Life-Cycle Consumption Plans and Portfolio Policies in a Heath-Jarrow-Morton Economy

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Abstract

This paper applies the methods of Detemple, Garcia, and Rindisbacher (2003, 2005) to derive optimal lifetime consumption-portfolio plans in an economy characterized by a N-factor Heath-Jarrow-Morton (1992) bond sector that is Markovian with respect to 3N state variables. The Detemple-Garcia-Rindisbacher methodology is reviewed and its flexibility is further demonstrated.

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1 Introduction

The derivation of optimal lifetime consumption plans and portfolio policies has a rich history in the financial economics literature, starting with the pathbreaking works of Merton (1969, 1971) and Samuelson (1969). Recently, Detemple, Garcia, and Rindisbacher (2003, 2005) have made significant advances in the implementation of optimal portfolios in some of the most general and flexible economic environments to date. In Detemple, Garcia, and Rindisbacher, however, long-lived risky assets are restricted to be stocks and a single mutual fund of generic long-term bonds. Because trading in bonds of distinct maturities is not allowed, the authors are free to assume a process for the evolution of the instantaneous risk-free rate.

The present paper further generalizes the framework of Detemple, Garcia, and Rindisbacher by introducing an explicit term-structure of interest and thus, allowing one to trade in a complete set of bonds. For this introduction to be fruitful, however, a no-arbitrage environment must be guaranteed. Most importantly, the process describing the evolution of the risk-free spot rate must be derived carefully to prevent arbitrages. Otherwise, in the presence of arbitrage opportunities, one would be incapable of deriving an optimal (monotone-utility-maximizing) lifetime consumption plan, let alone determining an asset-trading policy. The Heath-Jarrow-Morton (1992) model of no-arbitrage in the bond sector offers a natural framework for this purpose. However, as is well-known, unrestricted Heath-Jarrow-Morton models for the forward-rate volatilities engender non-Markovian dynamics for the risk-free rate. Since the methods of Detemple, Garcia, and Rindisbacher rely on a finite state-variable diffusion system for their application of Malliavin Calculus, the Heath-Jarrow-Morton model must be specialized to admit a Markovian representation. Although more general volatility specifications may be preferred in certain contexts, the Markovian model of Ritchken and Sanakarasubramanian (1995) and Inui and Kijima (1998) is adopted for its familiarity. Once the methodology is demonstrated in this case, a number of alternatives can be considered.

This paper is organized as follows: Section 2 reviews the methodology of Detemple, Garcia, and Rindisbacher. Section 3 introduces the bond economy in the way of a Markovian Heath-Jarrow-Morton model. The ready connections between the two models are shown. Section 4 uses the methods reviewed earlier to obtain optimal consumption plans and portfolio policies in the context of an economy with equities and a bond sector characterized by a Markovian Heath-Jarrow-Morton model. In addition to the general case, an illustration of low dimensions is provided for clarity. Section 5 concludes and discusses additional issues for financial firms to consider in the development of life-cycle plans and retirement products that take an integrated approach to lifetime economic planning. Section 6 is an Appendix that contains a set of needed rules of Malliavin Calculus.

2 The Methodology of Detemple, Garcia, and Rindisbacher

This Section reviews the methodology developed by Detemple, Garcia, and Rindisbacher (2003, 2005) for the derivation of explicit life-cycle consumption plans and portfolio policies. This methodology makes the economic approach of Cox and Huang (1989) applicable to arguably the most general environments to date. Because this novel methodology has not yet been applied to the extent allowed by its flexibility, the intellectual investment in a somewhat lenghty review may well be highly rewarded.

One starts with determining the individual's optimal lifetime consumption plan in the tradition of Arrow (1953) and Debreu (1959). The solution to this static problem is the *ex-ante privately-optimal* blueprint to be implemented, regardless of institutional surroundings. Implementing this solution in a pure Arrow-Debreu manner requires markets for contingent contracts that do not exist. Therefore, the hope is that the existing market environment allows financial engineers to turn this consumption plan into reality for the agent despite deviations in institutions.[1]

2.1 The Optimal Consumption Plan

2.1.1 The Environment

Before one can derive an optimal consumption plan, the economic environment must be stated. For Detemple, Garcia, and Rindisbacher – as is standard in Continuous-Time Finance – all relevant uncertainty is encapsulated in a D-dimensional vector of Brownian Motions

$$W(t) = [W_1(t), W_2(t), ..., W_D(t)]'.$$

This economic uncertainty is translated into a state-variable vector of finite dimensions,

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[1] In traditional microeconomics, this is the step taken when one studies Radner (1972) economies. A Radner economy is one with securities and active trading over time. Under certain conditions, basic theory guarantees that whatever plan is implementable in an Arrow-Debreu economy is also achievable in the environment posited by Radner. Because of the relative simplicity of the discrete-time, finite-state models in standard microeconomics, the existence result above translates promptly into the determination of trading policies (See the standard textbook treatment by Mas-Colell, Whinston, and Green, Chapter 19, 1995). There is little distinction at the conceptual level between the above progression borrowed from microeconomics and what follows.

denoted $Y(t)$. Jointly, these state variables follow diffusion processes

$$dY(t) = \mu_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dW(t). \tag{1}$$

There are N risky stocks and one instantaneously risk-free asset. The risk-free asset evolves according to

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \tag{2}$$

where $r(t)$ is the instantaneous risk-free rate of return. Each stock, S_n, evolves according to

$$\frac{dS_n(t)}{S_n(t)} = (\mu_{S_n}(t) - \delta_{S_n}(t))dt + \Sigma_{d=1}^{D}\sigma_{S_n,d}(t)dW_d(t), \tag{3}$$

where $\mu_{S_n}(t)$ is the drift, $\delta_{S_n}(t)$ is the dividends rate and $\sigma_{S_n,d}(t)$ is the volatility of stock n with respect to Brownian Motion d at time t. All parameters above are possibly random and time-dependent but their uncertain character is caused exclusively by their functional dependence on the values of state variables $Y(.)$ at time t.

The financial market is *standard and complete*. That is, no-arbitrage prevails, there are exactly as many stocks as there are Brownian Motions ($N = D$), and the D-by-D matrix $\sigma(t)$ is invertible. Technical restrictions, such as the Novikov condition, hold. One can thus define the market price of risk, $\theta(t)$,[2] as well as the unique pricing kernel, $\xi(t)$,

$$\xi(t) = \exp\left\{-\int_0^t r(s)ds - \int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\}. \tag{4}$$

Heuristically speaking, the pricing kernel of Equation (4) acts as a price generator for state-contingent contracts traded in pure Arrow-Debreu markets, in which all financial activity takes place at time 0. The cost at time 0 of purchasing one unit of consumption in any particular state is equal to the probability of this state's occurrence multiplied by the value of the above pricing kernel $\xi(t)$ in this particular state. The resulting value is known to general economists as an Arrow-Debreu price and to option theorists as the state's risk-neutral probability.[3] Given these Arrow-Debreu prices, the financial firm can determine its customer's optimal contingent consumption plan, subject to a static Arrow-Debreu budget constraint in the vein of Cox and Huang (1989).[4] This is the topic of the following Subsection.

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[2] That is, $\theta(t)$ solves uniquely the system

$$\begin{bmatrix} \mu_{S_1}(t) - r(t) \\ \vdots \\ \mu_{S_D}(t) - r(t) \end{bmatrix} = \begin{bmatrix} \sigma_{S_1,1}(t) & ... & \sigma_{S_1,D}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{S_D,1}(t) & ... & \sigma_{S_D,D}(t) \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \vdots \\ \theta_D(t) \end{bmatrix},$$

[3] Samuelson and Merton's (1969) Utili-Prob density function represents the crucial methodological step towards risk-neutral, Arrow-Debreu, pricing in derivatives theory. Breeden and Litzenberger (1978) offer valuable insights in the fundamental connections between Arrow-Debreu theory and the prices of options. For a good introductory treatment, the reader is directed to Baz and Chacko (2004).

[4] This constraint requires that the current cost of the contingent consumption plan does not exceed the present value of the consumer's wealth. The standard abstraction is that the individual's endowment consists of a single lump-sum of k dollars available at time 0 or that, abstracting from moral hazard issues, the value of the customer's future earnings can be capitalized at the present date using the Arrow-Debreu prices to value the uncertain flow of earnings. Building on the results of He and Pages (1993) and El Karoui and Jeanblanc-Picque (1998), Bodie, Detemple, Otruba, and Walter (2004) take steps toward needed realism with respect to restrictions on financial wealth in the context of moral hazard. These are not introduced here for added fluidity.

2.1.2 The Static Problem

Following Cox and Huang (1989), the financial intermediary derives the individual's optimal consumption plan at time 0. This plan will exactly determine the stochastic flow of consumption, $c^*(.)$, enjoyed by the agent throughout the *planning horizon* $[0, T]$ as well as the stochastic lump-sum terminal wealth payment at time T, $X^*(T)$. This plan, denoted $(c^*(.), X^*(T))$, maximizes the von Neumann-Morgenstern utility function

$$E\left[\int_0^T U_1(t, c(t))dt + U_2(X(T))\right], \tag{5}$$

subject to the static Arrow-Debreu budget constraint

$$k \geq E\left[\int_0^T \xi(t)c(t)dt + \xi(T)X(T)\right], \tag{6}$$

where k is the individual's initial endowment. The utility functions, $U_1(.)$ and $U_2(.)$, are strictly concave, non-decreasing, continuous, and differentiable, with marginal utilities that are continuous, positive, strictly decreasing, and that satisfy standard Inada's conditions.[5]

At an interior solution, optimality requires that expected marginal utility per dollar be equated across all commodities in the sense of basic Consumer Choice Theory. This can be seen from writing out the Lagrangian problem with λ as the Lagrange multiplier. The

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[5] Since the utility functions $U_1(.)$ and $U_2(.)$ may go to $-\infty$ below certain floor consumption levels, the initial wealth k is assumed to be sufficient to provide this floor consumption throughout the planning horizon. That is, using Arrow-Debreu pricing,

$$E\left[\int_0^T \xi(t)\underline{c}(t)dt + \xi(T)\underline{X}(T)\right] < k,$$

where $(\underline{c}(.), \underline{X}(T))$ are those floors.

resulting unconstrained problem is

$$\begin{aligned}
&\max_{(c(.),X(T))} E\left[\int_0^T U_1(t,c(t))dt + U_2(X(T))\right] + \lambda\left(k - E\left[\int_0^T \xi(t)c(t)dt + \xi(T)X(T)\right]\right) \\
= &\max_{(c(.),X(T))} E\left[\int_0^T \left(U_1(t,c(t)) - \lambda\xi(t)c(t)\right)dt + U_2(X(T)) - \lambda\xi(T)X(T)\right] + \lambda k.
\end{aligned}$$

The first-order conditions can be rewritten as

$$\frac{\underbrace{EU_1'(t,c(t))}_{\text{expected marginal utility}}}{\underbrace{E\xi(t)}_{\textit{price}}} = \frac{\underbrace{EU_2'(X(T)}_{\text{expected marginal utility}}}{\underbrace{E\xi(T)}_{\textit{price}}} = \lambda,$$

where the expectation operator E is momentarily retained to add clarity to the interpretation of the denominators as Arrow-Debreu prices. To capture these optimality conditions, one defines the inverse marginal utility function $I_1(.)$ and the corresponding terminal-wealth inverse marginal utility function $I_2(.)$.[6] For any particular value of λ, the optimal consumption policy is $c(t) = I_1(\lambda\xi(t))$ and $X(T) = I_2(\lambda\xi(T))$. When functional forms for the utility functions are specified, one can determine the value of λ that is consistent with the budget constraint's being exactly binding. In the present general exposition, one must be satisfied with denoting this particular λ by $\lambda(k)$. Thus, the optimal lifetime consumption plan for this customer is given by

$$(c^*(t), X^*(T)) = (I_1(\lambda(k)\xi(t)), I_2(\lambda(k)\xi(T))). \tag{7}$$

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[6] That is, $I_1(y) = c$, if $U_1'(c) = y$ and $I_2(y) = X$, if $U_2'(X) = y$.

2.2 The Supporting Portfolio Policy

2.2.1 The Wealth Process

In the absence of a complete set of fully-functioning Arrow-Debreu markets, the provision of the optimal consumption plan $(c^*(t), X^*(T))$ becomes a matter of financial engineering. In particular, a policy of dynamic trading must substitute for the inexistent Arrow-Debreu markets. Associated with any portfolio policy, $\pi(t)$, is a wealth process, $X(t)$, that corresponds to the financial wealth generated by trading according to $\pi(t)$. For this active portfolio policy to implement $(c^*(t), X^*(T))$, the wealth process at time t must provide exactly the amount necessary to purchase the remaining part of the optimal consumption plan over $[t, T]$. Mathematically,

$$X(t) = \xi(t)^{-1} E_t \left[\int_t^T \xi(s) c^*(s) ds + \xi(T) X^*(T) \right]. \tag{8}$$

In addition, a wealth process driven by dynamic trading in the financial markets is characterized by

$$dX(t) = (r(t)X(t) - c(t))dt + X(t)\pi(t)'[(\mu(t) - r(t))dt + \sigma(t)dW(t)].$$

For ease, one considers the discounted wealth process, $\xi(t)X(t)$. A standard application of Ito's Lemma yields

$$d\left(\xi(t)X(t)\right) = -\xi(t)c(t)dt + \xi(t)X(t)\{\pi(t)'\sigma(t) - \theta(t)'\}dW(t). \tag{9}$$

From the developments of this Subsection, one should take away that the volatility term associated with the discounted wealth process at time t is $\xi(t)X(t)\{\pi(t)'\sigma(t) - \theta(t)'\}$. Examining the wealth process from a different perspective will provide an additional expression for the volatility term in Equation (9). This additional expression, which relies on Malliavin Calculus and the Clark-Ocone formula, is sufficient to generate the explicit portfolio policy. The following Subsection succintly provides some needed concepts, including the notion of Malliavin derivative and the Clark-Ocone formula. Much greater depth of treatment can be found in the papers by Detemple, Garcia, and Rindisbacher as well as in the textbook by Nualart (1995).

2.2.2 The Clark-Ocone Formula

Let $F(W)$ be a particular terminal value of a Brownian functional F, where W represents a complete realized path of a one-dimensional Brownian Motion over the horizon $[0, S]$. The date of this terminal value, denoted S, need not coincide with the end time of the individual's *planning horizon*, which is denoted T in the economic problem taken up in this paper. For concreteness, the reader may wish to keep in mind the case of $F(W) = \xi(t)X(t)$, to which the results of this Subsection are applied. In this case, the terminal date S is t.

The Malliavin derivative is defined as

$$\mathcal{D}_s F(W) = \lim_{\varepsilon \to 0} \frac{F(W + \varepsilon 1_{[s,S]}) - F(W)}{\varepsilon}. \tag{10}$$

As a matter of economics, Equation (10) summarizes a comparative-statics exercise regarding $F(W)$. The question is the following. How would an ε-disturbance of the Brownian

Motion past time s affect the terminal value $F(W)$? The Malliavin derivative provides the answer in the limit, as the perturbation becomes infinitesimal. As noted by Detemple, Garcia, and Rindisbacher, this derivative is a close relative of what econometricians and macroeconomists know as an impulse response function in the standard state-space framework. When there are more than one Brownian Motion, $\mathcal{D}_s F(W)$ is a vector of derivatives. Each element of this vector captures the change in $F(W)$ caused by a perturbation in the path of a single Brownian Motion, taking the trajectories of all other Brownian Motions as given,

$$\mathcal{D}_s F(W) = \left[\lim_{\varepsilon \to 0} \frac{F(W + \varepsilon \vec{1}^{(1)}_{[s,S]}) - F(W)}{\varepsilon}, ..., \lim_{\varepsilon \to 0} \frac{F(W + \varepsilon \vec{1}^{(D)}_{[s,S]}) - F(W)}{\varepsilon} \right]', \tag{11}$$

where $\vec{1}^{(d)}_{[s,S]}$ is the zero vector up to time s and $\vec{1}^{(d)}_{[s,S]} = (0, 0, ..., \underbrace{1}_{d^{th}\ entry}, ..., 0)$ over $[s, S]$.

With this definition of the Malliavin derivative, the Clark-Ocone Theorem, presented in Ocone and Karatzas (1991), can be stated.

Theorem 1 *(Clark-Ocone Formula) Any Brownian functional $F(W)$ with existing Malliavin derivatives can be written as*

$$F(W) = EF(W) + \int_0^S [E_s \mathcal{D}_s F(W)]' dW(s).$$

Intuitively, the formula is an economic statement about how one goes from *not knowing* to *knowing*. Consider $F(W) = \xi(t)X(t)$. One can view $\xi(t)X(t)$ as a *realized* terminal value for the process that exclusively describes *discounted wealth at time t*. This realization was determined long ago when a particular trajectory for the vector of Brownian Motions was randomly drawn. However, for the uninformed individual, knowledge of this realized terminal value has to be slowly reached. The uninformed individual thus comes to *knowing* $\xi(t)X(t)$ as the sum of an unconditional expected value, $E\xi(t)X(t)$, plus the accumulation of forecast updates as the path of the Brownian Motions is incrementally revealed, $\int_0^t E_s\mathcal{D}_s[\xi(t)X(t)]'dW(s)$, or

$$\xi(t)X(t) = E[\xi(t)X(t)] + \int_0^t E_s\mathcal{D}_s[\xi(t)X(t)]'dW(s). \tag{12}$$

As all uncertainty is finally resolved over the last instant, the value of the discounted wealth process at time t obtained by the uninformed observer must converge to the value obtained by the active trading described in the previous Subsection. Thus, the time-t volatility term of the Clark-Ocone formula provides an alternative expression for the volatility term in Equation (9)

$$E_t\mathcal{D}_t[\xi(t)X(t)]' = \mathcal{D}_t[\xi(t)X(t)]' = \xi(t)X(t)\{\pi(t)'\sigma(t) - \theta(t)'\}. \tag{13}$$

Re-arranging Equation (13) yields

$$\pi(t) = [\sigma(t)']^{-1}\left[\theta(t) + \frac{1}{\xi(t)X(t)}\mathcal{D}_t[\xi(t)X(t)]\right]. \tag{14}$$

For Equation (14) to be an explicit description of the portfolio policy $\pi(t)$, the financial firm must obtain values for $\mathcal{D}_t[\xi(t)X(t)]$. This remaining task is accomplished as follows. First, one uses the rules of Malliavin Calculus (compiled in the Appendix) to expand this term into the conditional expectation of functions of the state variables' Malliavin derivatives,

$\mathcal{D}_t Y(s)$. The second and last step is to use the diffusion nature of the state variables to generate a set of values for $\mathcal{D}_t Y(s)$ and use these as the basis for a simulation-based estimate of the conditional expectations. This amounts to applying Monte Carlo methods to a system of standard linear stochastic differential equations.

2.2.3 Expanding $\mathcal{D}_t[\xi(t)X(t)]$

Applying the rules of Malliavin calculus (See the Appendix) to

$$\xi(t)X(t) = E_t[\int_t^T \xi(s)c(s)ds + \xi(T)X(T)], \tag{15}$$

the term $\mathcal{D}_t[\xi(t)X(t)]$ can be re-expressed as

$$\begin{aligned}
\mathcal{D}_t[\xi(t)X(t)] &= \mathcal{D}_t\left[E_t\left[\int_t^T \xi(s)c(s)ds + \xi(T)X(T)\right]\right] \\
&= E_t\left[\mathcal{D}_t\left[\int_t^T \xi(s)c(s)ds\right] + \mathcal{D}_t\left[\xi(T)X(T)\right]\right] \\
&= E_t\left[\mathcal{D}_t\left[\int_t^T \xi(s)I_1(\lambda\xi(s))ds\right] + \mathcal{D}_t\left[\xi(T)I_2(\lambda\xi(T))\right]\right] \\
&= E_t\left[\int_t^T \{I_1(\lambda\xi(s)) + I_1'(\lambda\xi(s))\lambda\xi(s)\}\mathcal{D}_t\xi(s)ds\right] \\
&\quad + E_t\left[\{I_2(\lambda\xi(T)) + I_2'(\lambda\xi(T))\lambda\xi(T)\}\mathcal{D}_t\xi(T)\right].
\end{aligned} \tag{16}$$

Finally, the rules of Malliavin Calculus yield

$$\begin{aligned}
\mathcal{D}_t\xi(s) = -\xi(s)&\left[\int_t^s \{\partial r(Y(v),v) + \theta(Y(v),v)'\partial\theta(Y(v),v)\}\,\mathcal{D}_tY(v)dv\right] \\
-\xi(s)&\left[\int_t^s \partial\theta(Y(v),v)\mathcal{D}_tY(v)dW(v) + \theta(Y(t),t)\right].
\end{aligned} \tag{17}$$

Equation (17) reveals $\mathcal{D}_t Y(v)$ to be the building blocks for the implementation of the optimal portfolio policy $\pi(t)$.

2.2.4 Malliavin Derivatives of State Variables

Recall that the economy of Detemple, Garcia, and Rindisbacher is characterized by the state-variable vector $Y(t)$'s satisfying

$$dY(t) = \mu_Y(t, Y(t))dt + \sigma_Y(t, Y(t))dW(t). \tag{1}$$

When integrated subject to the set of initial conditions $Y(0)$, Equation (1) becomes

$$Y(s) = Y(0) + \int_0^s \mu_Y(v, Y(v))dv + \int_0^s \sigma_Y(v, Y(v))dW(v). \tag{18}$$

The same rules as those used above yield

$$\mathcal{D}_t Y(s) = \int_t^s \partial\mu_Y(v, Y(v))\mathcal{D}_t Y(v)dv + \int_t^s \partial\sigma_Y(v, Y(v))\mathcal{D}_t Y(v)dW(v) + \sigma_Y(t, Y(t)), \tag{19}$$

which, in differential form, is equivalent to the system of linear stochastic differential equations

$$d(\mathcal{D}_t Y(s)) = \mathcal{D}_t Y(s)\{\partial\mu_Y(s, Y(s))ds + \partial\sigma_Y(s, Y(s))dW(s)\} \tag{20}$$

subject to initial conditions $\mathcal{D}_t Y(t) = \sigma_Y(t, Y(t))$. Using these equations, simulation techniques provide a set of values for $\mathcal{D}_t Y(v)$ and thus, for Equation (17)'s $\mathcal{D}_t \xi(s)$. These simulated values for $\mathcal{D}_t \xi(s)$ are used to provide Monte-Carlo-based estimates of the conditional expectation terms of Equation (16). This allows for immediate implementation.[7] This review being complete, the next Section takes on the task of introducing a Markovian Heath-Jarrow-Morton bond economy. This will allow for a generalized application of the Detemple-Garcia-Rindisbacher approach to a model of the economy characterized by an explict term-structure of interest and, correspondingly, trading in a rich set of assets that includes bonds of various maturities.

3 A Markovian Heath-Jarrow-Morton Bond Economy

3.1 Background

This Section builds on the Markovian approach to the Heath-Jarrow-Morton (1992) model of the term-structure developed in Ritchken and Sanakarasubramanian (1995) and Inui and Kijima (1998). Most of the derivations are closely related to those in Inui and Kijima. These are sketched because, unlike in Inui and Kijima, the market prices of risk on bonds cannot be subsumed in a risk-neutral form of the Brownian Motions. Indeed, the necessity to maintain an explicit expression for the pricing kernel is an essential difference between the *utility-based* determination of optimal consumption-portfolio policies to which Heath, Jarrow, and Morton's model is applied in this paper and the *preference-free* derivatives-pricing methodology, a context in which the Heath-Jarrow-Morton model has been widely used over the past decade. Thus, in the present context, the Markovian representation of an

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[7] Detemple, Garcia, and Rindisbacher (esp. 2003, Section III) provide some algorithms for this Monte Carlo. For in-depth treatments of Monte Carlo methods and their properties, the reader is invited to consult Glasserman (2003) for a thorough presentation of Monte Carlo methods in the context of Financial Engineering and Train (2003) for a general-econometrics presentation of simulation-based methods.

N-factor term-structure of interest over the *trading horizon* $[0, \tau]$[8] requires $3N$ state variables. This is a departure from the traditional $2N$ representation afforded by the absorption of the market prices of risk into the equivalent measure.

The initial forward-rate curve $f(0, S)$ – for all maturities $S \in [0, \tau]$ – is given. This curve summarizes the rate of interest imbedded in debt contracts written at time 0 for instantaneous lending at time S. The N-factor evolution of this forward-rate curve is modeled by

$$df(t, S) = \alpha(t, S)dt + \Sigma_{i=1}^{N} \sigma_{f,i}(t, S)dW_i(t). \tag{21}$$

Heath, Jarrow, and Morton derive conditions on the structure of Equation (21) that bar arbitrage opportunities from the bond sector. In particular, *uniqueness* of an equivalent martingale measure across *all* bonds requires the drift term in Equation (21) to be

$$\alpha(t, S) = \Sigma_{i=1}^{N} \sigma_{f,i}(t, S) \left[-\phi_i(t) + \int_t^S \sigma_{f,i}(t, v)dv \right], \tag{22}$$

where $\phi(t) = [\phi_1(t), ..., \phi_N(t)]'$ is the vector of (minus) the market prices of risk on bonds at time t.[9] This vector is common for any set of N bonds ordered by increasing maturity dates

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[8] The end point of trading activities, time τ, is *strictly* beyond the end time of the customer's *planning horizon*, T. Technically, this is needed so that the financial firm does not to run out of hedging instruments towards the end of the individual's planning horizon. In the real-world, this assumption is readily satisfied since new bonds of later maturities are automatically issued over time. In effect, in life, the investor's finite planning horizon nests itself into the infinite institutional horizon.

[9] That is, as in Heath, Jarrow and Morton (1992, esp. pp.82-83 and Proposition 3), $\phi(t)$ solves the system

$$\begin{bmatrix} \mu_{B_1}(t) - r(t) \\ \vdots \\ \mu_{B_N}(t) - r(t) \end{bmatrix} = - \begin{bmatrix} \sigma_{B_1,1}(t) & ... & \sigma_{B_1,N}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{B_N,1}(t) & ... & \sigma_{B_N,N}(t) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_N(t) \end{bmatrix},$$

where $[B_1, ...B_N]$ is an ordered sequence of bonds of distinct maturities that all mature at a date strictly later than t, $\mu_{B_i}(t)$ is the drift of bond B_i and $\sigma_{B_i,j}(t)$ naturally corresponds to the volatility of bond B_i with respect to Brownian Motion j at time t.

(Heath-Jarrow-Morton, 1992, Proposition 3).[10]

Thus, in integral form, Equation (21) is

$$f(t,S) = f(0,S) + \Sigma_{i=1}^{N} \int_0^t \sigma_{f,i}(s,S) \left[\int_s^S \sigma_{f,i}(s,v)dv - \phi_i(s) \right] ds + \Sigma_{i=1}^{N} \int_0^t \sigma_{f,i}(s,S)dW_i(s). \tag{23}$$

3.2 Dynamics of The Spot Rate of Interest

Since the risk-free rate, $r(t)$, is the instantaneous forward rate, $f(t,t)$, Equation (23) yields

$$r(t) = f(0,t) + \Sigma_{i=1}^{N} \int_0^t \underbrace{\sigma_{f,i}(s,t) \left[\int_s^t \sigma_{f,i}(s,v)dv - \phi_i(s) \right]}_{b_i(s,t)} ds + \Sigma_{i=1}^{N} \int_0^t \sigma_{f,i}(s,t)dW_i(s). \tag{24}$$

In differential form, Equation (24) above becomes

$$dr(t) = \left[\frac{\partial f(t,S)}{\partial S}_{S=t} - \Sigma_{i=1}^{N} \sigma_{r,i}(r(t),t)\phi_i(t) \right] dt + \Sigma_{i=1}^{N} \sigma_{r,i}(r(t),t)dW_i(t), \tag{25}$$

Remember that these are the negatives of the market prices of risk since the bonds volatilities are $\sigma_{B_i,j}(t) = -\int_t^{B_i} \sigma_j(t,v)dv$. Intuition can be maintained from recalling that as forward rates increase, the price of bonds is lowered.

[10] This invariance of the vector of market prices of risk on bonds with respect to the set of ordered maturities has important implications. One is that it allows great discretion on the part of the financial intermediary producing term-structure-related life-cycle products. At any given point in time, the trading policy $\pi(t)$ is altered by the chosen set of bonds only via the bonds' volatilities that populate the inverted matrix, $\sigma(t)'$. The freedom of choice afforded by this invariance should be mastered by real-world financial firms to direct their trading towards least-cost (e.g., transaction-costs-minimizing) policies.

where

$$\left.\frac{\partial f(t,S)}{\partial S}\right|_{S=t} = \frac{\partial f(0,t)}{\partial t} + \Sigma_{i=1}^{N}\int_0^t \frac{\partial b_i(s,t)}{\partial t}ds + \Sigma_{i=1}^{N}\int_0^t \frac{\partial \sigma_{f,i}(s,t)}{\partial t}dW_i(s) \tag{26}$$

and

$$\sigma_{r,i}(r(t),t) = \sigma_{f,i}(t,t) \tag{27}$$

is the instantaneous forward-rate – or risk-free spot-rate – volatility.

If one supposes, as do Ritchken and Sanakarasubramanian (1995) and Inui and Kijima (1998), that[11]

$$\frac{\partial \sigma_{f,i}(t,S)}{\partial S} = -\kappa_i(S)\sigma_{f,i}(t,S), \tag{28}$$

one gets that

$$\frac{\partial b_i(t,S)}{\partial S} = -\kappa_i(S)b_i(t,S) + \sigma_{f,i}^2(t,S). \tag{29}$$

After substitution of Equations (28) and (29) for the appropriate terms, Equation (26) becomes

$$\left.\frac{\partial f(t,S)}{\partial S}\right|_{S=t} = \frac{\partial f(0,t)}{\partial t} - \Sigma_{i=1}^{N}\kappa_i(t)\left[\int_0^t b_i(s,t)ds + \int_0^t \sigma_{f,i}(s,t)dW_i(s)\right] + \Sigma_{i=1}^{N}\int_0^t \sigma_{f,i}^2(s,t)ds.$$

This leads the dynamic equation for the spot rate to be

---

[11] In integral form, this is equivalent to $\sigma_{f,i}(t,S) = \sigma_{r,i}(r(t),t)\exp\{-\int_t^S \kappa_i(u)du\}$, where $\kappa_i(u)$ may be a deterministic function of time and, $\sigma_{r,i}(r(t),t)$ remains to be modeled, thus allowing some flexibility. As noted in Inui and Kijima, even this arguably spartan class of volatilities allows for rich patterns in the forward-rate curve such as twists, using a two-factor model.

$$
\begin{aligned}
dr(t) &= \left[\frac{\partial f(0,t)}{\partial t} - \Sigma_{i=1}^{N}\kappa_i(t)\left[\int_0^t b_i(s,t)ds + \int_0^t \sigma_{f,i}(s,t)dW_i(s)\right] + \Sigma_{i=1}^{N}\int_0^t \sigma_{f,i}^2(s,t)ds\right]dt \\
&\quad -\Sigma_{i=1}^{N}\sigma_{r,i}(r(t),t)\phi_i(t)dt + \Sigma_{i=1}^{N}\sigma_{r,i}(r(t),t)dW_i(t). \qquad (30)
\end{aligned}
$$

Following Inui and Kijima (1998) with the addition of the market prices of the risk, Equation (30) can be re-expressed as

$$
\begin{aligned}
dr(t) &= \frac{\partial f(0,t)}{\partial t}dt + \kappa_N(t)\left[f(0,t) - r(t)\right]dt + \Sigma_{i=1}^{N}\gamma_i(t)dt - \Sigma_{i=1}^{N}\sigma_{r,i}(r(t),t)\phi_i(t)dt \\
&\quad +\Sigma_{i=1}^{N}\sigma_{r,i}(r(t),t)dW_i(t) + \Sigma_{i=1}^{N-1}\left[\kappa_N(t) - \kappa_i(t)\right]\chi_i(t)dt, \qquad (31)
\end{aligned}
$$

where

$$
\gamma_i(t) = \int_0^t \sigma_{f,i}^2(s,t)ds, \qquad (32)
$$

and

$$
\chi_i(t) = \int_0^t b_i(s,t)ds + \int_0^t \sigma_{f,i}(s,t)dW_i(s). \qquad (33)
$$

3.3 Diffusion Processes for State Variables in the Markovian Model

Equation (31) shows that the spot rate obeys a diffusion process that is Markovian with respect to the $3N$-dimensional state-variable system composed of $r(t)$, $\{\gamma_i(t), \phi_i(t)\}_{i=1,\ldots N}$ and $\{\chi_i(t)\}_{i=1,\ldots N-1}$. For this representation of the spot-rate process to permit the application of the Detemple-Garcia-Rindisbacher methodology, this $3N$-dimensional state-variable vector must follow a joint diffusion process. At this point, the task remains to guarantee that $\{\gamma_i(t), \phi_i(t)\}_{i=1,\dots N}$ and $\{\chi_i(t)\}_{i=1,\dots N-1}$ conform.

Following Inui and Kijima (esp. p.431), one shows readily that

$$d\gamma_i(t) = \left[\sigma_{r,i}^2(r(t),t) - 2\kappa_i(t)\gamma_i(t)\right] dt \tag{34}$$

and

$$\begin{aligned} d\chi_i(t) &= \left[\int_0^t \left(-\kappa_i(t)b_i(s,t) + \sigma_{f,i}^2(s,t)\right) ds - \int_0^t \kappa_i(t)\sigma_{f,i}(s,t)dW_i(s)\right] dt \\ &\quad -\sigma_{r,i}(r(t),t)\phi_i(t)dt + \sigma_{r,i}(r(t),t)dW_i(t) \\ &= \left[\gamma_i(t) - \sigma_{r,i}(r(t),t)\phi_i(t) - \kappa_i(t)\chi_i(t)\right] dt + \sigma_{r,i}(r(t),t)dW(t). \end{aligned} \tag{35}$$

Thus, the no-arbitrage structure produced by the Markovian Heath-Jarrow-Morton framework automatically guarantees that state variables $\{\gamma_i(t)\}_{i=1,\dots N}$ and $\{\chi_i(t)\}_{i=1,\dots N-1}$ follow Markovian diffusions processes. One may take this to be a sign that Markovian Heath-Jarrow-Morton models are natural fits for the methods of Detemple, Garcia, and Rindisbacher. However, as in Detemple, Garcia, and Rindisbacher, one cannot escape the need to impose a structure on market prices of risk, $\{\phi_i(t)\}_{i=1,\dots N}$. For this purpose, Section 4 will adopt an amended version of the functional form for the evolution of the market prices of risk assumed in their paper. With the above structure in place for the Markovian Heath-Jarrow-Morton bond economy, the next Section turns to the derivation of optimal consumption-portfolio rules in an economy with equities and a complete bond sector.

4 Optimal Lifetime Plans with Stocks and an Explicit Term-Structure of Interest

4.1 The Amended Environment

As before, all uncertainty is captured by the D-dimensional vector of Brownian Motions

$$W(t) = [W_1(t), W_2(t), ..., W_D(t)]'.$$

The first K Brownian Motions affect equities exclusively (and no other source of uncertainty impacts the equity sector) while the remaining $D - K$ Brownian Motions drive all developments in the bond sector.[12]

Using the notation of Section 2, each of the K stocks, S_n, follows

$$\frac{dS_n(t)}{S_n(t)} = (\mu_{S_n}(t) - \delta_{S_n}(t))dt + \Sigma_{d=1}^{K} \sigma_{S_n,d}(t)dW_d(t), \tag{36}$$

and the risk-free asset obeys

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt. \tag{2}$$

Associated with the equities is a well-defined process for the market prices of risk on stocks, $\theta(t)$ (See footnote 2).

In the bond sector, an initial forward-rate curve, $f(0, S)$, is given for all maturities S over the *trading horizon* $[0, \tau]$ and the $(D - K)$-factor evolution of this forward-rate curve

---

[12] The requirement that no Brownian Motion may affect both stocks and bonds is maintained in this paper for conceptual and notational ease.

is dictated by

$$df(t,S) = \alpha(t,S)dt + \Sigma_{i=K+1}^{D}\sigma_{f,i}(t,S)dW_i(t). \tag{37}$$

In light of Equation (31), the spot-rate process is driven by

$$\begin{aligned} dr(t) &= \left[\frac{\partial f(0,t)}{\partial t} + \kappa_D(t)\left[f(0,t) - r(t)\right] + \Sigma_{i=K+1}^{D}\gamma_i(t) - \Sigma_{i=K+1}^{D}\sigma_{r,i}(r(t),t)\phi_i(t)\right] dt \\ &\quad + \Sigma_{i=K+1}^{D-1}\left[\kappa_D(t) - \kappa_i(t)\right]\chi_i(t)dt + \Sigma_{i=K+1}^{D}\sigma_{r,i}(r(t),t)dW_i(t). \end{aligned} \tag{38}$$

where $\gamma_i(t)$ and $\chi_i(t)$ are defined as in Equations (32) and (33), respectively, and whose dynamics are presented at the end of the previous Section.

In the notation of Section 3, $\phi_i(t)$, for $i \in \{K+1, K+2, ..., D\}$, are (the negatives of) the market prices of risk in the bond sector (See footnotes 8 and 9). A diffusion structure is imposed on these market prices of risk, as well as those that characterize equities.[13] A modified version of Detemple, Garcia, and Rindisbacher's process is adopted. The K-dimensional process associated with equities, $\theta(t)$, thus obeys, for each $n \in \{1, 2, ..., K\}$,

$$d\theta_n(t) = \kappa_{\theta_n}(\overline{\theta_n} - \theta_i(t))dt + \mu_{\theta_n}^{r}(r(t),\theta_n(t))dt + \Sigma_{d=1}^{K}\sigma_{\theta_n,d}(\theta_n(t))dW_d(t), \tag{39}$$

---

[13] In the notation of Footnotes 2 and 8 above, these market prices of risk uniquely solve the linear system

$$\begin{bmatrix} \mu_{S_1}(t) - r(t) \\ \vdots \\ \mu_{S_K}(t) - r(t) \\ \mu_{B_{K+1}}(t) - r(t) \\ \mu_{B_{K+2}}(t) - r(t) \\ \vdots \\ \mu_{B_D}(t) - r(t) \end{bmatrix} = \begin{bmatrix} \sigma_{S_1,1}(t) & \ldots & \sigma_{S_1,K}(t) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & 0 & 0 & 0 \\ \sigma_{S_K,1}(t) & \ddots & \sigma_{S_K,K}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_{B_{K+1},K+1}(t) & \ldots & \ldots & -\sigma_{B_{K+1},D}(t) \\ 0 & 0 & 0 & -\sigma_{B_{K+2},K+1}(t) & \ddots & \ddots & -\sigma_{B_{K+2},D}(t) \\ 0 & 0 & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -\sigma_{B_D,K+1}(t) & \ldots & \ldots & -\sigma_{B_D,D}(t) \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \vdots \\ \theta_K(t) \\ \phi_{K+1}(t) \\ \phi_{K+2}(t) \\ \vdots \\ \phi_D(t) \end{bmatrix}$$

and the $(D-K)$-dimensional process that corresponds to the bond market, $\phi(t)$, follows, for each $i \in \{K+1, K+2, ..., D\}$,

$$d\phi_i(t) = \kappa_{\phi_i}(\overline{\phi_i} - \phi_i(t))dt + \mu^r_{\phi_i}(r(t), \phi_i(t))dt + \Sigma^D_{d=K+1}\sigma_{\phi_i,d}(\phi_i(t))dW_d(t), \tag{40}$$

where

$$\mu^r_{\theta_n}(r(t), \theta_n(t)) = \kappa_{\theta_{n,r}}(f(0,t) - r(t))(\theta_{n,l} + \theta_n(t)) \left[1 - \frac{\theta_{n,l} + \theta_n(t)}{\theta_{n,l} + \theta_{n,u}}\right], \tag{41}$$

$$\sigma_{\theta_n,d}(\theta_n(t)) = \sigma_{\theta_{n,d}}(\theta_{n,l} + \theta_n(t))^{\rho_{n,1}} \left[\left[1 - \frac{\theta_{n,l} + \theta_n(t)}{\theta_{n,l} + \theta_{n,u}}\right]^{1-\rho_{n,1}}\right]^{\rho_{n,2}}, \tag{42}$$

$$\mu^r_{\phi_i}(r(t), \phi_i(t)) = \kappa_{\phi_{i,r}}(f(0,t) - r(t))(\phi_{i,l} + \phi_i(t)) \left[1 - \frac{\phi_{i,l} + \phi_i(t)}{\phi_{i,l} + \phi_{i,u}}\right], \tag{43}$$

and

$$\sigma_{\phi_i,n}(\phi_i(t)) = \sigma_{\phi_i,n}(\phi_{i,l} + \phi_i(t))^{\rho_{i,1}} \left[\left[1 - \frac{\phi_{i,l} + \phi_i(t)}{\phi_{i,l} + \phi_{i,u}}\right]^{1-\rho_{i,1}}\right]^{\rho_{i,2}}. \tag{44}$$

In the expressions above, κ_{θ_n} and κ_{ϕ_i} are constant parameters that capture mean-reversion tendencies toward central values $\overline{\theta_n}$ and $\overline{\phi_i}$, respectively. Coefficients $\kappa_{\theta_{n,r}}$ and $\kappa_{\phi_{i,r}}$ measure interest-rate sensitivities, relative to initial forward rates. $\sigma_{\theta_{n,d}}$ and $\sigma_{\phi_i,n}$ are constant base volatility parameters and all other parameters are constants, described in Detemple, Garcia, and Rindisbacher (2003, p.413). This functional form implies that processes $\theta_n(t)$ and $\phi_i(t)$ remain at all times in $[-\theta_{n,l}, \theta_{n,u}]$ and $[-\phi_{i,l}, \phi_{i,u}]$, respectively.

In this economy with stocks and bonds, the complete vector of relevant state variables, denoted $Y(t)$ in Section 2, is therefore $(3D-2K)$-dimensional. It is composed of the spot rate $r(t)$, the K market prices of risk on equities $\{\theta_n(t)\}_{n=1,...,K}$, the $D-K$ market prices on bonds $\{\phi_i(t)\}_{i=K+1,...,D}$, and auxiliary state variables $\{\gamma_i(t)\}_{i=K+1,...D}$ and $\{\chi_i(t)\}_{i=K+1,...D-1}$.

4.2 Consumption Plans and Portfolio Policies

The environment presented in the preceding Subsection provides the adequate expression for $\xi(t)$, the pricing kernel in this stocks-and-bonds economy.[14] It is

$$\begin{aligned}\xi(t) \;&=\; \exp\left\{-\int_0^t r(s)ds - \Sigma_{n=1}^{K}\int_0^t \theta_n(s)dW_n(s) - \frac{1}{2}\Sigma_{n=1}^{K}\int_0^t (\theta_n(s))^2\, ds\right\} \\ &\quad \times \exp\left\{\Sigma_{i=K+1}^{D}\int_0^t \phi_i(s)dW_i(s) - \frac{1}{2}\Sigma_{i=K+1}^{D}\int_0^t (\phi_i(s))^2\, ds\right\}. \qquad (45)\end{aligned}$$

Recall from Section 2 that the optimal consumption plan for an individual whose preferences over *planning horizon* $[0,T]$ – with $T < \tau$ – are represented by the von Neuman-Morgenstern expected utility function of Equation (5) subject to the static Arrow-Debreu budget constraint of Equation (6) is captured by

$$(c^*(t), X^*(T)) = (I_1(\lambda(k)\xi(t)), I_2(\lambda(k)\xi(T))),$$

where $I_1(.)$ and $I_2(.)$ are the respective inverse marginal utility functions.

In the stocks-and-bonds economy of this Section, the discounted wealth process, $\xi(t)X(t)$, follows

$$\begin{aligned}d\left(\xi(t)X(t)\right) \;&=\; -\xi(t)c(t)dt + \xi(t)X(t)\Sigma_{d=1}^{K}\left[\Sigma_{n=1}^{K}\pi_n(t)\sigma_{S_n,d}(t) - \theta_d(t)\right]dW_d(t) \\ &\quad +\xi(t)X(t)\Sigma_{d=K+1}^{D}\left[\Sigma_{i=K+1}^{D}\pi_i(t)\sigma_{B_i,d}(t) + \phi_d(t)\right]dW_d(t). \qquad (46)\end{aligned}$$

---

[14] Recall that bond premia are decreasing in the forward-rate-driving Brownian Motions. Correspondingly, unlike in the case of stocks, positive realizations of the Brownian increments indicate added scarcity in current investor-wealth and thus, higher state prices. Hence, no negative sign accompanies the $\phi_i(t)$ terms in the pricing kernel. This economic reasoning is confirmed by a simple application of Ito's Lemma to check that this pricing kernel is indeed consistent with bond prices' being martingales when $\xi(t)$ is used as the discount factor.

In matrix form, its volatility term is

$$\xi(t)X(t)\left\{\underbrace{\begin{bmatrix} \sigma_{S_1,1}(t) & \ldots & \sigma_{S_K,1}(t) & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & 0 & 0 \\ \sigma_{S_1,K}(t) & \ldots & \sigma_{S_K,K}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{B_{K+1},K+1}(t) & \ldots & \sigma_{B_D,K+1}(t) \\ 0 & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_{B_{K+1},D}(t) & \ldots & \sigma_{B_D,D}(t) \end{bmatrix}}_{\sigma(t)'}\underbrace{\begin{bmatrix} \pi_1(t) \\ \vdots \\ \pi_K(t) \\ \vdots \\ \pi_D(t) \end{bmatrix}}_{\pi(t)} + \underbrace{\begin{bmatrix} -\theta_1(t) \\ \vdots \\ -\theta_K(t) \\ \phi_{K+1}(t) \\ \vdots \\ \phi_D(t) \end{bmatrix}}_{\Theta(t)}\right\}.$$

Thus, in this slightly amended notation, Equation (14) for $\pi(t)$, the portfolio policy at time t becomes

$$\pi(t) = [\sigma(t)']^{-1}\left[-\Theta(t) + \frac{1}{\xi(t)X(t)}\mathcal{D}_t[\xi(t)X(t)]\right], \tag{47}$$

where $\mathcal{D}_t[\xi(t)X(t)]$ is the D-dimensional vector of Malliavin derivatives. From this point on, all developments follow those of Section 2. $\mathcal{D}_t[\xi(t)X(t)]$ is expanded to obtain conditional expectations whose functional dependence on the state variables' Malliavin derivatives is explicit. Simulation-based methods are used to generate a large number of realizations of these building-block Malliavin derivatives and Monte Carlo estimates of the conditional expectations provide the bridge to the explicit portfolio policy and ready implementation. Little insight may be gained from working through the algebra of the general D-Brownian-Motion case. However, an example of modest dimensions suffices to make the approach more transparent. After being exposed to this low-dimension illustration, heftier cases can be handled with relative ease. This example leads to the conclusion of the paper.

4.3 An Illustration: A Two-Factor Economy

In this Subsection, the vector of Brownian Motions is *two-dimensional.* The first Brownian Motion, $W_1(t)$, impacts an equity-index, denoted $S(t)$. This index follows

$$\frac{dS(t)}{S(t)} = (\mu_S(t) - \delta_S(t))dt + \sigma_{S,1}(t)dW_1(t).$$

In the bond sector, the forward-rate curve $f(t, S)$ follows a single-factor process driven by $W_2(t)$. The forward-rate volatility is characterized by

$$\sigma_{f,2}(s, v) = \underbrace{\sigma_r[r(s)]^\beta}_{\sigma_{r,2}(r(s),s)} \exp\left\{-\int_s^v \kappa_2(u)du\right\}. \tag{48}$$

In integral form, the Markovian spot-rate process – See Equation (38), whose structure is somewhat simplified because of the low-dimensionality in this Subsection – is

$$\begin{aligned} r(s) \;&=\; r(0) + \int_0^s \left[\frac{\partial f(0,u)}{\partial u} + \kappa_2(u)\left[f(0,u) - r(u)\right] + \gamma_2(u) - \sigma_r[r(u)]^\beta \phi_2(u)\right] du \\ &\quad + \int_0^s \sigma_r[r(u)]^\beta dW_2(u), \end{aligned} \tag{49}$$

where

$$\gamma_2(u) = \int_0^u \sigma_{f,2}^2(v, u)dv, \tag{50}$$

The market-price-of-risk process associated with the equity index, $\theta(t)$, is characterized by

$$\theta(s) = \theta(0) + \int_0^s \left[\kappa_\theta(\overline{\theta} - \theta(u)) + \mu_\theta^r(r(u), \theta(u))\right] du + \int_0^s \sigma_{\theta,1}(\theta(u)) dW_1(u), \tag{51}$$

and the market-price-of-risk process corresponding to the bond market, $\phi(t)$, follows

$$\phi(s) = \phi(0) + \int_0^s \left[\kappa_\phi(\overline{\phi} - \phi(u)) + \mu_\phi^r(r(u), \phi(u))\right] du + \int_0^s \sigma_{\phi,2}(\phi(u)) dW_2(u), \tag{52}$$

where the short-hand notation – $\mu_\theta^r(r(u), \theta(u))$, $\mu_\phi^r(r(u), \phi(u))$, $\sigma_{\theta,1}(\theta(u))$, and $\sigma_{\phi,2}(\phi(u))$ – stands for the same expressions as above.

Now that integral forms for all four state variables – $r(s)$, $\gamma_2(s)$, $\theta(s)$, and $\phi(s)$ – have been established, the task is to obtain their Malliavin derivatives.

4.3.1 Malliavin Derivatives of the Four State Variables

Consider the spot rate, $r(t)$. Following the rules of Malliavin Calculus presented in the Appendix, its Malliavin derivative is $\mathcal{D}_t r(s) = [\mathcal{D}_t^1 r(s), \mathcal{D}_t^2 r(s)]'$,[15] whose element, $\mathcal{D}_t^i r(s)$, $i \in \{1, 2\}$, is

$$\begin{aligned} \mathcal{D}_t^i r(s) \quad &= \quad \int_t^s \left[\left\{-\kappa_2(u) - \sigma_r \beta [r(u)]^{\beta-1} \phi_2(u)\right\} \mathcal{D}_t^i r(u) + \mathcal{D}_t^i \gamma_2(u) - \sigma_r [r(u)]^\beta \mathcal{D}_t^i \phi_2(u)\right] du \\ &\quad + \int_t^s \sigma_r \beta [r(u)]^{\beta-1} \mathcal{D}_t^i r(u) dW_2(u) + 1_{\{i=2\}} \sigma_r [r(t)]^\beta, \end{aligned} \tag{53}$$

---

[15] Recall that when the vector of Brownian Motions is multi-dimensional, as is the case here ($D = 2$), Malliavin derivatives are vectors of identical dimensions. In the notation of Section 2, $\mathcal{D}_t^i r(s) = \lim_{\varepsilon \to 0} \frac{r(s, W + \varepsilon \vec{1}^{(i)}_{[t,s]}) - r(s, W)}{\varepsilon}$, where the dependence of $r(s)$ on Brownian Motion i is emphasized. Although in the simple example of this Subsection, certain elements may be identically zero, the possibility of cross-correlations in more general settings makes it imperative to keep in mind the multiple dimensions of each derivative.

which, in differential form, is

$$\begin{aligned} d(\mathcal{D}_t^i r(s)) &= \left[\left\{-\kappa_2(s) - \sigma_r\beta[r(s)]^{\beta-1}\phi_2(s)\right\}\mathcal{D}_t^i r(s) + \mathcal{D}_t^i\gamma_2(s) - \sigma_r[r(s)]^{\beta}\mathcal{D}_t^i\phi_2(s)\right] ds \\ &\quad +\sigma_r\beta[r(s)]^{\beta-1}\mathcal{D}_t^i r(s)dW_2(s), \end{aligned} \tag{54}$$

subject to initial condition $\mathcal{D}_t^i r(t) = 1_{\{i=2\}}\sigma_r[r(t)]^{\beta}$.

Next, the same set of rules is applied to $\gamma_2(t)$ to obtain

$$\begin{aligned} \mathcal{D}_t^i\gamma_2(s) &= \int_t^s 2\beta\left[\sigma_r^2[r(u)]^{2\beta-1}\exp\{-2\int_u^s \kappa_i(v)dv\}\right]\mathcal{D}_t^i r(u)du \\ &= \int_t^s 2\beta\frac{1}{r(u)}\left[\sigma_r^2[r(u)]^{2\beta}\exp\{-2\int_u^s \kappa_i(v)dv\}\right]\mathcal{D}_t^i r(u)du. \end{aligned} \tag{55}$$

In differential form, Equation (55) is

$$d(\mathcal{D}_t^i\gamma_2(s)) = \frac{2\beta\sigma_r^2}{r(s)}[r(s)]^{2\beta}\mathcal{D}_t^i r(s)ds, \tag{56}$$

whose initial condition is $\mathcal{D}_t^i\gamma_2(t) = 0$.

For $\theta(t)$, one gets

$$\begin{aligned} \mathcal{D}_t^i\theta(s) &= \int_t^s \left[\left\{\frac{\partial\mu_\theta^r(r(u),\theta(u))}{\partial\theta} - \kappa_\theta\right\}\mathcal{D}_t^i\theta(u) + \frac{\partial\mu_\theta^r(r(u),\theta(u))}{\partial r}\mathcal{D}_t^i r(u)\right] du \\ &\quad + \int_t^s \frac{\partial\sigma_{\theta,1}(\theta(u))}{\partial\theta}\mathcal{D}_t^i\theta(u)dW_1(u) + 1_{\{i=1\}}\sigma_{\theta,1}(\theta(t)), \end{aligned} \tag{57}$$

or,

$$
\begin{aligned}
d(\mathcal{D}_t^i\theta(s)) &= \left[\left\{\frac{\partial\mu_\theta^r(r(s),\theta(s))}{\partial\theta} - \kappa_\theta\right\}\mathcal{D}_t^i\theta(s) + \frac{\partial\mu_\theta^r(r(s),\theta(s))}{\partial r}\mathcal{D}_t^i r(s)\right] ds \\
&\quad + \frac{\partial\sigma_{\theta,1}(\theta(s))}{\partial\theta}\mathcal{D}_t^i\theta(s)dW_1(s), \qquad (58)
\end{aligned}
$$

subject to $\mathcal{D}_t^i\theta(t) = 1_{\{i=1\}}\sigma_{\theta,1}(\theta(t))$, where the derivatives of the μ- and σ-terms in the above expressions are standards derivatives of the relevant expression in Section 4.1. Equivalent expressions are easily derived for the elements of $d(\mathcal{D}_t\phi(s))$, the Malliavin derivative of the market price of risk on bonds. The above system of linear stochastic differential equations is hoped to be transparent enough to indicate how standard Monte Carlo methods apply.

4.3.2 The Explicit Portfolio Policy

In any particular application, specific functional forms for the utility functions must be provided. Utility functions that display either constant relative risk aversion or, more generally, hyperbolic absolute risk aversion have been widely used since the pioneering works of Merton (1969, 1971) and Samuelson (1969) on optimal lifetime consumption-portfolio policies. For the present illustration, the exposition's clarity does not seem to suffer excessively from retaining the generic utility functions of the earlier Sections. In the current environment, Equation (45) simplifies to

$$
\xi(t) = \exp\left\{-\int_0^t r(s)ds - \int_0^t \theta(s)dW_1(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds + \int_0^t \phi(s)dW_2(s) - \frac{1}{2}\int_0^t \phi(s)^2 ds\right\}. \qquad (59)
$$

Along the same lines as above, one gets

$$\begin{bmatrix} \pi_1(t) \\ \pi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\sigma_B(t)}{\sigma_B(t)\sigma_S(t)} & 0 \\ 0 & \frac{\sigma_S(t)}{\sigma_B(t)\sigma_S(t)} \end{bmatrix} \left[\frac{1}{\xi(t)X(t)}\mathcal{D}_t[\xi(t)X(t)] - \begin{bmatrix} -\theta(t) \\ \phi(t) \end{bmatrix} \right], \tag{60}$$

where

$$\begin{aligned} \mathcal{D}_t[\xi(t)X(t)] &= E_t \int_t^T \{I_1(\lambda\xi(s)) + I_1'(\lambda\xi(s))\lambda\xi(s)\}\mathcal{D}_t\xi(s)ds \\ &\quad + E_t\{I_2(\lambda\xi(T)) + I_2'(\lambda\xi(T))\lambda\xi(T)\}\mathcal{D}_t(\xi(T)), \end{aligned} \tag{61}$$

and $\mathcal{D}_t\xi(s)$ is a two-dimensional vector with element $\mathcal{D}_t^i\xi(s)$

$$\begin{aligned} \mathcal{D}_t^i\xi(s) &= -\xi(s)\left[\int_t^s \mathcal{D}_t^i r(v)dv + \int_t^s \mathcal{D}_t^i\theta(v)dW_1(v) + 1_{\{i=1\}}\theta(t) + \int_t^s \theta(v)\mathcal{D}_t^i\theta(v)dv\right] \\ &\quad + \xi(s)\left[\int_t^s \mathcal{D}_t^i\phi(v)dW_2(v) + 1_{\{i=2\}}\phi(t) - \int_t^s \phi(v)\mathcal{D}_t^i\phi(v)dv\right]. \end{aligned} \tag{62}$$

The generalization to larger problems follows the same route as the example provided in this Subsection.

5 Conclusion

After reviewing the methodology developed by Detemple, Garcia, and Rindisbacher for the determination of explicit lifetime consumption-portfolio plans, this paper generalized the applicability of their results to economies with a Markovian Heath-Jarrow-Morton bond sector characterized by $3N$ state variables. In doing so, the flexibility of the Detemple-Garcia-Rindisbacher approach was further demonstrated. A two-dimensional example was presented to expose as transparently as possible the steps to be taken for implementation.

It goes without saying that excellent numerical work and econometric work must be accomplished in parallel to render the theory effective. However, the author believes that these theoretical advances are highly relevant to the development by financial firms of a rich menu of optimal life-cycle and retirement products. Going in this direction, the generalized results presented above should be combined with restrictions on wealth processes presented in Bodie, Detemple, Otruba, and Walter (2004) that are required in an environment prone to moral hazard. As in Bodie, Detemple, Otruba, and Walter (2004) and Bodie, Merton, and Samuelson (1992), issues of habit formation and work-supply flexibility should also be incorporated. These considerations were not introduced here to allow for a clear and focused introduction of a bond sector in the Detemple-Garcia-Rindisbacher methodology. Nevertheless, these are needed elements for the development of integrated life-cycle financial products.

6 Appendix: Needed Rules of Malliavin Calculus

These formulas appear in Detemple, Garcia, and Rindisbacher (2003, 2005). They are compiled in this Appendix for ease. Here, the Brownian Motion is taken to be one-dimensional. The generalization to the multi-dimensional case is direct from Equation (11).

1. (Wiener Integral) If $h(t)$ is a non-stochastic function of time and $F(W) = \int_0^T h(s)dW(s)$, then $\mathcal{D}_t F(W) = h(t)$.

2. (Riemann Integral) If $h(t)$ is a function of time and of the path of the Brownian Motion up to time t and $F(W) = \int_0^T h(s)ds$, then $\mathcal{D}_t F(W) = \int_t^T \mathcal{D}_t h(s)ds$.

3. (Ito Integral) If $h(t)$ is a path-dependent function of time and $F(W) = \int_0^T h(s)dW(s)$,

$\mathcal{D}_t F(W) = h(t) + \int_t^T \mathcal{D}_t h(s) dW(s)$.

4. (Chain Rule) If $F = (F_1, ..., F_n)'$ are Brownian functionals and $\phi(F)$ is a differentiable function of F, then $\mathcal{D}_t \phi(F) = \Sigma_{i=1}^n \phi_i'(F) \mathcal{D}_t F_i(W)$, where $\phi_i'(F)$ is the derivative with respect to the ith argument of ϕ.

5. (Commuting Operators) The Conditional Expectation operator E_s and the Derivative Operator $\mathcal{D}_v$ commute.

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