{\gamma} \right]. \quad (52)$$

Proof. With a worthless early exercise premium, Equation (10) becomes

$$W_t(EP, H, t_2) = \lim_{T \to \infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} dH_v + e^{\{-r(T-t)\}} [EP_T - I]^+ \right] \quad (53)$$

Consider the second term in Equation (53):

$$\begin{aligned}
& e^{\{-r(T-t)\}} \tilde{E}_t \left[[EP_T - I]^+ \right] \\
= \; & e^{\{-r(T-t)\}} \int_{\ln I}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} [EP_T - I] \, e^{\left(\frac{-[\ln EP_T - \ln EP_t - \mu_{T-t}]^2}{2\sigma^2(T-t)} \right)} d\ln EP_T \\
= \; & e^{\{-r(T-t)\}} \int_{\ln I}^{\infty} \frac{EP_T}{\sqrt{2\pi\sigma^2(T-t)}} e^{\left(\frac{-[\ln EP_T - \ln EP_t - \mu_{T-t}]^2}{2\sigma^2(T-t)} \right)} d\ln EP_T \\
& - e^{\{-r(T-t)\}} \int_{\ln I}^{\infty} \frac{I}{\sqrt{2\pi\sigma^2(T-t)}} e^{\left(\frac{-[\ln EP_T - \ln EP_t - \mu_{T-t}]^2}{2\sigma^2(T-t)} \right)} d\ln EP_T, \quad (54)
\end{aligned}$$

which is a variant of the Black-Scholes formula (for derivations along probabilistic lines, see Baz and Chacko, 2004, pp.57-61). Under the risk-neutral measure,

$$\ln EP_T = \ln EP_t + \left(r - \frac{1}{2}\sigma^2 \right)(T-t) + \gamma\delta \int_t^T \frac{e^{\{\delta(t_1-u)\}}}{e^{\{\delta(t_1-u)\}} + 1} du + \sigma(z_T - z_t),$$

and, denoting

$$\mu_{T-t} = \left(r - \frac{1}{2}\sigma^2 \right)(T-t) + \gamma\delta \int_t^T \frac{e^{\{\delta(t_1-u)\}}}{e^{\{\delta(t_1-u)\}} + 1} du,$$

we find $\ln EP_T \sim N\left(\ln EP_t + \mu_{T-t}, \sigma^2(T-t) \right)$.

We can rewrite Equation (54) above as

$$\begin{aligned}
e^{\{-r(T-t)\}}\tilde{E}_t\left[[EP_T - I]^+\right] &= EP_t e^{\left\{-\gamma \ln\left(\frac{e^{\{\delta(t_1-T)\}}+1}{e^{\{\delta(t_1-t)\}}+1}\right)\right\}} \Phi\left(\frac{\ln(\frac{EP_t}{I}) + \mu_{T-t} + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -Ie^{-r(T-t)}\Phi\left(\frac{\ln(\frac{EP_t}{I}) + \mu_{T-t}}{\sigma\sqrt{T-t}}\right) \\
&= EP_t\left(\frac{e^{\{\delta(t_1-t)\}}+1}{e^{\{\delta(t_1-T)\}}+1}\right)^{\gamma} \Phi\left(\frac{\ln(\frac{EP_t}{I}) + \mu_{T-t} + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -Ie^{-r(T-t)}\Phi\left(\frac{\ln(\frac{EP_t}{I}) + \mu_{T-t}}{\sigma\sqrt{T-t}}\right). \qquad (55)
\end{aligned}$$

Consider the following change of variables. Let $e^{\{\delta(t_1-u)\}} = q$, so that $dq = -\delta q du$. Then we have

$$\begin{aligned}
\mu_{T-t} &= \left(r - \frac{1}{2}\sigma^2\right)(T-t) - \gamma\delta\int_{e^{\{\delta(t_1-t)\}}}^{e^{\{\delta(t_1-T)\}}} \frac{q}{q+1}\frac{1}{\delta q}dq \\
&= \left(r - \frac{1}{2}\sigma^2\right)(T-t) - \gamma\int_{e^{\{\delta(t_1-t)\}}}^{e^{\{\delta(t_1-T)\}}} \frac{1}{q+1}dq \\
&= \left(r - \frac{1}{2}\sigma^2\right)(T-t) - \gamma\ln\left(\frac{e^{\{\delta(t_1-T)\}}+1}{e^{\{\delta(t_1-t)\}}+1}\right).
\end{aligned}$$

Thus, taking the limit as $T \to \infty$ in Equation (55) and using the expression for μ_{T-t} derived above, we have

$$\lim_{T\to\infty} \tilde{E}_t\left[e^{\{-r(T-t)\}}[EP_T - I]^+\right] = EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}. \qquad (56)$$

Now let us turn to the first term in Equation (53)

$$
\begin{aligned}
& \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} dH_v \right] \\
= \;& E_t \left[\int_t^T \xi_{t,v} X_v k_{2,v} D(\Sigma_{i=1}^2 k_{i,v}) dv \right] \\
= \;& X_t \left[\int_t^T E_t e^{\left(-\left(r+\frac{1}{2}\theta^2\right)(v-t)-\theta(z_v-z_t)+\left(\alpha-\frac{1}{2}\sigma^2\right)(v-t)+\sigma(z_v-z_t)\right)} k_{2,v} D(\Sigma_{i=1}^2 k_{i,v}) dv \right] \\
= \;& X_t \left[\int_t^T E_t e^{\left((\sigma-\theta)(z_v-z_t)+\left(\alpha-r-\frac{1}{2}\sigma^2-\frac{1}{2}\theta^2\right)(v-t)\right)} k_{2,v} D(\Sigma_{i=1}^2 k_{i,v}) dv \right] \\
= \;& X_t \left[\int_t^T \kappa e^{\{-\delta v\}} \left(\kappa e^{\{-\delta(v-t_1-\Delta)\}} + \kappa e^{\{-\delta v\}} \right)^{-\gamma} dv \right] \\
= \;& X_t \kappa^{1-\gamma} \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma} \int_t^T e^{\{(\gamma-1)\delta v\}} dv \\
= \;& X_t \kappa^{1-\gamma} \frac{\left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{(\gamma-1)\,\delta} \left(e^{\{(\gamma-1)\delta T\}} - e^{\{(\gamma-1)\delta t\}} \right), \qquad (57)
\end{aligned}
$$

which in the limit as $T \to \infty$, is equal to

$$
\lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} dH_v \right] = \frac{X_t \kappa^{1-\gamma} e^{\{\delta(\gamma-1)t\}} \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{(1-\gamma)\,\delta}. \qquad (58)
$$

Putting together Equations (2) and (4) we obtain the value of the option

$$
\begin{aligned}
W_t(EP, H, \infty) \;=\;& \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} dH_v \right] + \lim_{T\to\infty} \tilde{E}_t \left[e^{\{-r(T-t)\}} \left[EP_T - I \right]^+ \right] \\
=\;& \frac{X_t \kappa^{1-\gamma} e^{\{\delta(\gamma-1)t\}} \left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma}}{(1-\gamma)\,\delta} \\
&+ \left[\lambda \kappa^{1-\gamma} X_t \frac{e^{\{\delta\gamma t\}} \left(e^{\{\delta t_1\}} + e^{\{\delta t\}} \right)^{-\gamma}}{(1-\gamma)\,\delta} \right] \left(e^{\{\delta(t_1-t)\}} + 1 \right)^{\gamma} \\
=\;& \frac{X_t \kappa^{1-\gamma 1-\gamma} e^{\{\delta(\gamma-1)t\}}}{(1-\gamma)\,\delta} \left[\left(e^{\{\delta(t_1+\Delta)\}} + 1 \right)^{-\gamma} + \lambda e^{\{\delta t\}} \left(\frac{e^{\{\delta(t_1-t)\}} + 1}{e^{\{\delta t_1\}} + e^{\{\delta t\}}} \right)^{\gamma} \right] \qquad (59)
\end{aligned}
$$

In order to emphasize the relation between the price of the investment option and the flow of payoffs upon exercise EP_t, we can rearrange Eq. (59) as follows

$$
\begin{aligned}
W_t(EP,H,\infty) &= \frac{X_t\kappa^{1-\gamma}e^{\{\delta(\gamma-1)t\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{(1-\gamma)\,\delta}+EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma} \\
&= \frac{EP_t}{\lambda e^{\{\delta\gamma t\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma}}e^{\{\delta(\gamma-1)t\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}+EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma} \\
&= EP_t\left[\frac{e^{\{\delta(\gamma-1)t\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}}{\lambda}+\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}\right] \\
&= EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}\left[\frac{e^{\{\delta(\gamma-1)t\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda}+1\right]. \qquad (60)
\end{aligned}
$$

■

6.4 Derivation of Firm 1's EP_t

$$
\begin{aligned}
EP_t &= E_t\left[\int_{t+\Delta}^{\infty}\xi_{t,v}k_{1,v}\left[\lambda X_v D(\Sigma_{i=1}^{2}k_{i,v})\right]dv\right] \\
&= \lambda X_t E_t[\int_{t+\Delta}^{\infty}e^{\left\{-\left[r+\frac{1}{2}\theta^2\right](v-t)-\theta(z_v-z_t)\right\}}\kappa e^{\{-\delta(v-t-\Delta)\}} \\
&\quad \cdot e^{\left\{\left[\alpha-\frac{1}{2}\sigma^2\right](v-t)+\sigma(z_v-z_t)\right\}}\left(\kappa e^{\{-\delta(v-t-\Delta)\}}+\kappa e^{\{-\delta v\}}\right)^{-\gamma}dv] \\
&= \lambda\kappa^{1-\gamma}X_t\left(e^{\{\delta(t+\Delta)\}}+1\right)^{-\gamma}e^{\{\delta(t+\Delta)\}}[\int_{t+\Delta}^{\infty}e^{\{\delta(\gamma-1)v\}}dv] \\
&= \lambda\kappa^{1-\gamma}X_t\frac{e^{\{\delta\gamma(t+\Delta)\}}\left(e^{\{\delta(t+\Delta)\}}+1\right)^{-\gamma}}{\delta(1-\gamma)}.
\end{aligned}
$$

6.5 Proposition 7

The leader will not exercise its capital-replacement option.

Proof. Using Equation (28) we derive the early exercise premium for the leader firm.

$$
\begin{aligned}
& EEP_t \\
= & \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - EP_v \left[r + \delta\gamma \left(\frac{e^{\{-\delta(v+\Delta)\}}}{1+e^{\{-\delta(v+\Delta)\}}} \right) \right] dv \right) \right] \\
& - \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} 2^{-\gamma} X_v \kappa^{1-\gamma} e^{\{\delta(\gamma-1)v\}} dv \right] \\
= & \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \left(r\left[EP_v - I\right] dv - EP_v \left[r + \delta\gamma \left(\frac{e^{\{-\delta(v+\Delta)\}}}{1+e^{\{-\delta(v+\Delta)\}}} \right) \right] dv \right) \right] \\
& - \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} EP_v \frac{\delta(1-\gamma) e^{\{\delta(\gamma-1)v\}} 2^{-\gamma}}{\lambda (e^{\{\delta(v+\Delta)\}}+1)^{-\gamma} e^{\{\delta\gamma(v+\Delta)\}}} dv \right] \\
= & \lim_{T\to\infty} \tilde{E}_t \left[\int_t^T e^{\{-r(v-t)\}} 1_{\{\tau_v = v\}} \phi_v dv, \right]
\end{aligned}
$$

which leads to the non-exercise of the capital-replacement option by the leader since $\phi_v = -rI - EP_v \left[\delta\gamma \left(\frac{e^{\{-\delta(v+\Delta)\}}}{1+e^{\{-\delta(v+\Delta)\}}} \right) + \frac{2^{-\gamma}\delta(1-\gamma)e^{\{\delta(\gamma-1)v\}}}{\lambda e^{\{\delta\gamma(v+\Delta)\}}(e^{\{\delta(v+\Delta)\}}+1)^{-\gamma}} \right] < 0.$ ■

6.6 Proposition 10

If $h > rI$, the value of the American capital-replacement option held by Firm 2 beyond time $t_1 + \Delta$ is given by

$$
\begin{aligned}
& W_t(EP, H, t_2) \\
= & EP_t \left[\frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta t\}} \left(e^{\{\delta t_1\}} + e^{\{\delta t\}}\right)^{-\gamma}} + \left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma} \right] - \frac{h}{r} \\
& + (h - rI) \left[\lim_{T\to\infty} \int_t^T \left[e^{\{-r(v-t)\}} \Phi \left(\frac{\ln\left(\frac{EP_t}{B_v}\right) + \mu_{v-t}}{\sigma\sqrt{v-t}} \right) \right] dv \right] \\
& - \left[\lim_{T\to\infty} \int_t^T EP_t \left[e^{\{-r(v-t)\}} \left(\frac{e^{\{\delta(t_1-t)\}}+1}{e^{\{\delta(t_1-v)\}}+1} \right)^{\gamma} \Phi \left(\frac{\ln(\frac{EP_t}{B_v}) + \mu_{v-t} + \sigma^2(v-t)}{\sigma\sqrt{v-t}} \right) \right] dv \right]
\end{aligned} \tag{61}
$$

with $\mu_{v-t} = \left(r - \frac{1}{2}\sigma^2\right)(v-t) - \gamma \ln\left(\frac{e^{\{\delta(t_1-v)\}}+1}{e^{\{\delta(t_1-t)\}}+1}\right)$ and B_v is the immediate exercise boundary.

Proof. We derive a pseudo closed-form solution for the value of the early exercise premium as a function of the unknown exercise boundary B_v as in Kim (1990), Jacka (1991), and

Carr-Jarrow-Myneni (1992). For any particular T, we have that

$$\begin{aligned} EEP_t &= \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}}1_{\{EP_v \geq B_v\}}\phi_v dv\right] \\ &= \left[\int_t^T \left[e^{\{-r(v-t)\}}\int_{\ln B_v}^{\infty}\frac{\phi_v}{\sqrt{2\pi\sigma^2(v-t)}}e^{\left(\frac{-[\ln EP_v - \ln EP_t - \mu_{v-t}]^2}{2\sigma^2(v-t)}\right)}d\ln EP_v\right]dv\right] \\ &= (h-rI)\left[\int_t^T \left[e^{\{-r(v-t)\}}\int_{\ln B_v}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2(v-t)}}e^{\left(\frac{-[\ln EP_v - \ln EP_t - \mu_{v-t}]^2}{2\sigma^2(v-t)}\right)}d\ln EP_v\right]dv\right] \\ &\quad - \left[\int_t^T \left[e^{\{-r(v-t)\}}\varphi_v\int_{\ln B_v}^{\infty}\frac{EP_v}{\sqrt{2\pi\sigma^2(v-t)}}e^{\left(\frac{-[\ln EP_v - \ln EP_t - \mu_{v-t}]^2}{2\sigma^2(v-t)}\right)}d\ln EP_v\right]dv\right] \end{aligned} \tag{62}$$

with

$$\varphi_v = \left[\frac{\gamma\delta e^{\{\delta(t_1-v)\}}}{e^{\{\delta(t_1-v)\}}+1} + \frac{\delta(1-\gamma)\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta v\}}\left(e^{\{\delta t_1\}}+e^{\{\delta v\}}\right)^{-\gamma}}\right]. \tag{63}$$

Thus, we find that

$$\begin{aligned} EEP_t &= (h-rI)\left[\int_t^T \left[e^{\{-r(v-t)\}}\int_{\frac{\ln\left(\frac{B_v}{EP_t}\right)-\mu_{v-t}}{\sigma\sqrt{v-t}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\left\{\frac{-S_v^2}{2}\right\}}dS_v\right]dv\right] \\ &\quad - \left[\int_t^T \left[e^{\{-r(v-t)\}}\varphi_v\int_{\ln B_v}^{\infty}\frac{EP_v}{\sqrt{2\pi\sigma^2(v-t)}}e^{\left(\frac{-[\ln EP_v - \ln EP_t - \mu_{v-t}]^2}{2\sigma^2(v-t)}\right)}d\ln EP_v\right]dv\right] \\ &= (h-rI)\left[\int_t^T \left[e^{\{-r(v-t)\}}\Phi\left(\frac{\ln\left(\frac{EP_t}{B_v}\right)+\mu_{v-t}}{\sigma\sqrt{v-t}}\right)\right]dv\right] \\ &\quad - \left[\int_t^T EP_t\left[e^{\{-r(v-t)\}}\left(\frac{e^{\{\delta(t_1-t)\}}+1}{e^{\{\delta(t_1-v)\}}+1}\right)^{\gamma}\Phi\left(\frac{\ln(\frac{EP_t}{B_v})+\mu_{v-t}+\sigma^2(v-t)}{\sigma\sqrt{v-t}}\right)\right]dv\right] \end{aligned} \tag{64}$$

with $S_v = \frac{\ln EP_v - \ln EP_t - \mu_{v-t}}{\sigma\sqrt{v-t}}$ and $\mu_{v-t} = \left(r - \frac{1}{2}\sigma^2\right)(v-t) - \gamma\ln\left(\frac{e^{\{\delta(t_1-v)\}}+1}{e^{\{\delta(t_1-t)\}}+1}\right)$.

As regards the value of the pure European option, its value is given by

$$\lim_{T\to\infty}\tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}}dH_v\right] + \lim_{T\to\infty}\tilde{E}_t\left[e^{\{-r(T-t)\}}\left[EP_T - I\right]^+\right]. \tag{65}$$

Consider the second term in Equation (65). By the proof of Corollary 4 we find that

$$\lim_{T\to\infty}\tilde{E}_t\left[e^{\{-r(T-t)\}}\left[EP_T - I\right]^+\right] = EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}. \tag{66}$$

Consider now the first term in Equation (65).

$$
\begin{aligned}
&\tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} dH_v\right] \\
=\ & E_t\left[\int_t^T \xi_{t,v}\left[\kappa^{1-\gamma} X_v e^{\{\delta(\gamma-1)v\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}-h\right] dv\right] \\
=\ & X_t\left[\int_t^T E_t\left[e^{\left(-\left(r+\frac{1}{2}\theta^2\right)(v-t)-\theta(z_v-z_t)+\left(\alpha-\frac{1}{2}\sigma^2\right)(v-t)+\sigma(z_v-z_t)\right)}\right]\kappa^{1-\gamma} e^{\{\delta(\gamma-1)v\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} dv\right] \\
& -h\int_t^T e^{-r(v-t)} dv \\
=\ & X_t\left[\int_t^T E_t\left[e^{\left((\sigma-\theta)(z_v-z_t)+\left(\alpha-r-\frac{1}{2}\sigma^2-\frac{1}{2}\theta^2\right)(v-t)\right)}\right]\kappa^{1-\gamma} e^{\{\delta(\gamma-1)v\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} dv\right]-\frac{h}{r}\left(1-e^{-r(T-t)}\right) \\
=\ & X_t\left[\int_t^T \kappa^{1-\gamma} e^{\{\delta(\gamma-1)v\}}\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} dv\right]-\frac{h}{r}\left(1-e^{-r(T-t)}\right) \\
=\ & \kappa^{1-\gamma} X_t \frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}\left(e^{\{\delta(\gamma-1)T\}}-e^{\{\delta(\gamma-1)t\}}\right)}{\delta(\gamma-1)}-\frac{h}{r}\left(1-e^{-r(T-t)}\right) \qquad (67)
\end{aligned}
$$

which in the limit as $T \to \infty$, is equal to

$$
\lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} dH_v\right] = \kappa^{1-\gamma} X_t \frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} e^{\{\delta(\gamma-1)t\}}}{\delta(1-\gamma)} - \frac{h}{r}. \qquad (68)
$$

Putting Equations (66) and (68) together we obtain that the European option is worth

$$
\begin{aligned}
&\lim_{T\to\infty} \tilde{E}_t\left[\int_t^T e^{\{-r(v-t)\}} dH_v\right] + \lim_{T\to\infty} \tilde{E}_t\left[e^{\{-r(T-t)\}}\left[EP_T-I\right]^+\right] \\
=\ & \kappa^{1-\gamma} X_t \frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma} e^{\{\delta(\gamma-1)t\}}}{\delta(1-\gamma)} + EP_t\left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma} - \frac{h}{r} \\
=\ & EP_t\left[\frac{\left(e^{\{\delta(t_1+\Delta)\}}+1\right)^{-\gamma}}{\lambda e^{\{\delta t\}}\left(e^{\{\delta t_1\}}+e^{\{\delta t\}}\right)^{-\gamma}} + \left(e^{\{\delta(t_1-t)\}}+1\right)^{\gamma}\right] - \frac{h}{r} \qquad (69)
\end{aligned}
$$

so that the value of the American option is the sum of Equations (69) and (64). ■